Division **M**

Mathematical Olympiads

November 13, 2013

for Elementary & Middle Schools

Contest **1**

1A *Time: 3 minutes*

A circle and a square are drawn on the same flat surface. What is the greatest number of points in their intersection?

1B *Time: 4 minutes*

The letters in each word on the sign "MY CANDY" are cycled separately as shown.

The next correct spelling of both words appears on line n. What is the smallest value of n?

1. MY CANDY
2. YM ANDYC
3. MY NDYCA

⋮

n. MY CANDY

1C *Time: 6 minutes*

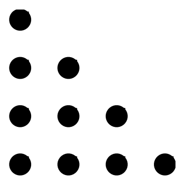
A palindrome is a whole number which reads the same forward as backwards. For example: 17571 is a palindrome, but 1234 is not a palindrome. Find the least sum of two palindromic years after the year 2013, such that this sum is not a palindrome.

1D *Time: 6 minutes*

The equation $8(Ax-7)-5(2x-1)=14x+B$ is true for every number that can replace the letter x. Find $A + B$.

1E *Time: 7 minutes*

In the figure shown, 10 points lie in four equally spaced columns, and four equally spaced rows. [Consecutive rows are 1 cm apart, and consecutive columns are 1 cm apart.] Find the probability, as a fraction in lowest terms, that three points selected at random lie in a straight line.



Please fold over on line. Write answers on back.

Division	Mathematical Olympiads	Contest
M	**November 13, 2013**	**1**
	for Elementary & Middle Schools	

1A *Student Name and Answer*

1B *Student Name and Answer*

1C *Student Name and Answer*

1D *Student Name and Answer*

1E *Student Name and Answer*

Please fold over on line. Write answers in these boxes.

Division **M**

Mathematical Olympiads

December 11, 2013

for Elementary & Middle Schools

Contest **2**

2A *Time: 4 minutes*

Lynda can run a mile in 8 minutes, while Noelle can run a mile in 10 minutes. At these rates, how many minutes will it take Noelle to complete a race that takes Lynda 20 minutes to complete?

2B *Time: 4 minutes*

A rectangle is partitioned into four smaller rectangles of areas 6, 10, 21, and 35 square inches by drawing only two straight lines. Find the number of inches in the perimeter of the original rectangle.

2C *Time: 5 minutes*

Pencils cost 15 cents each and erasers cost 8 cents each. Stephanie buys some pencils and some erasers for a total cost of $1.53. How many erasers did she purchase?

2D *Time: 6 minutes*

Using scientific notation, the difference $\left(2.3\times10^{12}\right)-\left(1.7\times10^{10}\right)=a\times10^{N}$, where $1 \le a < 10$ and N is an integer. Find the value of a.

2E *Time: 7 minutes*

The improper fraction $\frac{N}{6}$ is expressed as the sum of several distinct proper fractions each in lowest terms and each with denominator 6 or less. What is the greatest possible value of the whole number N?

Please fold over on line. Write answers on back.

Division	Mathematical Olympiads	Contest
M	**December 11, 2013** *for Elementary & Middle Schools*	**2**

2A *Student Name and Answer*

2B *Student Name and Answer*

2C *Student Name and Answer*

2D *Student Name and Answer*

2E *Student Name and Answer*

Please fold over on line. Write answers in these boxes.

Division **M** — Mathematical Olympiads **January 15, 2014** for Elementary & Middle Schools — Contest **3**

3A *Time: 3 minutes*

At its last competition, the 12 students of the Centerville Math Team had the following distribution of scores: 1 student scored 5 points, 3 each scored 4 points, 4 each scored 3 points, 1 student scored 2 points, 1 student scored 1 point, and 2 students scored 0 points. To the nearest tenth, what was the mean (average) individual score?

3B *Time: 5 minutes*

Write as a fraction in lowest terms: $\dfrac{\left(\frac{2}{9}\right)^3+\left(\frac{2}{9}\right)^2}{\left(\frac{1}{3}-\frac{1}{9}\right)^2\times 1\frac{5}{6}}$.

3C *Time: 5 minutes*

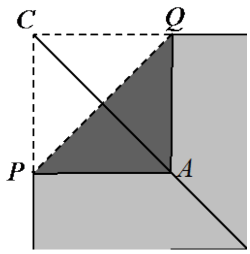
Find the greatest prime factor of the sum of 10! and 8!.

