

MATHCOUNTS®

2026 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide detailed solutions to the Sprint, Target and Team Rounds of the 2026 MATHCOUNTS State Competition. These solutions show creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

*Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions
with the MATHCOUNTS community.*

Sprint 1

We see that 40 is even, thus divisible by 2, and that the sum of the digits of 75 is $7 + 5 = 12$, which is divisible by 3, so 75 is likewise divisible by 3, enabling us to use the entire field for plots in a very simple manner. Align the 3 ft sides of the plots parallel to the 75 ft sides of the field and the 2 ft sides of the plots parallel to the 40 ft sides of the field so as to have

$\frac{75 \cancel{\text{ft}}}{3 \cancel{\text{ft}}} = 25$ plots lengthwise and $\frac{40 \cancel{\text{ft}}}{2 \cancel{\text{ft}}} = 20$ plots widthwise, for a total of $25 \times 20 = \mathbf{500}$ plots.

Sprint 2

Maxwell has Tuesday, Thursday, and Friday available, thus $6 + 3 + 5 = \mathbf{14}$ clubs to choose from.

Sprint 3

$29^2 - 17^2 = (29 + 17)(29 - 17) = 46 \times 12 = (2 \times 23)(2^2 \times 3)$, with 2, 3, and 23 as the distinct prime divisors, summing to $2 + 3 + 23 = \mathbf{28}$.

Sprint 4

Sally at the end kept half of the remaining candies after eating 2, leaving 24 candies for herself, and the friend had the other half, thus 24 candies. Thus, Sally had 48 candies after eating 2 but before dividing them with her friend; adding the 2 that she ate means that Sally started with **50** candies.

Sprint 5

$15 = x^2 - y^2 = (x + y)(x - y) = 5(x - y)$, so $(x - y) = 15/5 = 3$. Adding $x + y = 5$ and $x - y = 3$ yields $2x = 8$, so $x = \mathbf{4}$.

Sprint 6

Let a and b be the two distinct positive integers; let $m = \max(a; b)$ and $n = \min(a; b)$, so $m + n = a + b$ and $m > n$. Then $m + n = m - n + 8$, which means $2n = 8$, so $n = 4$. The desired sum is $m + n = m + 4$, which is minimized when m is minimized. Because m is an integer satisfying $m > n = 4$, the minimum allowed value of m is 5, so the minimum sum is $5 + 4 = \mathbf{9}$.

Sprint 7

The distance d , speed [rate] r , and time duration t are related by $d = rt$. Here $r = 55 \frac{\text{mi}}{\text{h}}$ and $t = 3 \text{ min} = 3\left(\frac{1}{60} \text{ h}\right) = \frac{1}{20} \text{ h}$, so $d = \left(55 \frac{\text{mi}}{\text{h}}\right)\left(\frac{1}{20} \cancel{\text{h}}\right) = \frac{11}{4} \text{ mi} = \mathbf{2.75 \text{ mi}}$.

Sprint 8

[NOTE: The word “weight” is used in this answer in conformance with its standard meaning in the context of commerce (the context of the problem) and law, matching the use of “mass” in the context of physics and engineering, where “weight” is used contrarily as a gravitation-based force.]

These cookies are modeled as right circular cylinders, with the weight W being the product of the cylinder’s volume $\pi(D/2)^2 h$, where D is the diameter and h is the height (here thickness), times the

density ρ of the cookie dough. Thus, $W = \pi\left(\frac{D}{2}\right)^2 h \rho = \frac{\pi}{4} h \rho D^2$. We can set up a proportion for the

weight W_{16} of a 16 in cookie and W_{10} of a 10 in cookie: $\frac{W_{16}}{W_{10}} = \frac{\frac{\pi}{4} h \rho D_{16}^2}{\frac{\pi}{4} h \rho D_{10}^2}$. Because we are given that the

thickness h has the same value regardless of diameter and the dough is identical for both cookies, thus the same density, the factor $\frac{\pi}{4} h \rho$ is a fixed value independent of diameter, and its presence in

both the numerator and denominator cancels out. Thus, $\frac{W_{16}}{W_{10}} = \frac{D_{16}^2}{D_{10}^2} = \frac{(16 \cancel{\text{in}})^2}{(10 \cancel{\text{in}})^2} = \frac{256}{100} = 2.56$. Therefore,

$W_{16} = 2.56 W_{10} = 2.56 \times 10 \text{ oz} = \mathbf{25.6 \text{ oz}}$.

Sprint 9

For the mean, we need $1/10$ of the sum of terms 1 through 10 of the Fibonacci sequence. The sum of terms m through n of the Fibonacci sequence is given by the value of term $(n + 2)$ minus the value of term $(m + 1)$. Term 12 is the sum of term 9 plus twice term 10, thus $34 + 2 \times 55 = 144$, and term 2 is 1; the difference is $144 - 1 = 143$, which divided by the term count 10 yields 14.3 for the mean.

The median of 10 terms in increasing order is the average of term 5 and term 6, thus $\frac{5+8}{2} = \frac{13}{2} = 6.5$.

The only value that occurs more than once is 1, in the first two terms, so the mode is 1. The sum of these three parameters is $14.3 + 6.5 + 1 = \mathbf{21.8}$.

Sprint 10

Let t be the tens digit of a positive 2-digit integer (so t is an integer satisfying $1 \leq t \leq 9$) and u be the ones [units] digit of that same integer (so u is an integer satisfying $0 \leq u \leq 9$). Then the integer under consideration is $10t + u$, which is 36 greater than $t + u$. Thus, $10t + u = t + u + 36$, so $9t = 36$ and $t = 4$, with no additional constraints on u . This means that any integer in the closed interval 40 .. 49, but no other value, works, yielding **10** qualifying values.

