

Division

M

Mathematical Olympiads

November 19, 2014

for Elementary & Middle Schools

Contest

1

1A Time: 3 minutes

Suppose today is Monday. What day of the week will it be 2014 days later?

1B Time: 4 minutes

The product of some whole numbers is 40. Find their least possible sum.

1C Time: 5 minutes

The five points P , Q , R , S , and T lie on a straight line, though not necessarily in that order. Suppose $PT = 20$, and Q is $\frac{4}{5}$ of the way from P to T . Additionally, R (between P and Q) is 4 units closer to P than to Q , and S is twice as far from P as it is from R . Find the sum of the two possible lengths of line segment $\overline{RS}$.

1D Time: 6 minutes

In the square array shown, each row, column, and diagonal has the same sum. Find the numerical sum $a + b + c + d + e + f$.

4	a	b
c	7	d
e	4	f

1E Time: 7 minutes

There are exactly *three* fractions $\frac{a}{b}$ in lowest terms having both of the following properties: (1) $\frac{1}{5} < \frac{a}{b} < \frac{1}{4}$ and (2) b is odd, with $10 < b < 20$.

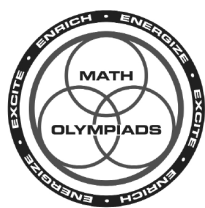
Find all *three* fractions that satisfy both conditions.

Please fold over on line. Write answers on back.

<i>Division</i> M	<i>Mathematical Olympiads</i> November 19, 2014 <i>for Elementary & Middle Schools</i>	<i>Contest</i> 1
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1A	<i>Student Name and Answer</i>
1B	<i>Student Name and Answer</i>
1C	<i>Student Name and Answer</i>
1D	<i>Student Name and Answer</i>
1E	<i>Student Name and Answer</i>

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Mathematical Olympiads

December 17, 2014

for Elementary & Middle Schools

Contest

2

2A Time: 3 minutes

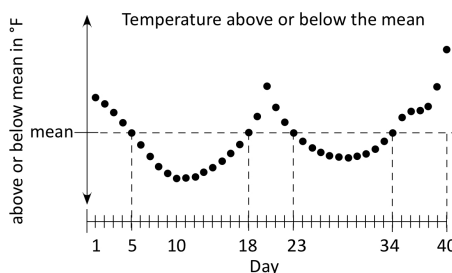
Write in simplest form: $\frac{7777-7-77-777}{7}$.

2B Time: 5 minutes

If $5x + 2y = 13$ and $7x + 9y = 12$, find the simplified value of $29x + 24y$.

2C Time: 6 minutes

The dotted wavy graph shows the temperatures relative to the mean temperature for a 40 day period. The dashed horizontal line represents the mean temperature for the 40 days. What percent of the 40 days was the temperature at or above the mean?



2D Time: 7 minutes

Three rectangular area rugs with dimensions: $6' \times 8'$, $8' \times 10'$, and $9' \times 12'$ are used to *entirely* cover the rectangular floor of a room $12' \times 16'$ with some overlap. Consider only the total area covered by exactly two rugs and not covered by one rug or three rugs. What is the least value of this area?

2E Time: 7 minutes

How many numbers in the set $\{1, 2, 3, \dots, 28, 29, 30\}$ cannot be represented by adding two or more different numbers in the set $\{0, 2, 3, 5, 8, 13\}$?
[Note that 4 cannot be represented as the sum of numbers from the above list.]

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December 17, 2014

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Contest

2

2A

Student Name and Answer

2B

Student Name and Answer

2C

Student Name and Answer

%

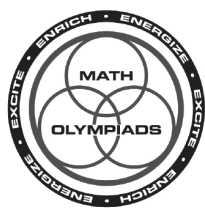
2D

Student Name and Answer

2E

Student Name and Answer

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January 14, 2015

for Elementary & Middle Schools

Contest
3

3A Time: 3 minutes

Using the incomplete division shown, find the sum of the missing digits in the four empty boxes.

$$\begin{array}{r}
 1 \square 3 \\
 3 \square \overline{) 5 \square 6 8} \\
 \underline{3 } \\
 2 \\
 \underline{2 } \\
 1 \\
 \underline{1 } \\
 0
 \end{array}$$

3B Time: 4 minutes

Soshana looks in a mirror and sees the reflection of a clock behind her as shown in figure A. How many **minutes** later will the reflection in the mirror of the same clock *next* show the image shown in figure B?

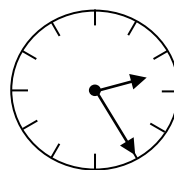


Figure A

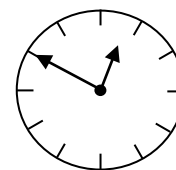
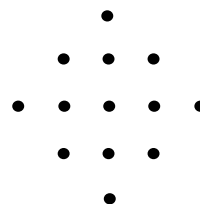


Figure B

3C Time: 5 minutes

Thirteen dots are arranged on a square grid in the pattern shown at the right.