[Note: $n! = n\times(n-1)\times(n-2)\times\ldots\times 2\times 1$ and is read as "*n* factorial".

For example: 6! = 6 × 5 × 4 × 3 × 2 × 1.]

3D *Time: 6 minutes*

A square sheet of paper has an area of 60 square units. The paper is folded so that corner *C* lands on point *A* as shown. The area of the lightly shaded L-shaped region is three times the area of the folded over darkly shaded region, ΔPAQ. The length of the "crease" $\overline{PQ}$ is $\sqrt{N}$. Find *N*.



3E *Time: 7 minutes*

Two trains, each traveling at 60 kph [1 kilometer/minute] are on parallel tracks heading toward one another. One train is one-half kilometer long and the other train is two-thirds kilometer long. The fronts of the two trains pass the same point *A* at the same time. How many seconds does it take for the two trains to completely pass each other?

Please fold over on line. Write answers on back.

Division	Mathematical Olympiads	Contest
M	January 15, 2014 *for Elementary & Middle Schools*	**3**

3A *Student Name and Answer*

3B *Student Name and Answer*

3C *Student Name and Answer*

3D *Student Name and Answer*

$N =$

3E *Student Name and Answer*

Seconds

Please fold over on line. Write answers in these boxes.

Division **M**

Mathematical Olympiads

February 12, 2014

for Elementary & Middle Schools

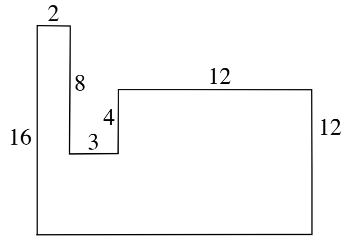
Contest **4**

4A *Time: 4 minutes*

The accompanying figure is composed entirely of straight line segments meeting at right angles with selected lengths shown.

Find the perimeter of this figure.

Note: the figure is NOT drawn to scale.



4B *Time: 4 minutes*

Express as a fraction in lowest terms: $\dfrac{3-6+9-12+ \ldots\ldots +27-30}{5-10+15-20+ \ldots\ldots +45-50}$.

4C *Time: 6 minutes*

Each of the digits 3, 5, 6, and 9 is used exactly once to in the following expression: $\dfrac{\square}{\square}-\dfrac{\square}{\square}$. What is the numeric value of the least positive difference of these two fractions in lowest terms?

4D *Time: 6 minutes*

What is the probability that a randomly selected three-digit positive integer has no repeated digits? Express your answer in decimal form.

4E *Time: 7 minutes*

A wood cube is painted white and cut into smaller $1 \times 1 \times 1$ "cubies". There are forty-eight $1 \times 1 \times 1$ "cubies" with exactly two faces painted white. How many of the $1 \times 1 \times 1$ "cubies" have exactly one face painted white?

Please fold over on line. Write answers on back.

Division **M** | **Mathematical Olympiads** for Elementary & Middle Schools | **February 12, 2014** | Contest **4**

4A Student Name and Answer

4B Student Name and Answer

4C Student Name and Answer

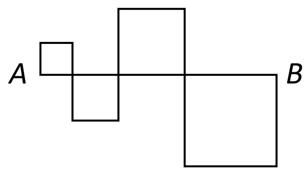
4D Student Name and Answer

4E Student Name and Answer

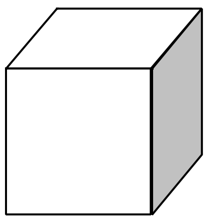
Please fold over on line. Write answers in these boxes.

5A *Time: 3 minutes*
Three vertices of a parallelogram have coordinates A(–3, – 5), B(– 1, 2), and C(11, 4). The fourth vertex lies in quadrant IV. Find its coordinates expressed as an ordered pair.

5B *Time: 5 minutes*
A single strand of wire is bent into four squares as seen in the diagram. If the distance from *A* to *B* is 12, find the length of the strand of wire used to construct the figure.



5C *Time: 6 minutes*
The cube in the diagram is cut into eight identical smaller cubes whose total surface area is *K* times the surface area of the original larger cube. Find *K*.



5D *Time: 6 minutes*
Find the sum of all integer values of *x* for which: $(x-5)^{x+2}=1$.