Sprint 11

The remainder upon dividing a nonnegative integer by 5 [just as with dividing by 2 or 10] depends only on the ones digit of the given integer. The ones digit of a power with positive integer exponent depends only on the exponent and the ones digit of the base. Thus, the ones digit of 23^{23} is the same as the ones digit of 3^{23} . For any digit d and any *positive* exponent n , the ones digit of d^n is the same as the ones digit of d^{n+4} . This means that we can divide n by 4, throwing away the quotient and keeping only the remainder r , and if $r = 0$ then change r to 4. Then the ones digit of d^n is the same as the ones digit of d^r . [NOTE: The cycle of 4 works for all digits, but the actual period is 1 when d is 0, 1, 5, or 6, is 2 when d is 4 or 9, and is 4 when d is 2, 3, 7, or 8. Some students prefer to memorize and keep straight these details of special cases in the heat of competition, whereas other students prefer to keep the rule simple by always using a cycle of 4 and potentially doing a small amount of extra work.] For 3^{23} , dividing 23 by 4 leaves a remainder of 3. Finally, $3^3 = 27$, whose ones digit is 7, which upon division by 5 leaves a remainder of **2**.

Sprint 12

$$30 = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right)x = \frac{8+4+2+1}{8}x = \frac{15}{8}x, \text{ which means } x = \frac{8 \times 30}{15} = 8 \times 2 = \mathbf{16}.$$

Sprint 13

Divide the equation by the constant coefficient 6 to get the two-intercept equation of the line:

$1 = \frac{1}{3}m + \frac{1}{6}b$; the m -intercept and the b -intercept are the reciprocals of coefficient of m and b , thus, 3 and 6, respectively. The vertices of the triangle are at $O(0; 0)$ where the m - and b -axes intersect, $M(3; 0)$ as the m -intercept of the given slant line, and $B(0; 6)$ as the b -intercept of the given slant line. $\overline{OM} \perp \overline{OB}$, so the area of the triangle is $\frac{1}{2} OM \cdot OB = \frac{1}{2} \times 3 \times 6 = \mathbf{9}$.

Sprint 14

There are two categories of solution: [1] more analysis effort, less plug and grind number crunching; [2] no analysis effort, just plug and grind number crunching. With 10 intervals to assess, the overall effort is similar for the two approaches.

Approach [1]:

The interval considered is $0.5n < i < 0.6n$. The easiest case is even n , meaning there is an integer $k = 0.5n$ such that $n = 2k$. Then $k < i < k + 0.2k$. Because of the strict inequality and because consecutive integers are separated by 1, we need, in order for such an i to exist, $0.2k > 1$, so $k > 5$, so $n = 2k > 10$. However, we want the complementary case that n is positive, even, and $n \leq 10$, so $n \in \{2; 4; 6; 8; 10\}$. For odd integers n , there is an integer $k = 0.5(n - 1)$ such that $n = 2k + 1$ and the interval is $k + 0.5 < i < k + (0.2k + 0.6)$. Because of the strict inequality and because consecutive integers are separated by 1, we need, in order for such an i to exist, $0.2k + 0.6 > 1$, so $k > 2$, so $n = 2k + 1 > 5$. However, we want the complementary case that n is positive, odd, and $n \leq 5$, so $n \in \{1; 3; 5\}$. Putting the two cases together, we have $n \in \{1; 2; 3; 4; 5; 6; 8; 10\}$, thus **8** qualifying values.

Approach [2]:

Integers are spaced 1 apart, so $\frac{3}{5}n - \frac{1}{2}n = \frac{1}{10}n > 1$ implies that for $n > 10$ there will be at least one integer in that interval. Let's rewrite the bounds as $0.5n < i < 0.6n$ and look for positive integers at most 10 for n such that no such integer i exists: $n = 1$: $0.5 < i < 0.6$ ✓; $n = 2$: $1.0 < i < 1.2$ ✓; $n = 3$: $1.5 < i < 1.8$ ✓; $n = 4$: $2.0 < i < 2.4$ ✓; $n = 5$: $2.5 < i < 3.0$ ✓; $n = 6$: $3.0 < i < 3.6$ ✓; $n = 7$: $3.5 < i < 4.2$ ✗; $n = 8$: $4.0 < i < 4.8$ ✓; $n = 9$: $4.5 < i < 5.4$ ✗; $n = 10$: $5.0 < i < 6.0$ ✓. There are **8** qualifying cases.

Sprint 15

$$A_{\text{old}} = lw = 60 \text{ ft}^2.$$

$$A_{\text{new}} = (l - 2 \text{ ft})(w + 4 \text{ ft}) = lw + 4 \text{ ft } l - 2 \text{ ft } w - 8 \text{ ft}^2$$

$$= 60 \text{ ft}^2 + 4 \text{ ft } l - 2 \text{ ft } \frac{60 \text{ ft}^2}{l} - 8 \text{ ft}^2$$

$$= 52 \text{ ft}^2 + 4 \text{ ft } l - 2 \text{ ft } \frac{60 \text{ ft}^2}{l}, \text{ which we wish to maximize over } l. \text{ As } l \text{ increases, the first term}$$

remains unchanged, the amount added in the second term increases, and the amount subtracted by the third term decreases—so that the value of the full expression is guaranteed to increase. Thus, the goal is to maximize l , which means to minimize w . Each of w and $l = \frac{60 \text{ ft}^2}{w}$ is specified to be a positive integer number of feet, for which the least of the qualifying values for w is 1 ft, so $l = 60$ ft. Therefore, $\max(A_{\text{new}}) = (60 \text{ ft} - 2 \text{ ft})(1 \text{ ft} + 4 \text{ ft}) = 58 \text{ ft} \times 5 \text{ ft} = \mathbf{290 \text{ ft}^2}$.

Sprint 16

For a triangle to be isosceles, at least two of the three sides must be congruent (same length). We are given two distinct lengths, 20 and 26, so the third length must be one of those two values. Thus, there are two qualifying triangles: 20-20-26 and 26-26-20 [hyphen-separated numbers representing side lengths]. The desired squares of the altitudes are the difference of the square of the first number minus the square of half the third number, thus $20^2 - 13^2 = 400 - 169 = 231$ and $26^2 - 10^2 = 676 - 100 = 576$ [with these computations being based on the desired altitude sitting on the axis of symmetry of the respective triangle so that it divides that triangle into two congruent right triangles, for which the Pythagorean theorem applies]. The sum of these two values is $231 + 576 = \mathbf{807}$.