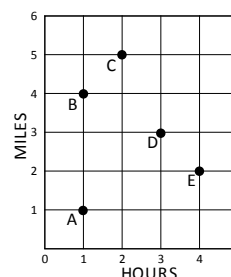
How many different squares can be formed by connecting at least four of these dots?



3D Time: 5 minutes

Five hikers, A, B, C, D, and E recorded their time and distance progress over various trails. The chart shown summarizes their results.

Find the difference [in miles per hour] between the fastest and slowest speeds recorded by the hikers.



3E Time: 7 minutes

In a pair of fair Wacky Dice, the red die has 1, 1, 1, 2, 3, and 3 on its six faces, while the green die has 2, 3, 3, 3, 5, and 5 on its six faces. If a pair of fair Wacky Dice is rolled, what is the probability the sum is odd? [Express your result as a fraction in lowest terms.]

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Contest

3

3A

Student Name and Answer

3B

Student Name and Answer

minutes

3C

Student Name and Answer

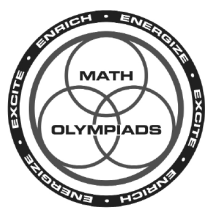
3D

Student Name and Answer

3E

Student Name and Answer

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Division

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Mathematical Olympiads

February 11, 2015

for Elementary & Middle Schools

Contest

4

4A Time: 3 minutes

If $(a \times b) + (a \times c) = 91$ and $b + c = 13$, then compute the value of $a + 2008$.

4B Time: 4 minutes

Set $A = \{21, 22, 23, 24\}$ and set $B = \{12, 13, 14, 15, 16\}$. Set C contains all possible differences when an element from set A and an element from set B are subtracted in either order. How many elements are contained in set C ?

4C Time: 5 minutes

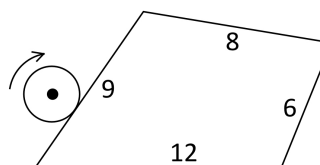
Each of a , b , and c are to be assigned one of the values 1, 2, or 4 without repetition. How many different values will be possible for the expression $(2a - b)^c$?

4D Time: 7 minutes

Find the least value of whole number N , with $N > 10$, so that the expression $2N - 7$ is both a perfect square and a perfect cube.

4E Time: 7 minutes

Without slipping, a circle with radius 1 inch rolls once completely around the outside of the quadrilateral shown. The sides of the quadrilateral have lengths 6, 8, 9, and 12 inches.



The center of the circle travels a path with length $P + Q\pi$. Find the sum $P + Q$.

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February 11, 2015

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Contest

4

4A

Student Name and Answer

4B

Student Name and Answer

4C

Student Name and Answer

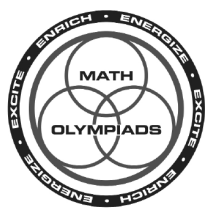
4D

Student Name and Answer

4E

Student Name and Answer

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Mathematical Olympiads

March 4, 2015

for Elementary & Middle Schools

Contest

5**5A** Time: 3 minutes

The first three terms of a sequence are 12, 25, and 17. The fourth term, and each term thereafter, is the arithmetic mean of the previous three terms. Compute the first *non-integer* term and write it as a mixed number.

5B Time: 4 minutes

Ingrid begins a game by placing a marker on 0 on the number line. She flips a fair coin and moves the marker one unit right if heads and one unit left if tails. After 85 flips, Ingrid's marker is on +17. How many of the flips were heads?

5C Time: 5 minutes

In the square array at the right, the sum of the numbers in each row, column, and diagonal is the same. Find the product $a \times b \times c$.

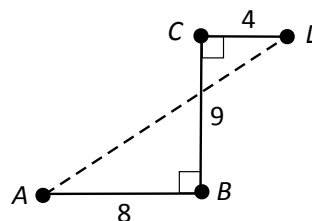
-3	+7	+5
a	+3	b
c	-1	+9

5D Time: 5 minutes

Find the value of the whole number N which satisfies $2^3 \times 4^N = 8^5$.

5E Time: 7 minutes

John travels from A to B to C to D along the solid straight paths shown with $\overline{AB} \perp \overline{BC}$ and $\overline{BC} \perp \overline{CD}$. If $AB = 8$ miles, $BC = 9$ miles, and $CD = 4$ miles, then how many miles would John *save* if he took the shorter direct route shown as the dotted straight-line segment $\overline{AD}$?