5E *Time: 7 minutes*
Seven cards are face-down on a table. Each of the cards has exactly one of the numbers 1, 2, 4, 6, 7, 8 and 10 facing down. No two cards have the same number on it. Two cards are randomly selected and turned over. What is the probability that their sum is a multiple of 3?

[Express your answer as a fraction in lowest terms]

Please fold over on line. Write answers on back.

Division	Mathematical Olympiads	Contest
M	**March 12, 2014**	**5**
	for Elementary & Middle Schools	

5A *Student Name and Answer*

5B *Student Name and Answer*

5C *Student Name and Answer*

5D *Student Name and Answer*

5E *Student Name and Answer*

Please fold over on line. Write answers in these boxes.

Division **M**

Mathematical Olympiads for Elementary & Middle Schools

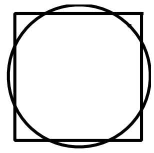
November 13, 2013

Contest **1**

SOLUTIONS AND ANSWERS

1A METHOD 1: *<u>Strategy</u>: Create a diagram.*

Count the intersections. **The diagram shows that the maximum number of intersections for a circle and a square is 8.**



METHOD 2: *<u>Strategy</u>: Consider intersections of a circle and a line.*

Each side of the square can intersect the circle in at most 2 points. Since there are 4 sides to the square the maximum number of intersections between the square and the circle is $2 \times 4 = 8$.

1B METHOD 1: *<u>Strategy</u>: Look for patterns.*

The word MY occurs in this form on every odd-numbered row. The word CANDY occurs in row 1 and then every 5 rows after that. Therefore it occurs in row 1, 6, 11, 16, and so on. The first time after row 1 that both occur in the correct order is row 11. **Thus n = 11.**

METHOD 2: *<u>Strategy</u>: Use the concept of LCM.*

MY has two letters and CANDY has 5 letters. The least common multiple (LCM) of 2 and 5 is 10. Therefore it will take 10 additional lines for the words to appear in the same order as the row 1. Therefore the row number when this occurs is $n = 11$.

FOLLOW-UP: The letters in the sign "FREE CHACHA LESSONS" are cycled separately within each word and placed in a numbered row as in the original problem. So row 2 reads "REEF HACHAC ESSONSL" and row 3 reads "EEFR ACHACH SSONSLE". What is the smallest row number after row 1 when the sign will once again correctly read "FREE CHACHA LESSONS"? [LCM(4, 6, 7) + 1 = 85]

1C *<u>Strategy</u>: Observe the characteristics when adding two palindromic numbers together.*

The sum of two palindromic numbers is also palindromic when the sum of each pair of digits is less than 10. Therefore we are looking for two palindromic years after 2013 for which there will be a sum for two corresponding digits that is greater than 10. The next few palindromic years are: 2112, 2222, 2332, 2442, 2552, 2662, 2772, 2882, 2992, etc. Observe that 2112 + 2992 = 2222 + 2882 = 2332 + 2772 = 2442 + 2662 = 5104. **Hence, the least non-palindromic sum for two palindromic years after 2013 is 5104.**

FOLLOW-UP: What is the smallest sum, greater than 10,000, formed by adding a 3-digit palindromic number and a 4-digit palindromic number? [10,001]

1A	8
1B	11
1C	5104
1D	-48
1E	$\frac{1}{8}$

1D *Strategy: Apply the distributive property and combine like terms.*

$8(Ax - 7) - 5(2x - 1) = 14x + B$

$8Ax - 56 - 10x + 5 = 14x + B$

$(8A - 10)x - 51 = 14x + B$

Since this last equation is true for all x, $8A - 10 = 14$ and $B = -51$. Solve for A by adding 10 to both sides of the equation and then divide both results by 14. This results in $8A = 24$ and $A = 3$.
Hence $A + B = 3 + (-51) = -48$.