Sprint 17

Fry does the odd-numbered serves; Amy does the even-numbered serves. Thus, serve #7 would be Fry's fourth serve, so, to meet the criterion on number of serves by Fry, a game must be complete within 6 serves. There is only one way to violate that—at the end of 6 serves, the score must be tied 3-3, and the game will finish 4-3 on the serve #7. The easiest solution involves calculating the probability of getting through 6 rounds with a 3-3 tie (the complement event) and subtracting that probability from 1. That situation is complicated by the probability of a particular player winning a point depending on who served. The odds are different for Fry getting 3 points out of 6 for the cases: [1] Fry wins the point on all 3 of his serves but none of Amy's serves; [2] Fry wins the point on 2 of his serves and 1 of Amy's serves; [3] Fry wins the point on 1 of his serves and 2 of Amy's serves; [4] Fry wins the point on all 3 of Amy's serves but none of his serves. The number of arrangements of Fry winning the point on f of his 3 serves is ${}_3C_f$ and of Fry winning on a of Amy's 3 serves is ${}_3C_a$, so combined is ${}_3C_f \cdot {}_3C_a$ —and, in our scenario, $f + a = 3$; the probability of any one such arrangement is $0.5^3 \times 0.3^a \times 0.7^{3-a}$, with the 0.5 being for each of Fry's 3 serves having probability on 0.5 regardless of outcome, the 0.3 for the probability for each of Amy's a serves that Fry won the point, and the 0.7 for the probability of each of Amy's $3 - a$ serves that Amy won the point. Thus, the probabilities are:

$$\begin{aligned} [1] {}_3C_3 \times {}_3C_0 \times 0.5^3 \times 0.3^0 \times 0.7^0 &= 1 \times 1 \times \frac{1}{8} \times \frac{27}{1000} \times 1 = \frac{27}{8000}; \\ [2] {}_3C_2 \times {}_3C_1 \times 0.5^3 \times 0.3^1 \times 0.7^1 &= 3 \times 3 \times \frac{1}{8} \times \frac{9}{100} \times \frac{7}{10} = \frac{567}{8000}; \\ [3] {}_3C_1 \times {}_3C_2 \times 0.5^3 \times 0.3^1 \times 0.7^2 &= 3 \times 3 \times \frac{1}{8} \times \frac{3}{10} \times \frac{49}{100} = \frac{1323}{8000}; \\ [4] {}_3C_0 \times {}_3C_3 \times 0.5^3 \times 0.3^0 \times 0.7^3 &= 1 \times 1 \times \frac{1}{8} \times 1 \times \frac{343}{1000} = \frac{343}{8000}. \end{aligned}$$

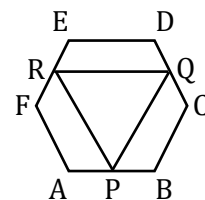
The sum of the probabilities for these 4 disjoint cases is $\frac{27 + 567 + 1323 + 343}{8000} = \frac{2260}{8000} = \frac{113}{400}$. However, this is the probability of the complement of our desired event, so we need to subtract this value from 1, thus $\frac{400 - 113}{400} = \frac{287}{400}$.

Sprint 18

$6^8 = 36^{x+1} = 6^{2(x+1)}$. Now, $6^y = 6^z$ implies $y = z$, so $8 = 2(x + 1) = 2x + 2$, and $x = \frac{8-2}{2} = 3$.

Sprint 19

This problem can be solved using proportions with no need to determine any areas. Given $DE = 2$ in regular hexagon $ABCDEF$ implies $FC = 2$ $ED = 4$. Q and R being midpoints of $\overline{EF}$ and $\overline{CD}$ means that $\overline{QR}$ is the median of trapezoid $DEFC$, so QR is the mean of $DE = 2$ and $FC = 4$, thus $QR = 3$. Thus, triangle PQR is an equilateral triangle with side length 3. Now, hexagon $ABCDEF$ can be subdivided into six congruent equilateral triangles of side length 2, which is equivalent in



area to one equilateral triangle of side length $2\sqrt{6}$. Thus, the ratio of the side length of triangle PQR to the side length of an equilateral triangle equivalent in area of the hexagon is $\frac{3}{2\sqrt{6}}$. Squaring this ratio yields the ratio of the area of PQR to the area of $ABCDEF$, thus $\left(\frac{3}{2\sqrt{6}}\right)^2 = \frac{9}{24} = \frac{3}{8} = 37.5\%$, which rounds to **38** %. [When two candidate rounded values are equally close—here 37 and 38—MATHCOUNTS rounds away from 0, unless explicitly specified otherwise.]

Sprint 20

The tokens (with count) being considered for rearrangement are A (1), B (2), I (2), L (1), PRO (1), T (1), Y (1)—total of $1 + 2 + 2 + 1 + 1 + 1 + 1 = 9$. Therefore, the number of distinct arrangements of these tokens is $\frac{9!}{1! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1!} = \frac{362880}{4} = \mathbf{90\,720}$.

Sprint 21

When there are $n \geq 2$ students in the row, the end 2 out of total n can so leave, and when $n = 1$, that 1 remaining student can so leave. Therefore, $\frac{2}{6} \times \frac{2}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{2}{2} \times \frac{4}{4} = \frac{1}{3} \times \frac{2}{5} \times \frac{1}{3} = \frac{2}{45}$.

Sprint 22

There are two cases: [1] one digit is 0, which cannot be the thousands digit due to requiring 4-digit integers; [2] none of the digits is 0, so no restriction on digit order. When we have chosen 4 distinct digits that sum to 11, in case [1] the 0 has 3 options for placement and the remaining 3 digits have $3!$ arrangements for the remaining 3 slots for a total of $3 \times 3! = 18$ arrangements of the 4 digits, and in case [2] all 4 digits can go in any of the 4 slots for a total of $4! = 24$ arrangements of the 4 digits. The required sum of 11 is achieved for case [1] with the 18 permutations of each of (in increasing order) 1028, 1037, 1046, 2036, 2045 for $5 \times 18 = 90$ distinct values. For case [2] the only possibility is the 24 permutations of 1235. Therefore, there are $90 + 24 = \mathbf{114}$ qualifying integers.