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Mathematical Olympiads

March 4, 2015

for Elementary & Middle Schools

Contest

5

5A

Student Name and Answer

5B

Student Name and Answer

5C

Student Name and Answer

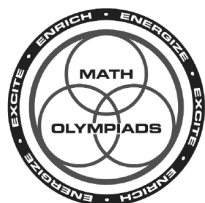
5D

Student Name and Answer

5E

Student Name and Answer

Please fold over on line. Write answers in these boxes.



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Mathematical Olympiads

November 19, 2014

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1

SOLUTIONS AND ANSWERS

1A

1A *Strategy:* Use the idea that the days of the week repeat after 7 days.

Divide 2014 by 7. The result is 287 with a remainder of 5. That means that 2014 days after Monday is the same as 5 days after Monday. The fifth day after Monday is **Saturday**.

FOLLOW-UP: Suppose the current month is January. What month will it be 2014 months from now? [November]

Saturday

1B

1B *Strategy:* Systematically examine the factors.

Make a table of products and examine sums of the factors. Be careful to include products with more than two factors as well.

	40	Sum		40	Sum
	1×40	41		$2 \times 2 \times 10$	14
<i>Two factors:</i>	2×20	22	<i>More than two factors:</i>	$2 \times 4 \times 5$	11 least sum
	4×10	14		$2 \times 2 \times 2 \times 5$	11 least sum
	5×8	13			

Therefore, **11** is the smallest sum for the factors of 40. Notice that including an additional factor of 1 only increases the required sum, and replacing a 4 with 2×2 has no effect on the sum.

[Exploration: Does the prime factorization always yield the minimum sum?]

FOLLOW-UPS: (1) The product of 4 whole numbers is 480. Compute the greatest possible sum of the numbers with that property. [483] (2) Compute the number of whole number factors of 1001. [8]

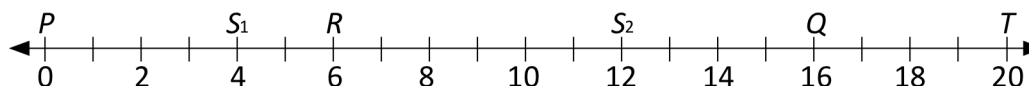
11

1C

1C *Strategy:* Draw a diagram that satisfies the conditions.

Draw a number line from 0 to 20 with P at 0 and T at 20. Since Q is $4/5$ of the way from P to T , place Q at 16. Now R is 4 units closer to P than to Q . If R were exactly in the middle of those two points it would be at 8. For each unit R moves toward P it is 2 units closer to P than to Q . Therefore, R must be at 6. Since S is twice as far from P as it is from R , S must be at 4 or 12.

The sum of the two possible lengths of segments RS is $2 + 6 = 8$.



1D

1E

8

48

$\frac{3}{13}, \frac{4}{17}, \frac{4}{19}$

1D METHOD 1: *Strategy:* Work systematically to find all the values of the variables.

Since $4 + a + b = a + 7 + 4$, $b = 7$. We know that $7 + 7 + e = e + 4 + f$ so $f = 10$. In addition we know that $4 + 7 + f = a + 7 + 4$ so $a = f = 10$. Now it is easy to see that $c = 10$ and $d = 4$. The sum $a + b + c + d + e + f = 10 + 7 + 10 + 4 + 7 + 10 = 48$.

4	10	7
10	7	4
7	4	10

METHOD 2: *Strategy:* Find the sum without finding the value of each variable.

This is a magic square by definition. The middle cell of any 3×3 magic square is the average of all 9 cells, so the sum of these nine cells is $7 \times 9 = 63$.

The sum of $a + b + c + d + e + f = 63 - (4 + 7 + 4) = 48$.

FOLLOW-UP: In the square array shown, each row and column has the same sum. The sum of the two diagonals differs by 3. Find the sum $a + b + c + d$.
[There are two solutions: 7 and 25]

1	a	b
c	2	1
2	2	d

1E METHOD 1: *Strategy:* Consider fractions with a common numerator.

Between $\frac{2}{10}$ and $\frac{2}{8}$ is $\frac{2}{9}$, but $b < 10$. Between $\frac{3}{5}$ and $\frac{3}{15}$ is $\frac{3}{13}$ which is one acceptable fraction.

Between $\frac{4}{20}$ and $\frac{4}{16}$ are $\frac{4}{19}$ and $\frac{4}{17}$ which are the other two acceptable fractions. All other possibilities done this way have $b > 20$. The three fractions that satisfy the conditions are $\frac{3}{13}$, $\frac{4}{17}$, and $\frac{4}{19}$.

METHOD 2: *Strategy:* Consider the fractions with odd denominators between 10 and 20.

The only possible values for b are 11, 13, 15, 17, and 19. Since $\frac{1}{5} < \frac{a}{b} < \frac{1}{4}$, we know that

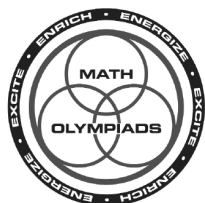
$.2 < \frac{a}{b} < .25$ and $.2b < a < .25b$. Create a table of values based upon the different values of b .