FOLLOW-UP: The equation $2(x - A) + 3(x + B) = Cx$ is true for all values of x. If A, B, and C are positive integers, what is the least possible value for A + B + C? [10]

1E METHOD 1: *Strategy: Count the cases using columns, rows, and diagonals.*

Columns: The first column contains 4 dots and there are ${}_4C_3 = 4$ ways to choose three of the dots to form a line. In the second column there are only 3 dots so only one way to select them to form a line. In the third and fourth columns there are only 2 dots and 1 dot so no lines of 3 dots are possible. Thus there are 5 ways to form a 3 dotted line in the columns.
Rows: The exact same patterns exist in the 4 rows. The top 2 rows have no possible 3 dotted lines. The row with 3 dots has 1 and the row with 4 dots has 4 for a total of 5 more ways.
Diagonals: There are the same number of dots in the 4 diagonals as there were in the columns and the rows. Thus there are 5 more ways to connect 3 dots to form a line.
The total number of lines is 5 + 5 + 5 = 15. The total number of ways to choose 3 dots out of 10 dots is ${}_{10}C_3 = (10 \times 9 \times 8)/(3 \times 2 \times 1) = 120$ ways. **Therefore the probability that 3 chosen dots lie in a straight line is 15/120 = 1/8.**

METHOD 2: *Strategy: Replace the dots with letters and make a list of all possible 3-dot lines.*

Use the diagram to verify this list of 15 sets of 3-dot lines:
Vertical Column: ABD, ABG, ADG, BDG, CEH
Horizontal Row: GHI, GHJ, GIJ, HIJ, DEF
Diagonal: ACF, ACJ, AFJ, CFJ, BEI

A			
B	C		
D	E	F	
G	H	I	J

There are 120 groups of 3 dots calculated as follows:
Any one of the 10 can be the first dot, followed by any of 9 remaining dots, followed by one of the remaining 8 dots. Therefore $10 \times 9 \times 8 = 720$ might appear to be the correct number. However choosing dot A then B then C is the same as choosing B, A, and C. So we must divide 720 by 6, all the different arrangements of any 3 given dots. 720/6 = 120.
The probability of choosing 3 dots that form a line is 15/120 = 1/8.

FOLLOW-UP: Use the table provided to select a pair of numbers that are adjacent in any row, column or along any diagonal. What is the probability the sum of the two numbers is even? [18/42 = 3/7]

1	2	1	2
2	1	2	1
1	2	1	2
2	1	2	1

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SOLUTIONS AND ANSWERS

2A METHOD 1: *Strategy: Set up a proportion.*

If Noelle can run the race in *n* minutes then 20/8 = *n*/10. Multiply both sides of the equation by 10 to get *n* = 200/8 = 25. **It takes 25 minutes for Noelle to finish the race.**

METHOD 2: *Strategy: Simplify the given rates.*

Running one-half mile takes Lynda 4 minutes and Noelle 5 minutes. At the same rates, in 20 minutes, Lynda can run 5 half-miles. To run 5 half-miles, Noelle needs 25 minutes.

FOLLOW-UP: A rectangular solid is 8 cm by 10 cm by 12 cm. Another rectangular solid having the same volume is x cm by 15 cm by 16 cm. Which solid has the larger surface area and by how much? [4 × 15 × 16, 136 sq cm]

2B *Strategy: Use the fact that areas that have common factors will share a side.*

Construct a rectangle and partition as given.
Areas of 6 and 10 have a common factor of 2.
Areas of 6 and 21 have a common factor of 3.
Areas of 21 and 35 have a common factor of 7.
Areas of 10 and 35 have a common factor of 5.
The perimeter of the rectangle is 2(8 + 9) = 34

	2	7
3	6	21
5	10	35

FOLLOW-UP: What is the difference between the largest and smallest possible areas of a rectangle with integral sides if the rectangle has a perimeter of 60 inches? [196 sq in]

2C METHOD 1: *Strategy: Use the fact that every multiple of 15 ends in 0 or 5.*

If Stephanie buys 1 eraser, she then would spend \$1.45 for pencils. However, \$1.45 is not a multiple of 3¢ and therefore not a multiple of 15¢. At 8¢ each for erasers, the cost of the pencils will always be odd, ending in 5. Buying 6 erasers leaves \$1.53 – \$.48 = \$1.05 for pencils, which is a multiple of 15¢. **Stephanie purchased 6 erasers.**

METHOD 2: *Strategy: Use a table to find multiples of 15 and 8 whose sum is 153.*

Multiples of 15	15	30	45	60	75	90	105
153 minus top row	138	123	108	93	78	63	48
Is the middle row a multiple of 8?	N	N	N	N	N	N	Y

Stephanie purchased 48¢ ÷ 8¢ = 6 erasers.