Sprint 23

For $x = 20$, $g(x) = f(x^2) = f(400) = f(1 + 57 \times 7) = f(1)$;

For $x = 22$, $g(x) = f(x^2) = f(484) = f(85 + 57 \times 7) = f(85)$,

and similarly for every value of x in between [namely $20 < x < 22$]:

$g(x) = f(x^2) = f((x^2 - 399) + 57 \times 7) = f(x^2 - 399)$. Therefore, we can work with $f(x)$ in the interval $1 \leq x \leq 85$ instead of $400 \leq x \leq 484$. With f having period 7, we can consider period 1 to be $1 \leq x < 8$, period 2 to be $8 \leq x < 15$, and so on [period k to be $7k - 6 \leq x < 7k + 1$] to period 12 being $78 \leq x < 85$. However, we still need to handle $x = 85$, which, due to the strict inequality on the upper end of our intervals, has been left out so far. Thus, we need to handle $x = 85$ as the one and only point participating in period 13. To avoid double counting points, assign each point to one and only one specific period [thus the " $\leq x <$ " pattern for the bounds of each period]. This means that it suffices to process points in only the first period and count each qualifying point 12 times, unless it is the first point in the period, in which case count it 13 times. We see from the graph that $f(x) = 2$ in the interval $1 \leq x < 8$ at $x \in \{1\frac{2}{5}; 3; 4\frac{1}{3}; 5\frac{1}{3}\}$. That is 4 points, each occurring in 12 periods [1, as the only value occurring in 13 periods, is not in the set], for a total of $4 \times 12 = \mathbf{48}$ points.

Sprint 24

The pattern is $a_n = \sqrt{400 + n}$ because $a_{n+1} = \sqrt{a_n^2 + 1} = \sqrt{400 + n + 1}$, so $a_{500} = \sqrt{900} = \mathbf{30}$.

Sprint 25

There are 5 cards, 3 of which are blue, 2 of which are "5"s, with no overlap of the two categories [because none of the cards is a blue "5"], so all 5 cards are accounted for. There are 9 blue cards other than the blue "5", of which Andrea has 3, thus ${}_9C_3 = 9 \times 8 \times 7 / (3 \times 2 \times 1) = 3 \times 4 \times 7 = 84$ combinations. Andrea has 2 of the 4 non-blue "5" cards, thus ${}_4C_2 = 4 \times 3 / (2 \times 1) = 6$ combinations. The two categories are independent, so altogether there are $84 \times 6 = \mathbf{504}$ combinations.

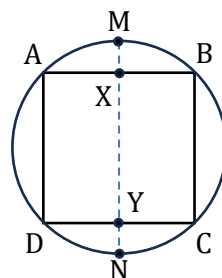
Sprint 26

$AB = 4$ in, X is the midpoint of $\overline{AB}$ so $AX = 2$ in, $AC = 4\sqrt{2}$ in is a diameter of the circle, $\overline{AX} \perp \overline{MN}$.

$$\begin{aligned}\text{Thus, } AM &= \sqrt{AX^2 + XM^2} = \sqrt{(2 \text{ in})^2 + [(2\sqrt{2} - 2) \text{ in}]^2} = \sqrt{4 + (8 + 4 - 8\sqrt{2})} \text{ in} \\ &= \sqrt{16 - 8\sqrt{2}} \text{ in}.\end{aligned}$$

$$\text{Similarly, } AN = \sqrt{AX^2 + XN^2} = \sqrt{(2 \text{ in})^2 + [(2\sqrt{2} + 2) \text{ in}]^2} = \sqrt{16 + 8\sqrt{2}} \text{ in}.$$

$$\begin{aligned}\text{Thus, } AM \cdot AN &= \sqrt{(16 - 8\sqrt{2})(16 + 8\sqrt{2})} \text{ in}^2 = \sqrt{16^2 - (8\sqrt{2})^2} \text{ in}^2 = \sqrt{128} \text{ in}^2 \\ &= 8\sqrt{2} \text{ in}^2.\end{aligned}$$



Sprint 27

$$x^{x^{x+1}} = x^{(x^x)x} = x^{3x} = (x^x)^3 = 3^3 = 27.$$

Sprint 28

Making heavy use of $a^2 - b^2 = (a + b)(a - b)$ and $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$:

$$\begin{aligned}2^{49} - 2 &= 2(2^{48} - 1) = 2(2^{24} + 1)(2^{24} - 1) = 2(2^8 + 1)(2^{16} - 2^8 + 1)(2^{12} + 1)(2^{12} - 1) \\ &= 2(2^8 + 1)(2^{16} - 2^8 + 1)(2^4 + 1)(2^8 - 2^4 + 1)(2^6 + 1)(2^6 - 1) \\ &= 2(2^8 + 1)(2^{16} - 2^8 + 1)(2^4 + 1)(2^8 - 2^4 + 1)(2^2 + 1)(2^4 - 2^2 + 1)(2^3 + 1)(2^3 - 1) \\ &= 2(2^8 + 1)(2^{16} - 2^8 + 1)(2^4 + 1)(2^8 - 2^4 + 1)(2^2 + 1)(2^4 - 2^2 + 1)(2^1 + 1)(2^2 - 1)(7)\end{aligned}$$

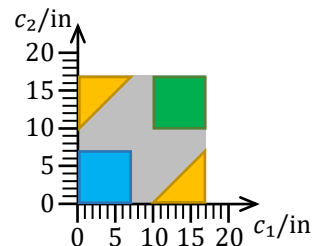
[NOTE: This is not a complete prime factorization, as $2^{16} - 2^8 + 1 = 97 \times 673$; however, it is adequate for answering the question.] We see explicitly factors of the form $2^n + 1$: $2^8 + 1 = 257$, which is prime (checked by attempting division by 2, 3, 5, 7, 11, 13); $2^4 + 1 = 17$, which is prime; $2^2 + 1 = 5$, which is prime; $2^1 + 1 = 3$, which is prime. However, we must be careful as there is one other qualifying value that is not explicitly in the form $2^n + 1$, and that is the prime number $2 = 2^0 + 1$. The sum of all of these values is $257 + 17 + 5 + 3 + 2 = 284$.