$b = 11$	$b = 13$	$b = 15$	$b = 17$	$b = 19$
$2.2 < a < 2.75$	$2.6 < a < 3.25$	$3 < a < 3.75$	$3.4 < a < 4.25$	$3.8 < a < 4.75$

There are no whole numbers for a when $b = 11$ or when $b = 15$. When $b = 13$, $a = 3$ and when b is either 17 or 19, a must be 4. The three fractions that satisfy the conditions are $\frac{3}{13}$, $\frac{4}{17}$, and $\frac{4}{19}$.

FOLLOW-UP: Compute the number of pairs (a, b) such that a and b are positive whole numbers and $\frac{a}{5} < \frac{2}{b} < 1$. [10]

NOTE: Other FOLLOW-UP problems related to some of the above can be found in our two contest problem books and in "Creative Problem Solving in School Mathematics."
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2**SOLUTIONS AND ANSWERS****2A****2A METHOD 1:** *Strategy: Simplify the numerator first.*

Combine $7777 - 7 - 77 - 777 = 7000 - 84 = 6916$. Divide to get $6916 \div 7 = 988$.

METHOD 2: *Strategy: First divide by 7.*

$$\frac{7777 - 7 - 77 - 777}{7} = 1111 - 1 - 11 - 111 = 988.$$

FOLLOW-UP: What is the value of $\frac{193}{7} - \frac{106}{14} + \frac{6}{21} - \frac{484}{28}$? [3]

988**2B****2B METHOD 1:** *Strategy: Express $29x$ in terms of $5x$ and $7x$.*

Notice that $29x = 3(5x) + 2(7x)$ and $24y = 3(2y) + 2(9y)$. Therefore, multiply the first equation by 3 and multiply the second equation by 2 and then add the results:

$$3(5x + 2y) = 3(13) \rightarrow 15x + 6y = 39$$

$$2(7x + 9y) = 2(12) \rightarrow 14x + 18y = 24$$

$$\text{Add the equations: } 29x + 24y = 63$$

METHOD 2: *Strategy: Solve separately for x and y .*

$$9(5x + 2y) = 9(13) \rightarrow 45x + 18y = 117$$

$$-2(7x + 9y) = -2(12) \rightarrow -14x - 18y = -24$$

$$\text{Add the equations: } 31x = 93$$

It follows that $x = 93 \div 31 = 3$.

Substitute into one of the original two equations to get that $y = -1$. Then substitute both values to compute that $29x + 24y = 29(3) + 24(-1) = 63$.

FOLLOW-UP: If $3x + 4y = 23$ and $7x - 5y = 11$, find the value of $29x + 10y$. [137]

63**2C****2C** *Strategy: Count the number of days the wavy line is at or above the dotted line.*

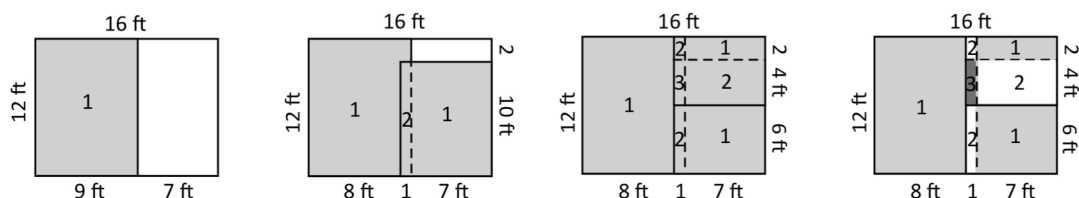
The temperature is at or above the mean for the first 5 days. Then it is below the mean for the next 12 days, at or above the mean for the following 6 days, below the mean for the next 10 days, and finally at or above the mean for the remaining 7 days. This is 18 out of 40 days or **45%** of the time that the temperature was at or above the mean.

FOLLOW-UP: The mean of a set of five numbers is 88. If one of the numbers is removed, the new mean is 85. What is the value of the number that was removed? [100]

45%**2D****36****2E****12**

2D *Strategy:* Place the 9 by 12 rug first to see how much space the smaller rugs need.

Place the 12 foot side of the 9×12 rug along the 12 foot wall. The smaller rugs must cover a 7×12 region. With no gaps allowed, the 8 foot side of the 8×10 rug lies along the 16 foot wall. Place the 8 foot side of the 6×8 rug along the other 16 foot wall. The two smaller rugs overlap each other by a 4×8 rectangle [see the progression of the 4 diagrams below]. The last picture shows three light grey regions where only one rug covers the floor, one dark grey region where all three rugs overlap and 3 white regions where exactly two rugs overlap. The minimum area covered by exactly two rugs is $1 \times 6 + 1 \times 2 + 7 \times 4 = 6 + 2 + 28 = 36$ sq ft. [The numbers in each region represent the number of rugs covering the region.]