FOLLOW-UP: Harry purchased some 12¢ stamps and some 39¢ stamps for a total of \$3.69. What are all of his possible purchases? [(21@12¢ & 3@39¢) or (8@12¢ & 7@39¢)]

2A	25
2B	34
2C	6
2D	2.283
2E	33

2D METHOD 1: *<u>Strategy</u>: Factor out the GCF and simplify.*

$(2.3 \times 10^{12}) - (1.7 \times 10^{10}) = 10^{10}(2.3 \times 10^2 - 1.7)$
$= 10^{10}(230 - 1.7)$
$= 10^{10}(228.3)$
$= 10^{12}(2.283)$

Since $1 \le a < 10$ and $a \times 10^N = 2.283 \times 10^{12}$, $a = 2.283$.

METHOD 2: *<u>Strategy</u>: Do the indicated arithmetic.*

$2.3 \times 10^{12} = 2.3 \times 1{,}000{,}000{,}000{,}000 = 2{,}300{,}000{,}000{,}000$
$1.7 \times 10^{10} = 1.7 \times 10{,}000{,}000{,}000 = 17{,}000{,}000{,}000$
Subtract to get $2{,}283{,}000{,}000{,}000 = 2.283 \times 10^{12}$, thus $a = 2.283$.

FOLLOW-UP: What is the value of $5^2 \times 5^6 \times 5^{10} \times 5^{14} \div 5^{16} \div 5^{18}$? [1/25]

2E *<u>Strategy</u>: List pairs of fractions whose denominators are 2, 3, 4, 5, and 6.*

Let N represent the sum of the numerators when the denominator is 6.

$$\frac{N}{6} = \frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{4}{5}\right) + \left(\frac{2}{5} + \frac{3}{5}\right) + \left(\frac{1}{6} + \frac{5}{6}\right)$$

$$= 5\frac{1}{2} = \frac{11}{2} = \frac{33}{6}$$

Therefore the largest value of the numerator, N, is 33.

FOLLOW-UP: If $\frac{1}{x} + \frac{1}{y} = \frac{2}{5}$ *and* $\frac{1}{x} - \frac{1}{y} = \frac{1}{5}$ *find the value of x divided by y.* [1/3]

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Division **M** — Mathematical Olympiads — January 15, 2014 — for Elementary & Middle Schools — Contest **3**

SOLUTIONS AND ANSWERS

3A METHOD 1: *Strategy: Use the definition of mean (average).*

Add the scores and divide by 12: (5 + 4 + 4 + 4 + 3 + 3 + 3 + 3 + 2 + 1 + 0 + 0)÷12 = 32÷12. **Since 32 ÷ 12 = 2.666…, the mean to the nearest tenth is 2.7.**

METHOD 2: *Strategy: Use a quick way to add the scores.*

1(5) + 3(4) + 4(3) + 1(2) + 1(1) + 2(0) = 32 and 32/12 = 2.666… = 2.7 to the nearest tenth.

FOLLOW-UP: If there were two additional mathletes who had identical scores, what would each of them scored to raise the mean to be exactly 3? [5 each]

3A 2.7

3B METHOD 1: *Strategy: Factor and simplify numerator and denominator separately.*

$$\frac{\left(\frac{2}{9}\right)^3+\left(\frac{2}{9}\right)^2}{\left(\frac{1}{3}-\frac{1}{9}\right)^2\times 1\frac{5}{6}}=\frac{\left(\frac{2}{9}\right)^2\left(\frac{2}{9}+1\right)}{\left(\frac{2}{9}\right)^2\left(\frac{11}{6}\right)}=\frac{\left(\frac{11}{9}\right)}{\left(\frac{11}{6}\right)}=\frac{11}{9}\times\frac{6}{11}=\frac{2}{3}$$. **The fraction in lowest terms is 2/3.**

METHOD 2: *Strategy: Simplify numerator and denominator and then divide.*

Numerator: $\left(\frac{2}{9}\right)^3+\left(\frac{2}{9}\right)^2=\frac{8}{729}+\frac{4}{81}=\frac{8}{729}+\frac{36}{729}=\frac{44}{729}$.

Denominator: $\left(\frac{1}{3}-\frac{1}{9}\right)^2\times 1\frac{5}{6}=\left(\frac{2}{9}\right)^2\times\frac{11}{6}=\frac{4}{81}\times\frac{11}{6}=\frac{44}{486}$.