Sprint 29

Similar to the triangle inequality for triangles, the sum of the lengths of the three shortest sides of a quadrilateral must be greater than the longest side. Thus, for a 20 in stick after the three cuts, the longest piece must be at least 5 in and less than 10 in long. The initial 3 in piece must be one of the three shortest sides. There are two cuts to be made in the remaining 17 in piece. Designate one end of that piece as position 0 in, the other end as position 17 in, c_1 as the position of the first cut, and c_2 as the position of the second cut, with the restrictions: $0 \text{ in} < c_1 < 17 \text{ in}$, $0 \text{ in} < c_2 < 17 \text{ in}$, $c_1 \neq c_2$, so that we end up with four pieces having lengths: 3 in, $\min(c_1, c_2)$, $\max(c_1, c_2) - \min(c_1, c_2)$, $17 \text{ in} - \max(c_1, c_2)$. To keep all three pieces from the last two cuts less than 10 in long, we must avoid these cases:

- [1] $\max(c_1, c_2) \leq 7 \text{ in}$ [the piece from the 17 in end is at least 10 in long];
- [2] $\min(c_1, c_2) \geq 10 \text{ in}$ [the piece from the 0 in end is at least 10 in long];
- [3] $|c_1 - c_2| \geq 10 \text{ in}$ [the middle piece is at least 10 in long].

The gray shaded region is the qualified area, which is the total shaded area $(17 \text{ in})^2 = 289 \text{ in}^2$ minus the area shaded other than gray. The blue and green squares are $(7 \text{ in})^2 = 49 \text{ in}^2$ each and the gold triangles together constitute another square of area 49 in^2 , leaving $289 \text{ in}^2 - 3 \times 49 \text{ in}^2 = 142 \text{ in}^2$. Therefore, the desired probability is the ratio of the gray shaded area to the total shaded area, $\frac{142 \text{ in}^2}{289 \text{ in}^2} = \frac{142}{289}$.



Sprint 30

[NOTE: Modern axioms of probability prevent choosing from an infinite set of integers in an equally likely manner, thus the “unusual” distribution.]

The k th interdimensional number is 1 less than the $(k + 1)$ th triangular number, thus

$\frac{(k+1)(k+2)}{2} - 1 = \frac{k(k+3)}{2}$, which is the n in the probability expression; $n + 1$ is $\frac{(k+1)(k+2)}{2}$. Thus, the

probability to pick $n = \frac{k(k+3)}{2}$ is $\frac{1}{n(n+1)} = \frac{4}{k(k+1)(k+2)(k+3)}$. The probability of picking a positive

integer and having it be interdimensional is the sum of this expression over all positive integer values of k —an infinite series. A commonly useful technique to handle such expressions is telescoping, which involves turning each term into a subtraction of two terms. Try having the first term of the subtraction to be the above single fraction but drop the last factor in the denominator and the second term be the above fraction but drop the first factor:

$\frac{4}{k(k+1)(k+2)} - \frac{4}{(k+1)(k+2)(k+3)} = \frac{4[(k+3)-k]}{k(k+1)(k+2)(k+3)} = \frac{4 \times 3}{k(k+1)(k+2)(k+3)}$, which is 3 times our probability

fraction above, so we have $\frac{4}{k(k+1)(k+2)(k+3)} = \frac{1}{3} \left[\frac{4}{k(k+1)(k+2)} - \frac{4}{(k+1)(k+2)(k+3)} \right]$. Let's write the first 3

terms [k being successively 1, 2, 3]: $\left[\frac{1}{3} \left(\frac{4}{6} \right) - \frac{1}{3} \left(\frac{4}{24} \right) \right] + \left[\frac{1}{3} \left(\frac{4}{24} \right) - \frac{1}{3} \left(\frac{4}{60} \right) \right] + \left[\frac{1}{3} \left(\frac{4}{60} \right) - \frac{1}{3} \left(\frac{4}{120} \right) \right] + \dots$. Note

how the third term cancels the second and the fifth term cancels the fourth. This cancellation

continues on forever, leaving only the first term, so the answer is $\frac{1}{3} \left(\frac{4}{6} \right) = \frac{2}{9}$.

Target 1

The bus ride is the vertical leg of the right triangle, so, by Pythagoras, $\sqrt{(100 \text{ mi})^2 - (30 \text{ mi})^2} = \sqrt{(10\,000 - 900) \text{ mi}^2} = \sqrt{9100} \text{ mi} = 95.3939\dots \text{ mi}$, which rounds to the nearest 0.1 mi as **95.4** mi.

Target 2

The figure shown here is drawn to scale (which is not required to be the case for handbook and competition materials and is not the case for the text of this problem).



The area A of any parallelogram and, therefore, any rhombus is the product bh , where

b is the base and h is the height, which must be perpendicular to one another. In this case, b is the length of a horizontal side of the rhombus and h is the vertical diameter of the circle, indicated by the blue line. We are given that the area πr^2 enclosed by the circle is 25π , so $r^2 = 25$ and $r > 0$,

making the radius 5 and the diameter double that, thus 10. Therefore, $b = \frac{A}{h} = \frac{120}{10} = 12$. In a rhombus, all 4 sides are congruent, so the perimeter is $4 \times 12 = \mathbf{48}$.

Target 3

The 6 is by far most effectively used as an exponent applied to a base of 2, leaving 0 and 2, for which 2 should be the base and 0 the exponent. Therefore, $2^6 + 2^0 = 64 + 1 = \mathbf{65}$.

Target 4

The least card values that can be chosen are 2 and 2 for a sum of 4, for which the die must be 2—the count of such events is $\frac{4 \times 3}{2} \times 1 = 6$. The next level up for cards is 2 and 3 for a sum of 5, for which the die must be 1—the count of such events is $\frac{8 \times 4}{2} \times 1 = 16$. Any higher sum on the cards added to any die value exceeds the desired sum of 6. That is a total of $6 + 16 = 22$ winning combinations. The total number of combinations $\frac{36 \times 35}{2} \times 6 = 3780$. Therefore, the desired probability is $\frac{22}{3780} = \frac{\mathbf{11}}{1890}$.