2E METHOD 1: *Strategy:* Count in an organized fashion.

Change the set to $\{2, 3, 5, 8, 13\}$ and also consider a single number. Any one number can be counted: 2, 3, 5, 8, and 13. The new sums using two numbers are: 7, 10, 15; 11, 16; 18, 21. The new sums using three numbers are 20 and 23. These can be found by adding the unused numbers to the sums found in the “sum of two numbers” group. The new sums using four numbers are 28 and 29. These can be found by adding the unused numbers to the sums found in the “sum of three numbers” group. The sum of all 5 numbers is 31 which is not an acceptable sum. Therefore there are $5 + 9 + 2 + 2 = 18$ sums that are possible so $30 - 18 = 12$ sums that are not possible.

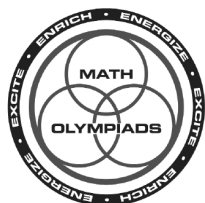
METHOD 2: *Strategy:* Since the sum of the five numbers is 31, if the sum S is possible the sum $31 - S$ is also possible.

It is only necessary to build “possible sums” half way to 30. The sums with their “co-sum pairs”:

$2 = 0 + 2$	$29 = 3 + 5 + 8 + 13$	(Since 2 is possible, $31 - 2$ is possible.)
$3 = 0 + 3$	$28 = 2 + 5 + 8 + 13$	
$5 = 0 + 5$	$26 = 5 + 8 + 13$	
$7 = 2 + 5$	$24 = 3 + 8 + 13$	
$8 = 0 + 8$	$23 = 2 + 8 + 13$	
$10 = 2 + 3 + 5$	$21 = 8 + 13$	
$11 = 3 + 8$	$20 = 2 + 5 + 13$	
$13 = 0 + 13$	$18 = 5 + 13$	
$15 = 2 + 5 + 8$	$16 = 3 + 5 + 8$	

Since 18 sums are possible, $30 - 18 = 12$ sums are not possible. The list of “impossible sums” contains: 1, 4, 6, 9, 12, 14, 17, 19, 22, 25, 27, and 30.

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SOLUTIONS AND ANSWERS

3A

3A *Strategy:* Use the rules for long division.

The 1 in the quotient multiplies the 3□ in the divisor to get 36. Thus the □ must be filled with a 6.

When 36 is subtracted from 5□ the result is 22. Thus the □ must be filled with 8.

When 2□6 is subtracted from 226 the result is 10. Thus the □ must be filled with a 1.

The number in the □ in the quotient multiplied by 36 results in 216. Thus the □ is filled with a 6.

The sum of the four unknown numbers is $6 + 8 + 1 + 6 = 21$.

FOLLOW-UP: Find the sum of the five missing numbers in the product shown. [17]

$$\begin{array}{r} \square 823 \\ \times \square 9 \\ \hline 704\square 7 \\ 15646 \\ \hline \square 268\square 7 \end{array}$$

21

3B

3B **METHOD 1:** *Strategy:* Since the clocks are mirror images of the real time, consider the hands of the clock moving counter-clockwise.

If the hands of the clock in figure A move counter-clockwise to end at the position in figure B, the time will change from 2:25 to 12:50. This will take **95** minutes.

[1 hour and 35 minutes is unacceptable.]

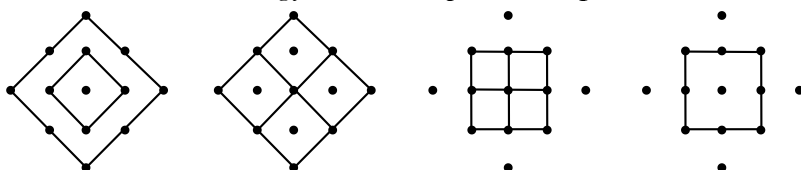
METHOD 2: *Strategy:* Reflect each clock over a vertical line connecting 12 and 6.

Reflecting the clock in figure A results in the time 9:35 and reflecting the clock in figure B results in 11:10. The number of minutes from 9:35 to 11:10 is 95 minutes.

95

3C

3C **METHOD 1:** *Strategy:* Draw all possible squares.



$$2 + 4 + 4 + 1 = 11 \text{ different squares}$$

METHOD 2: *Strategy:* Label the dots and make a chart.

Label the dots as seen in the diagram:

ABGD	AEMI	BCGF	BDLI	BEJG	CDHG
CFKH	DGLI	FGKJ	GHLK	GJML	

Therefore, there are a total of 11 squares that can be formed by the thirteen dots.