Divide: $\frac{44}{729}\div\frac{44}{486}=\frac{44}{729}\times\frac{486}{44}=\frac{2}{3}$

3B $\frac{2}{3}$

3C *Strategy: Apply the definition of factorial.*

Since 10! = 10 × 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 and 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1, we can deduce that 10! = 10 × 9 × 8!.

10! + 8! = 10 × 9 × 8! + 8!
= 8!(10 × 9 + 1)
= 8!(91)
= 8! × 7 × 13

Every prime factor of 8! is less than 8. **The greatest prime factor is 13.**

3C 13

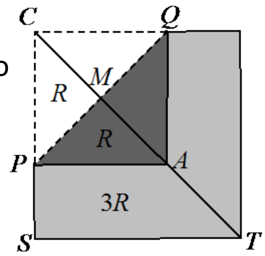
3D 48

3E 35

3D METHOD 1: *<u>Strategy</u>: Apply area formulas.*

Let K equal the area of ΔPAQ. Then the area of square $CPAQ$ is $2K$ and the areaa of the L-shaped region is $3K$. The total area of the big square is 60 so $60 = 2K + 3K = 5K$. K is 12 and the area of square $CPAQ = 24$. Since the area of a square is s^2, where s represents the length of a side of the square, $s = \sqrt{24}$. Use the Pythagorean theorem, $\left(\sqrt{24}\right)^2 + \left(\sqrt{24}\right)^2 = (PQ)^2$ so $(PQ)^2 = 48$ and $PQ = \sqrt{48}$. **Therefore $N = 48$.**

METHOD 2: *<u>Strategy</u>: Apply the concept that the ratio of areas of similar triangle equals the ratio of the squares of the corresponding sides.*



Let the area of $\Delta PMA = R$. Then the area of ΔCMP is also R and the area of quadrilateral $PATS$ is $3R$. Both ΔPMA and ΔCST are right isosceles triangles so they are similar to each other and the ratio of their areas equals the ratio of the squares of a pair of corresponding sides. That is $\frac{A_{\Delta PMA}}{A_{\Delta CST}} = \frac{(MP)^2}{(ST)^2}$. Thus

$\frac{R}{5R} = \frac{(MP)^2}{\left(\sqrt{60}\right)^2}$ and $(MP)^2 = \frac{60}{5} = 12$ so $MP = \sqrt{12}$. It follows that $PQ = 2\sqrt{12} = \sqrt{4}\sqrt{12} = \sqrt{48}$.

Consequently $N = 48$.

3E METHOD 1: *<u>Strategy</u>: Use the formula distance = rate $\times$ time.*

Since the shorter train is 1/2 kilometer long and it travels at 1 km/min, it will take 1/2 minute to completely pass point A. The longer train is 2/3 km long so it still has an additional 1/6 km (2/3 – 1/2 = 1/6) before it too passes point A. Since both trains are traveling at 1 km/min and each has to travel 1/2 of the 1/6 km they must each travel 1/12 km. This will take an additional 1/12 of a minute or 5 seconds. **The total amount of time needed for the two trains to pass is 1/2 minute + 5 seconds or 35 seconds.**

METHOD 2: *<u>Strategy</u>: Consider one train moving at 0 km/h and the other at 120 km/h.*

If we keep one train from moving and let the other move at the sum of their rates, 120 km/h or 2kn per minute (1 km per so seconds), it will take the same amount of time for the one train to pass the other. The total distance the one train must move is 1/2 km + 2/3 km = 7/6 km. This takes 7/6 of 30 seconds or 35 seconds.

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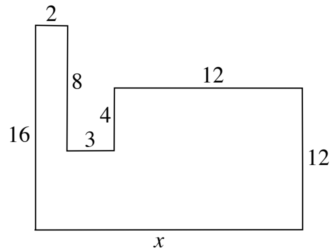
SOLUTIONS AND ANSWERS

4A *Strategy: Find the horizontal then vertical lengths.*

Label the missing length *x*. The top three horizontal lengths must be the same as the bottom length so
x = 2 + 3 + 12 = 17.



The entire perimeter is 16 + 2 + 8 + 3 + 4 + 12 + 12 + 17 = 74.