Target 5

$Q(n)$ is the sum of the first n terms of the underlying divergent infinite series $\sum_{k=1}^{\infty} \lfloor \sqrt{k} \rfloor$, which can be expressed also as $1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 4 + \dots$, with 3 copies of 1 summed with 5 copies of 2, 7 copies of 3, and generally $2k + 1$ copies of each positive integer k . The table below shows the progression of evaluations that lead to the answer of the given problem: Column 1 is the band of k values for which the floor of the square root of k that is given in Column 2. Because the terms in the underlying series above are piecewise constant, the sequence of values $Q(1), Q(2), Q(3), Q(4), Q(5), \dots [1, 2, 3, 5, 7, \dots]$ is piecewise linear as seen in Column 3 (which is provided for additional insight but is not needed to reach the answer)—more specifically piecewise arithmetic as seen in Column 4 that gives the initial value of the respective arithmetic sequence piece, the common difference d between consecutive terms of that arithmetic sequence piece, and the final value of the arithmetic sequence piece. Column 5 gives the familiar calculation of the sum of a finite arithmetic series. Column 6 gives the cumulative sum of the desired $Q(1) + \dots + Q(k_{\max})$.

$k_{\min} \dots k_{\max}$	$\lfloor \sqrt{k} \rfloor$	$Q(k)$	$Q(k_{\min}) [d] Q(k_{\max})$	$Q(k_{\min}) + \dots + Q(k_{\max})$	$Q(1) + \dots + Q(k_{\max})$
1 .. 3	1	k	1 [1] 3	$3 \times \frac{1+3}{2} = 6$	6
4 .. 8	2	$2k - 3$	5 [2] 13	$5 \times \frac{5+13}{2} = 45$	$6 + 45 = 51$
9 .. 15	3	$3k - 11$	16 [3] 34	$7 \times \frac{16+34}{2} = 175$	$51 + 175 = 226$
16 .. 24	4	$4k - 26$	38 [4] 70	$9 \times \frac{38+70}{2} = 486$	$226 + 486 = 712$
25 .. 35	5	$5k - 50$	75 [5] 125	$11 \times \frac{75+125}{2} = 1100$	$712 + 1100 = 1812$
36 .. 48	6	$6k - 85$	131 [6] 203	$13 \times \frac{131+203}{2} = 2171$	$1812 + 2171 = 3983$

The sum through $Q(35)$ is short of 2026 by 214. $Q(36) = 131$, which makes up a little over half of the shortfall, so $Q(37) = 137$ should finish it off:

$$[(Q(1) + \dots + Q(35)) + Q(36) + Q(37) = 1812 + 131 + 137 = 2080 \geq 2026, \text{ so, indeed, } n = 37.]$$

Target 6

Let the number of sides for the three regular polygons be n_1, n_2 , and n_3 in decreasing order. For those polygons to share sides with a common vertex, the combined interior angle measures around that vertex is 360° . The interior angle measures of regular polygon of n sides is $180^\circ - \frac{360^\circ}{n}$, so

$$360^\circ = \left(180^\circ - \frac{360^\circ}{n_1}\right) + \left(180^\circ - \frac{360^\circ}{n_2}\right) + \left(180^\circ - \frac{360^\circ}{n_3}\right) = 540^\circ - \left(\frac{360^\circ}{n_1} + \frac{360^\circ}{n_2} + \frac{360^\circ}{n_3}\right), \text{ and}$$

$$\frac{360^\circ}{n_1} + \frac{360^\circ}{n_2} + \frac{360^\circ}{n_3} = 180^\circ. \text{ Dividing both sides by } 360^\circ \text{ yields } \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2}. \text{ We have } \frac{1}{n_1} = \frac{1}{n_2} = \frac{1}{n_3}$$

if $n_1 = n_2 = n_3 = 6$ (and one of the figures given in the problem corresponds to this case);

otherwise, $\min(n_1, n_2, n_3) < 6$ and $\max(n_1, n_2, n_3) > 6$. Polygon 3 is the polygon with the fewest

sides, so $3 \leq n_3 \leq 5$. Letting $n_3 = 3$: $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{3} = \frac{1}{2}$, so $\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{6}$, which, in turn, means

$$n_1 = \frac{6n_2}{n_2-6} = 6 + \frac{36}{n_2-6}, \text{ with } n_1 \geq n_2 \geq 7. \text{ Minimize } n_2 \text{ to maximize } n_1, \text{ so try } n_2 = 7: n_1 = 6 + \frac{36}{7-6} =$$

$6 + 36 = 42$, which is an integer. Thus, $(n_1, n_2, n_3) = (42; 7; 3)$ yields the maximum count of sides to

be 42 so far. Try $n_3 = 4$ next: $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{4} = \frac{1}{2}$, so $\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{4}$, so $n_1 = \frac{4n_2}{n_2-4} = 4 + \frac{16}{n_2-4}$, so

$n_1 \geq n_2 \geq 5$. The optimum case $n_2 = 5$ yields $n_1 = 4 + \frac{16}{5-4} = 4 + 16 = 20$, which does not beat our

maximum of 42. Lastly, $n_3 = 5$: $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{5} = \frac{1}{2}$, so $\frac{1}{n_1} + \frac{1}{n_2} = \frac{3}{10}$, so $n_1 = \frac{10n_2}{3n_2-10}$, so $n_1 \geq n_2 \geq n_3 = 5$;

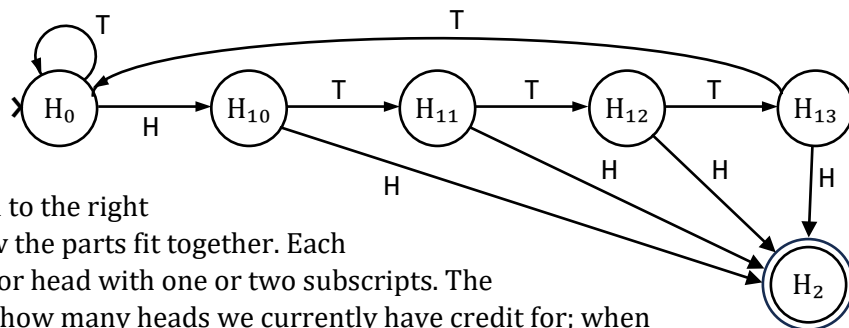
however, $n_2 = 5$ yields $n_1 = 10$ —also short. Therefore, the maximum count of sides is **42**.