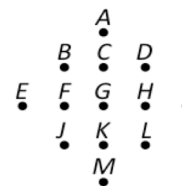
A
 •
 B C D
 • • •
 E F G H I
 • • •
 J K L
 • • •
 M

3E

5
18

METHOD 3: *Strategy: Simplify the problem.*

Label the dots as in Method 2. Consider the 9 dots *B, C, D, F, G, H, J, K, and L*. Four small squares can now be formed in addition to one larger square *CFKH* and the large square *BDLJ*, for a total of 6 squares. Next consider the 9 dots *E, B, A, J, G, D, M, L and I*. Again there are the four small squares but only one large square *AEMI*, for a total of 5 squares. Finally, $6 + 5 = 11$ squares total.



FOLLOW-UP: Use the same thirteen dots as in the question and find the number of equilateral triangles that can be formed by three of the dots. [0]

3D METHOD 1: *Strategy: Calculate each rate using the formula: rate = distance ÷ time.*

Hiker	A	B	C	D	E
Distance (miles)	1	4	5	3	2
Time (hours)	1	1	2	3	4
Rate (mph)	$1 \div 1 = 1$	$4 \div 1 = 4$	$5 \div 2 = 2.5$	$3 \div 3 = 1$	$2 \div 4 = .5$

Hiker B travels the fastest at 4 mph. Hiker E travels the slowest at .5 mph. The difference in their speeds is $4 - .5 = 3.5$ mph.

METHOD 2: *Strategy: See the rate of the hikers as the slope of a line.*

The slopes of the lines that connect the origin to each of the points give the rate of each hiker. The slope is rise over run or distance ÷ time. The steepest line has the greatest slope and the least steep line has the least slope. The steepest line connects the origin to hiker B and has a slope of 4 and the least steep line connects the origin to hiker E and has a slope of $1/2$. The difference in these two numbers is $4 - 1/2 = 3 \frac{1}{2}$.

3E METHOD 1: *Strategy: Apply the laws of probability.*

The sum of two whole numbers is odd when one number is odd and the other is even. The probability of an odd on the red die and an even on the green die is $5/6 \times 1/6 = 5/36$. The probability of an even on the red die and an odd on the green die is $1/6 \times 5/6 = 5/36$. Therefore the probability of an odd sum is $5/36 + 5/36 = 10/36$, reduced to **$5/18$** .

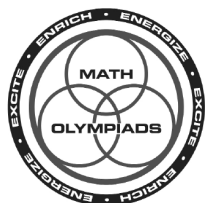
METHOD 2: *Strategy: Make a chart of all possibilities.*

Red Blue	2	3	3	3	5	5
1	X					
1	X					
1	X					
2		X	X	X	X	X
3	X					
3	X					

As shown there are 36 possible pairs for the two dice. The 10 odd sums are each indicated with an X. The probability that the sum is odd is $10/36 = 5/18$.

FOLLOW-UP: Two coins are drawn from a jar containing 3 nickels, 1 dime, and 2 quarters. What is the probability that the value is at least 35¢? [1/5]

NOTE: Other FOLLOW-UP problems related to some of the above can be found in our two contest problem books and in "Creative Problem Solving in School Mathematics." Visit www.moems.org for details and to order.



Division

M

Mathematical Olympiads

February 11, 2015

for Elementary & Middle Schools

Contest

4**SOLUTIONS AND ANSWERS****4A****4A METHOD 1:** *Strategy: Apply the distributive property and use substitution.*

The distributive property states that $(a \times b) + (a \times c) = a \times (b + c)$ so by substitution we have $a \times 13 = 91$. Divide by 13 to get $a = 7$. Thus $a + 2008 = 2015$.

METHOD 2: *Strategy: Substitute, distribute and then simplify.*

Since $b + c = 13$, $b = 13 - c$. Substitute this into the first given equation to get that $(a \times (13 - c)) + (a \times c) = 13a - a \times c + a \times c = 13a = 91$. Thus $a = 7$ and $a + 2008 = 2015$.

2015**4B****4B METHOD 1:** *Strategy: Calculate the largest and smallest positive differences.*

The largest positive difference between the numbers in the two sets is $24 - 12 = 12$. The smallest positive difference between the numbers in the two sets is $21 - 16 = 5$. Confirm that every number between 5 and 12 can be found by subtraction: $22 - 16 = 6$, $23 - 16 = 7$, $24 - 16 = 8$, $24 - 15 = 9$, $24 - 14 = 10$, $24 - 13 = 11$. This accounts for the 8 positive differences. There will be 8 negative differences found by reversing the subtrahend and minuend. There are **16** possible differences.

METHOD 2: *Strategy: Create a table of all possible differences.*

$A - B$	12	13	14	15	16
21	9	8	7	6	5
22	10	9	8	7	6
23	11	10	9	8	7
24	12	11	10	9	8

There are 8 different differences in this table and there will be 8 different differences in the table for $B - A$. Therefore there are 16 possible differences for numbers in the sets.