4A: 74

4B METHOD 1: *Strategy: Factor and cancel.*

$$\frac{3-6+9-12+\;.....\;+27-30}{5-10+15-20+\;.....\;+45-50}=\frac{3\times(1-2+3-4+\;.....\;+9-10)}{5\times(1-2+3-4+\;.....\;+9-10)}=\frac{3}{5}.$$

The fraction in lowest terms is 3/5.

METHOD 2: *Strategy: Add efficiently.*

Group the numbers in both numerator and denominator two at a time. The result is nine –3's in the numerator and nine –5's in the denominator or $\frac{9(-3)}{9(-5)}=\frac{3}{5}$.

4B: $\frac{3}{5}$

4C *Strategy: Find the LCD of all possible fractions.*

Write out the 4 × 3 = 12 possible fractions and then rewrite them with the same denominator: $\frac{3}{5},\frac{3}{6},\frac{3}{9},\frac{5}{6},\frac{5}{9},\frac{6}{9},\frac{9}{6},\frac{9}{5},\frac{6}{5},\frac{9}{3},\frac{6}{3},\frac{5}{3}$. These are the same as $\frac{54}{90},\frac{45}{90},\frac{30}{90},\frac{75}{90},\frac{50}{90},\frac{60}{90},\frac{135}{90},\frac{162}{90},\frac{108}{90},\frac{270}{90},\frac{180}{90},\frac{150}{90}$. When arranged in order we can see that $\frac{45}{90}$ and $\frac{50}{90}$ are the closest. **Therefore the least positive difference in simplest form is $\frac{5}{90}=\frac{1}{18}$.** [$\frac{54}{90}-\frac{50}{90}=\frac{3}{5}-\frac{5}{9}$ violates the problem constraints.]

FOLLOW-UP: Use each of the numbers 3, 5, 6, and 9 exactly once to form the difference of two fractions as in the original problem, but find the largest possible difference and the largest possible product. [8/3 and 6]

4C: $\frac{1}{18}$

4D: .72

4E: 96

4D **METHOD 1**: *Strategy: Use multiplication to generate outcomes.*

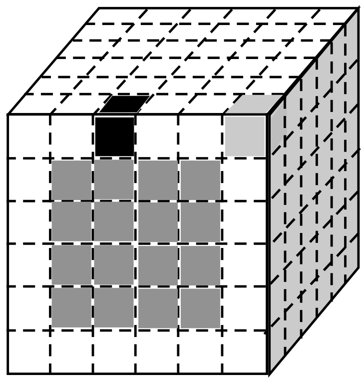
There are $9 \times 10 \times 10 = 900$ three-digit positive integers. There are 9 choices for the hundreds digit (cannot be 0), 9 choices for the tens digit (cannot be the same digit chosen for the hundreds place but it can be 0), and 8 remaining choices for the ones digit. There are $9 \times 9 \times 8 = 648$ non-repeating three-digit positive integers. **The probability of selecting a 3-digit number without repeats is** $\mathbf{\frac{\cancel{9} \times 9 \times 8}{\cancel{9} \times 10 \times 10} = \frac{72}{100} = .72}$**.**

METHOD 2: *Strategy: Apply the fundamental counting principle.*

The hundreds-digit can be any digit except zero. The tens-digit can be any digit but it must be different from the hundreds-digit and the probability of that occurring is $\frac{9}{10}$. The probability that the units-digit is different from the previous two digits is $\frac{8}{10}$. The probability that all three are digits differ is $\frac{9}{10} \cdot \frac{8}{10} = \frac{72}{100} = .72$.

FOLLOW-UP: How many 5-digit codes can be created if the first digit must be a 3, the last digit must be prime and no digits can repeat? [504]

4E *Strategy: Use symmetry and then consider corner, edge, and face (middle) cubies.*



A $1 \times 1 \times 1$ “cubie” with exactly two faces painted white must have come from an edge of the original cube but not a corner (see the very dark colored cubie). Since a cube has 12 edges; $\frac{48}{12} = 4$ of the smaller $1 \times 1 \times 1$ “cubies” must lie on each edge. Then there are 2 more cubies at each corner, so the original cube is $6 \times 6 \times 6$. For each face of the large cube, there are exactly 16 cubies with one white face (colored gray in the diagram). **Since there are 6 faces, there are $6 \times 16 = 96$ cubies with one white face.**

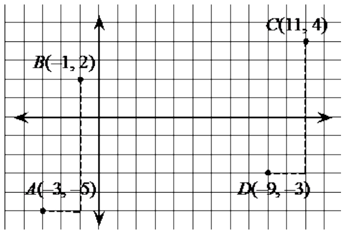
NOTE: Other FOLLOW-UP problems related to some of the above can be found in our two contest problem books and in “Creative Problem Solving in School Mathematics.” Visit www.moems.org for details and to order.