Target 7

$\frac{V_{\text{sph}} + V_{\text{con}}}{\text{ft}^3} = \frac{\frac{4}{3}\pi(r \text{ ft})^3 + \frac{1}{3}\pi(r \text{ ft})^2(2r \text{ ft})}{\text{ft}^3} = \frac{4}{3}\pi r^3 + \frac{2}{3}\pi r^3 = 2\pi r^3$ is the number of cubic feet in the stated sum;
 $\frac{A_{\text{cyl}}}{\text{ft}^2} = \frac{2\pi(3r \text{ ft})^2 + 2\pi(3r \text{ ft})(4r \text{ ft})}{\text{ft}^2} = 18\pi r^2 + 24\pi r^2 = 42\pi r^2$ is the number of square feet in the stated area. The equality of these two values $2\pi r^2 r = 21 \times 2\pi r^2$ means that $2\pi r^2(r - 21) = 0$, so $r^2 = 0$ or $r - 21 = 0$. Now, $r^2 = 0$ means $r = 0$, so that each shape is a single point and, thus, degenerate, not qualifying as truly of the specified shape. Therefore, $r = 21$ and $\frac{A_{\text{cyl}}}{\text{ft}^2} = 6r^2 = 6 \times 21^2 = \mathbf{2646}$ is the number of square feet in the cube's surface area.

Target 8

This answer is unusually long due to few students having the depth of experience with expected values needed for this problem. The state diagram to the right



shows what is going on and how the parts fit together. Each

of the six states is labeled as H for head with one or two subscripts. The

first or only subscript indicates how many heads we currently have credit for; when that subscript is 1, there is a second subscript indicating how many of the 4 allowed attempts to get a second H have been taken but T resulted. State H_0 indicates a status of not having any credit for a head (H). That is the starting state, and it is returned to whenever we get 4

consecutive tails (TTTT), requiring starting over from scratch in trying to get 2 heads within 5 flips—except that we do not get to reset our flip count back to 0. We stay there as long as we keep getting T flips, leaving only once we flip H, in which case we move to state H_{10} and get 4 chances to flip another H. We get just one flip at H_{10} , moving to the final state H_2 if that one flip is H, or to state H_{11} for a second chance if that one flip is T. Once we get to H_2 , that round of the game ends and the number of flips to get to the end is recorded. States H_{11} and H_{12} work similarly to H_{10} . If we get to H_{13} , that means we have made 3 attempts to get a second H, and all 3 attempts yielded T—you have one last attempt, for which an H gets you to the desired H_2 just as would happen in the three previous attempts, but if you get one more T, you go back to H_0 , losing credit that that one H. Each transition showing an H has probability $\frac{2}{3}$ of occurring, and each transition showing a T has

probability $\frac{1}{3}$. The idea of this problem is, after many rounds have been played, what is the average number of flips to finish a round? We can calculate the expected value E for the number of flips in a round. Because the probability of any one flip being an H at state H_0 is an unchanging $\frac{2}{3}$ no matter how many failed flips (T) we have had so far, the expected value of the average number of flips needed to get to H_{10} is the reciprocal of the probability of it happening of your first try, thus $\frac{1}{2/3} = \frac{3}{2}$; however, the remaining expected value calculations are more involved. The probability of flipping H at state H_{10} is $\frac{2}{3}$ and involves only 1 flip to get to H_2 , so the HH path to H_2 contributes expected value $\frac{2}{3} \times 1 = \frac{2}{3}$. The probability of flipping T at state H_{10} is $\frac{1}{3}$ and H at H_{11} is $\frac{2}{3}$ and involves 2 flips to get to H_2 , so the HTH path to H_2 contributes expected value $\frac{1}{3} \times \frac{2}{3} \times 2 = \frac{4}{9}$. The probability of flipping T at state H_{10} is $\frac{1}{3}$, T at H_{11} is $\frac{1}{3}$, and H at H_{12} is $\frac{2}{3}$ and involves 3 flips to get to H_2 , so the HTTH path to H_2 contributes expected value $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times 3 = \frac{2}{9}$. The probability of flipping T at state H_{10} is $\frac{1}{3}$, T at H_{11}

is $\frac{1}{3}$, T at H_{12} is $\frac{1}{3}$, and H at H_{13} is $\frac{2}{3}$ and involves 4 flips to get to H_2 , so the HTTTH path to H_2 contributes expected value $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} \times 4 = \frac{8}{81}$. Finally, the start-all-over-again move involves the probability of flipping T being $\frac{1}{3}$ at each of H_{10} , H_{11} , H_{12} , and H_{13} , and requiring 4 flips that must be counted even though wasted, and on top of all of that, we are not at H_2 , but starting over again at H_0 , which adds another E (with a still unknown value) flips, for an expected value contribution of $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times (4 + E) = \frac{4}{81} + \frac{1}{81}E$. All combined: $E = \frac{3}{2} + \frac{2}{3} + \frac{4}{9} + \frac{2}{9} + \frac{8}{81} + \frac{4}{81} + \frac{1}{81}E = \frac{483}{162} + \frac{1}{81}E$, so $\frac{80}{81}E = \frac{483}{162}$ and $E = \frac{94}{80} \times \frac{483}{2 \times 94} = \frac{483}{160}$.

Team 1

The only digits that divide 100 evenly are 1, 2, 4, and 5. The most important goal to achieve the answer to the problem is to minimize the number of digits: 1 accomplishes nothing to get to a product of 100 but adds to the digit count; 4 and 2×2 contribute identically to a product but 4 uses one less digit. $100 = 4 \times 5 \times 5$ minimizes the digit count. Next in importance is to sequence the digits in increasing order from left to right to achieve the desired goal, thus **455**.

Team 2

In 1 h = 90 min, the minute hand moves one full turn of 360° , for a rate of $360^\circ/(90 \text{ min}) = 4^\circ/\text{min}$. In 8 h, the hour hand moves one full turn of 360° , for a rate of $360^\circ/(8 \text{ h}) = 45^\circ/\text{h} = 45^\circ/(90 \text{ min}) = 0.5^\circ/\text{min}$. Assuming 0° is straight up toward the “8” with angle increasing clockwise, at 2:00 the hour hand is at $2 \text{ h} \times 45^\circ/\text{h} = 90^\circ$ while the minute hand is at 0° ; Then 24 min later, the hour hand is at $90^\circ + 24 \text{ min} \times 0.5^\circ/\text{min} = 102^\circ$, and the minute hand is at $24 \text{ min} \times 4^\circ/\text{min} = 96^\circ$. Thus, the angular separation between the hands is $|102^\circ - 96^\circ| = 6^\circ$.

