

9th United States of America Junior Mathematical Olympiad

Day 1. 12:30 PM – 5:00 PM EDT

April 18, 2018

Note: For any geometry problem whose statement begins with an asterisk (*), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAJMO 1. For each positive integer n , find the number of n -digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal, and
- the last digit is a prime.

USAJMO 2. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

USAJMO 3. (*) Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC} \perp \overline{BD}$. Let E and F be the reflections of D over lines BA and BC , respectively, and let P be the intersection of lines BD and EF . Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q , and the circumcircle of $\triangle FPD$ meets ω at D and R . Show that $EQ = FR$.

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Day 2. 12:30 PM – 5:00 PM EDT

April 19, 2018

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USAJMO 4. Triangle ABC is inscribed in a circle of radius 2 with $\angle ABC \geq 90^\circ$, and x is a real number satisfying the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$, where $a = BC, b = CA, c = AB$. Find all possible values of x .

USAJMO 5. Let p be a prime, and let $a_1, a_2, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \dots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p .

USAJMO 6. Karl starts with n cards labeled $1, 2, 3, \dots, n$ lined up in a random order on his desk. He calls a pair (a, b) of these cards *swapped* if $a > b$ and the card labeled a is to the left of the card labeled b . For instance, in the sequence of cards $3, 1, 4, 2$, there are three swapped pairs of cards, $(3, 1)$, $(3, 2)$, and $(4, 2)$.

He picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i cards to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards $1, 2, \dots, n$ exactly once in that order. (For example, the process starting at $3, 1, 4, 2$ would be $3, 1, 4, 2 \rightarrow 3, 4, 1, 2 \rightarrow 2, 3, 4, 1 \rightarrow 2, 4, 3, 1 \rightarrow 2, 3, 4, 1$.)

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.