

47th United States of America Mathematical Olympiad

Day 1. 12:30 PM – 5:00 PM EDT

April 18, 2018

Note: For any geometry problem whose statement begins with an asterisk (*), the first page of the solution must be a large, in-scale, clearly labeled diagram. Failure to meet this requirement will result in an automatic 1-point deduction.

USAMO 1. Let a, b, c be positive real numbers such that $a + b + c = 4\sqrt[3]{abc}$. Prove that

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

USAMO 2. Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

USAMO 3. For a given integer $n \geq 2$, let $\{a_1, a_2, \dots, a_m\}$ be the set of positive integers less than n that are relatively prime to n . Prove that if every prime that divides m also divides n , then $a_1^k + a_2^k + \dots + a_m^k$ is divisible by m for every positive integer k .

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Day 2. 12:30 PM – 5:00 PM EDT

April 19, 2018

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USAMO 4. Let p be a prime, and let $a_1, a_2, \dots, a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1 + k, a_2 + 2k, \dots, a_p + pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p .

USAMO 5. (*) In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E , lines AB and CD intersect at F , and lines BC and DA intersect at G . Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P , and that the circumcircle of $\triangle ADE$ intersects line CD at D and Q , where C, B, P, G and C, Q, D, F are collinear in this order. Prove that if lines FP and GQ intersect at M , then $\angle MAC = 90^\circ$.

USAMO 6. Let a_n be the number of permutations $(x_1, x_2, \dots, x_n)$ of the numbers $(1, 2, \dots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \leq k \leq n$ are all distinct. Prove that a_n is odd for all $n \geq 1$.