Division M | Mathematical Olympiads for Elementary & Middle Schools | March 12, 2014 | Contest 5

SOLUTIONS AND ANSWERS

5A *Strategy: Make a sketch.*

To move from B to A move down 7 units and to the left 2 units. Repeat the process starting at C to find D. When we move down 7 from C and then 2 to the left we end at (9 –3). **The fourth point of the parallelogram is (9, –3).**

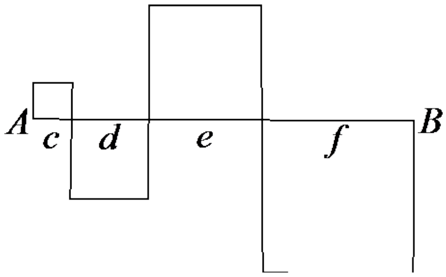


FOLLOW-UP: Given three non-collinear points in a plane what is the number of spots to place a fourth point to form a parallelogram? [3]

5A answer: (9, –3)

5B *Strategy: Consider the whole length as the sum of its parts.*

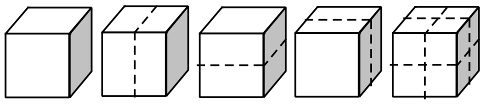
Let the lengths of the sides of the four squares be *c*, *d*, *e*, and *f* as seen in the diagram. The perimeter of each square is 4*c*, 4*d*, 4*e*, and 4*f*. The wire is bent to exactly form the perimeters of the four squares. We know that $AB = c + d + e + f = 12$. **Therefore the length of wire needed to construct the figure is $4(c + d + e + f) = 4(12) = 48$.**



5B answer: 48

5C *Strategy: Choose a convenient value.*

In order to cut a cube into 8 identical smaller cubes we must split (bisect) the cube along each edge. The length of each side of the small cube will be one-half the length of the sides of the original cube. Let the length of a side of the original cube be 10. The surface area of the cube would be $6(s)^2 = 6(10)^2 = 600$. Each side of the 8 identical smaller cubes would be 5. The surface area of each smaller cube is $6(5)^2 = 150$. The total surface area of all 8 cubes is $8(150) = 1200$. **The total surface area of the smaller cubes is two times the surface area of the original cube so $K = 2$.**



[Note: Since all cubes are similar, the ratio of the areas of two cubes equals the ratio of the squares of their sides. The sides are in a ratio of 1:2, their areas are in the ratio 1:4.]

5C answer: 2

5D answer: 8

5E answer: $\frac{8}{21}$

5D *Strategy: Organized search by considering different cases.*

Let $a = x - 5$ and $b = x + 2$. When a number in the form $a^b = 1$, the left side of the equation must be in one of the forms: a^0 and $a \neq 0$, $1^{\text{ANY NUMBER}}$ or $(-1)^{\text{EVEN NUMBER}}$.

Case I: $b = 0 \Rightarrow x = -2$ so $a = -7$ and $(-7)^0 = 1$.

Case II: $a = 1 \Rightarrow x = 6$ so $b = 8$ and $(1)^8 = 1$.

Case III: $a = -1 \Rightarrow x = 4$ so $b = 6$ and $(-1)^6 = 1$.

Thus the three solutions are –2, 6, and 4 and the sum of these numbers is 8.

FOLLOW-UP: How many different three-digit positive numbers of the form A5B are divisible by 11? Note that A and B could be the same digit. [8]

5E *Strategy: Make an organized list.*

Create a list of multiples of 3 (3, 6, 9, 12, 15, 18, …). Form a list of pairs of numbers that sum to a multiple of 3 using all the seven cards: 1 + 2, 1 + 8, 2 + 4, 2 + 7, 2 + 10, 4 + 8, 7 + 8, and 8 + 10.

There are 8 such pairs. The number of possible pairs chosen from 7 objects is ${}_7C_2 = \frac{7 \times 6}{2} = \mathbf{21}$.

Therefore the probability the sum of two randomly chosen cards is a multiple of 3 is $\frac{8}{21}$.

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