4C**16****4D****5****4C** *Strategy: Calculate every possible outcome.*

When $a = 1$, $b = 2$, and $c = 4$, we have $(2 \times 1 - 2)^4 = 0^4 = 0$.

When $a = 1$, $b = 4$, and $c = 2$, we have $(2 \times 1 - 4)^2 = (-2)^2 = 4$.

When $a = 2$, $b = 1$, and $c = 4$, we have $(2 \times 2 - 1)^4 = 3^4 = 81$.

When $a = 2$, $b = 4$, and $c = 1$, we have $(2 \times 2 - 4)^1 = 0^1 = 0$.

When $a = 4$, $b = 1$, and $c = 2$, we have $(2 \times 4 - 1)^2 = 7^2 = 49$.

When $a = 4$, $b = 2$, and $c = 1$, we have $(2 \times 4 - 2)^1 = 6^1 = 6$.

There are **5** different possible values for the expression.

FOLLOW-UPS: (1) How many solutions are possible if the given values for a , b , and c are changed to 1, 3, and 5? Why is the solution different than in the original problem? [6]

(2) Find 3 different values for a , b , and c so that there will only be 4 different solutions. [-1, 0, and 1]

4E**368****37**

4D METHOD 1: *Strategy: Compute numbers that are perfect squares and perfect cubes.*

For a number to be both a perfect square and a perfect cube, it must be a perfect sixth power since $a^6 = (a^2)^3$ or $a^6 = (a^3)^2$. Some of the small perfect powers of 6 are $1^6 = 1$, $2^6 = 64$, and $3^6 = 729$. Let $2N - 7$ equal each of these values and then solve for N .

If $2N - 7 = 1$, $2N = 8$ so $N = 4$. This violates the condition that $N > 10$.

If $2N - 7 = 64$, $2N = 71$ and N is a fraction. This violates the condition that N is a whole number.

In the third case $2N - 7 = 729$, $2N = 736$, so $N = 368$. This is the least value of N that satisfies all the conditions.

METHOD 2: *Strategy: Make a list of perfect cubes and explore those that are perfect squares.*

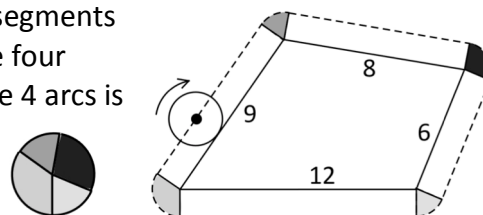
The perfect cubes are $1^3 = 1$ (also a perfect square), $2^3 = 8$, $3^3 = 27$, $4^3 = 64$ (perfect square), $5^3 = 125$, $6^3 = 216$, $7^3 = 343$, $8^3 = 512$, $9^3 = 729$ (perfect square), and so on.

Since $2N - 7$ must equal the number that is both a perfect square and a perfect cube and N must be a whole number greater than 10, $2N - 7 = 729$ so $2N = 736$ and $N = 368$.

FOLLOW-UP: The number 1 is both a perfect square and a triangular number. Find the next two greater values that are both perfect squares and triangular numbers. [36 and 1225]

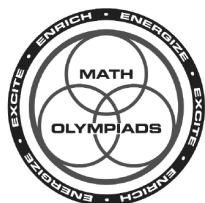
4E *Strategy: Draw a diagram.*

The path of the center of the circle is comprised of 4 line segments and 4 arcs. The four segments are the same lengths as the four sides of the quadrilateral and the sum of the lengths of the 4 arcs is equal to the circumference of the circle as seen in the small insert. Since the radius of the circle is 1, the circumference is 2π . The total distance the center of the circle travels is $9 + 8 + 6 + 12 + 2\pi = 35 + 2\pi = P + Q\pi$. Therefore $P + Q = 35 + 2 = 37$.



FOLLOW-UP: Compute the area covered by the circle as it makes one complete revolution around the quadrilateral. [70 + 4π]

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Division

M

Mathematical Olympiads

March 4, 2015

for Elementary & Middle Schools

Contest

5

SOLUTIONS AND ANSWERS

5A

5A *Strategy:* Calculate each new term.

The fourth term is $(12 + 25 + 17) \div 3 = 54 \div 3 = 18$.

The fifth term is $(25 + 17 + 18) \div 3 = 60 \div 3 = 20$.

The sixth term is $(17 + 18 + 20) \div 3 = 55 \div 3 = 18 \frac{1}{3}$.

18 1/3

5B **METHOD 1:** *Strategy:* Pair each tail with a head.

Of the 85 flips, 17 were heads that did not match any of the tails. Of the remaining 68 flips, half were heads and half were tails. The total number of heads was $34 + 17 = 51$.

METHOD 2 *Strategy:* Use algebra.

Let H = the number of heads flipped so $85 - H$ is the number of tails. Since the number of heads was greater than the number of tails by 17, $H - (85 - H) = 17$. Thus $2H = 102$ and $H = 51$.

FOLLOW-UPS: (1) Use all the conditions in the original problem to find the longest possible string of consecutive heads. [51] (2) What is the leftmost point on the number line that could have been reached if all the same conditions were applied? [-34]

5B

51

5C

5C *Strategy:* Consider the results based upon the first row.

The sum of the numbers in the first row is $-3 + 7 + 5 = 9$. Therefore every row, column, and diagonal totals to 9. In column three, $5 + b + 9 = 9$, so $b = -5$.

In row three, $c - 1 + 9 = 9$ so $c = 1$.

To find a use column one or row two to set up an equation. The value of a is 11.

The product of $a \times b \times c = 11 \times -5 \times 1 = -55$.

-55

5D

5D **METHOD 1:** *Strategy:* Express each side of the equation as a power of 2.

The expression 4^N can be expressed as $(2^2)^N = 2^{2N}$.

8^5 can be expressed as $(2^3)^5 = 2^{15}$.

$2^3 \times 4^N = 2^3 \times 2^{2N} = 2^{3+2N} = 2^{15}$. Since the bases of the two expressions are the same the exponents must be equal. Solve $3 + 2N = 15$ to get $N = 6$.

METHOD 2: *Strategy:* Expand both sides of the equation.

Notice that $2^3 = 8$. Therefore, $8 \times 4^N = 8 \times 8 \times 8 \times 8 \times 8$.

Divide by 8 to get $4^N = 8 \times 8 \times 8 \times 8$. Replace each 8 with $2 \times 2 \times 2$ and the 4 with 2^2 .

$(2^2)^N = (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (2 \times 2 \times 2)$. It follows that

$2^{2N} = 2^{12}$ so $2N = 12$ and $N = 6$.

FOLLOW-UP: Given $5^M \times 2^5 \times 5^2 = 10^N$ for some whole numbers M and N . What are the values for M and N ? [$M = 3$ and $N = 5$]

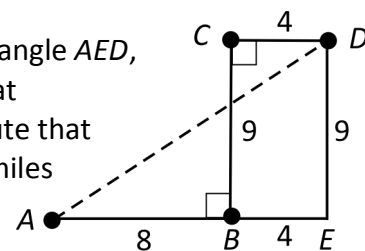
5E

6

6

5E METHOD 1: *Strategy: Use the Pythagorean theorem.*

Complete rectangle $BCDE$. Side $BE = 4$ and side $DE = 9$. In right triangle AED , $AE = 12$ and $DE = 9$. Apply the Pythagorean theorem that says that $AD^2 = 12^2 + 9^2 = 144 + 81 = 225$. Therefore $AD = 15$ miles. The route that John took was $8 + 9 + 4 = 21$ miles. John would save $21 - 15 = 6$ miles if he took the shorter direct route.

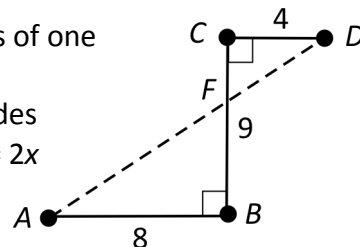


METHOD 2: *Strategy: Use similar triangles.*

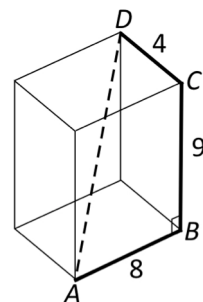
Right triangle ABF and right triangle CDF are similar since the angles of one triangle are the same as the angles in the second triangle.

Since side AB is 8 and side CD is 4, the ratio of the corresponding sides of the two triangles is 2 to 1. Since $BC = 9$, if we let $CF = x$, then $BF = 2x$ and $x + 2x = 9$ so $x = 3$. This means that $CF = 3$ and $BF = 6$. Apply the Pythagorean theorem to find DF . $DF^2 = 3^2 + 4^2 = 9 + 16 = 25$.

Thus $DF = 5$. [It should be noted that the numbers 3, 4, and 5 are commonly used to represent the lengths of the sides of a right triangle.] Since the sides of triangle ABF are twice the lengths of the sides of triangle CDF , $AF = 10$ and $AD = 5 + 10 = 15$. Since the distance John traveled was 21 miles he could have saved 6 miles by using the direct route.



FOLLOW-UP: In the original problem, let points A, B, and C be along the edges of one face of a rectangular solid, and points B, C, and D be along the edges of an adjacent side as shown. To the nearest whole number, how many miles would John save by taking the even shorter route directly from point A to point D? [8]



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