Team 3

$S \neq 0$ implies $I \geq 2$. Any odd I multiplied by 50 yields an integer ending in 50. Any integer I leaving a remainder of 1 when divided by 4 can be multiplied by 25 or 75 to yield an integer ending with 25 or 75, respectively. Then 6 times any even digit ends with that same digit, so 6 can be multiplied by 20, 40, 60, or 80 yielding a value that has the same last two digits as the number being multiplied by 6. Those are all the candidates. To maximize AIMS, we want the maximum A, which is 8 from $6 \times 80 = 480$, so $M = 0$, $I = 6$, and $S = 4$, resulting in AIMS = **8604**.

Team 4

Reorganizing List A in increasing order yields the 6 values: 11, 11, 13, 23, 23, 104. The median of these 6 values is the mean of the third and fourth values, thus $\frac{13+23}{2} = 18$. The added values 17 and 23 in List B can be inserted right after the 13 as the new fourth and fifth values, the mean of which is the median of 8 values, thus $\frac{17+23}{2} = 20$, so the median increases by $q = 20 - 18 = 2$. The mean of List A is $\frac{11+11+13+23+23+104}{6} = \frac{185}{6}$. For the mean of List B and the amount of decrease from List A, $p = \frac{185}{6} - \frac{185+17+23}{6+2} = \frac{185}{6} - \frac{225}{8} = 2.708\ldots$. Finally, we add $p + q = 2.708\ldots + 2 = 4.708\ldots$ and round to the nearest tenth to obtain the answer **4.7**.

Team 5

Let $MA = 12 - 7 = 5$ along the y -axis be the base of $\triangle MAP$. Then to have an area of at least 15 for $\triangle MAP$, a height of at least 6 is required, meaning the absolute value of the x -coordinate of P must be at least 6. Because P lies on $\overline{TH}$, with T at $(3; 2)$ and H at $(10; 8)$, choosing a point randomly on $\overline{TH}$ is equivalent to choosing randomly an x -coordinate in the interval $3 \leq x \leq 10$, with total length of $10 - 3 = 7$ but only $6 \leq x \leq 10$, with length $10 - 6 = 4$, satisfies the triangle area requirements, so the probability of such is $\frac{4}{7}$.

Team 6

The altitude being 48 in means each side is $\frac{2}{\sqrt{3}} \times 48 \text{ in} = 32\sqrt{3} \text{ in}$ and the area of the window is $768\sqrt{3} \text{ in}^2$. When $0 < h < 48$, the shaded region is a trapezoid of height h in, lower base length $32\sqrt{3} \text{ in}$, and upper base length $32\sqrt{3} \left(1 - \frac{h}{48}\right)$ in (based on taking advantage of proportions of similar shapes and of complementarity). The area of a trapezoid is the product of the height and the average of the base lengths, thus $32\sqrt{3} \left(1 - \frac{h}{96}\right) h \text{ in}^2 = \left(-\frac{\sqrt{3}}{3} h^2 + 32\sqrt{3}\right) \text{ in}^2$, so $p = -\frac{\sqrt{3}}{3}$, $q = 32\sqrt{3}$, and $pq = -32$.

Team 7

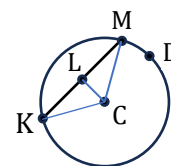
The prime factorization of 18 is $2 \times 3 \times 3$, so a positive integer with exactly 18 positive divisors must have 1, 2, or 3 distinct prime divisors in a total of 4 forms:

p^{17}	whose square is	p^{34}	that has	$34 + 1 = 35$	divisors;
$p^8 q$	whose square is	$p^{16} q^2$	that has	$(16 + 1)(2 + 1) = 51$	divisors;
$p^5 q^2$	whose square is	$p^{10} q^4$	that has	$(10 + 1)(4 + 1) = 55$	divisors;
$p^2 q^2 r$	whose square is	$p^4 q^4 r^2$	that has	$(4 + 1)(4 + 1)(2 + 1) = 75$	divisors.

The greatest count of factors is **75**.

Team 8

The figure shows the given points C and D as well the portion, notated as $\overline{KM}$, of the given line that is on or interior to the circle. A circle with center $C(11; 4)$ passing through $D(17; 10)$ has radius $\sqrt{(17 - 11)^2 + (10 - 4)^2} = \sqrt{6^2 + 6^2} = 6\sqrt{2}$. There is no further need of D . Let L be the point on the given line that is closest to C ; $\overline{CL} \perp \overline{KM}$. The shortest distance between the circle center and the line $x - y - 1 = 0$ is given by



$CL = \left| \frac{11 - 4 - 1}{\sqrt{1^2 + (-1)^2}} \right| = \frac{6}{\sqrt{2}} = 3\sqrt{2}$, with L being at $(-3; 3)$ relative to C , thus at $L(8; 7)$. CKL and CML are congruent right triangles with common leg CL being $\frac{1}{2}$ of the length of the respective hypotenuses CK and CM (both of which are radii of circle C). That means that the sides of both triangles CKL and CML have length ratios in increasing order of $1: \sqrt{3}: 2$, which means angle measures of 30° - 60° - 90° , so $m\angle KCM = m\angle KCL + m\angle LCD = 60^\circ + 60^\circ = 120^\circ$. That means that the area of sector KCM is $\frac{1}{3}$ the area of the whole circle, which, in turn, means the desired area, the area of the smaller segment of circle C formed by line segment KM , is the area of the sector KCM minus the area of triangle KCM . The area enclosed by the circle is $\pi(6\sqrt{2})^2 = 72\pi$, $\frac{1}{3}$ of which is 24π . We know that $CL = 3\sqrt{2}$, $KL = LM = \sqrt{3} CL = 3\sqrt{6}$, and $KM = KL + LM = 6\sqrt{6}$, so the area of KCM is $\frac{(3\sqrt{2})(6\sqrt{6})}{2} = 9\sqrt{12} = 18\sqrt{3}$. Therefore, the desired area is $24\pi - 18\sqrt{3} = 44.221\dots$, which rounds to the nearest tenth as **44.2**.

Team 9

The table below itemizes cases based on number of 20s and 26s in the sum. The “probability per arrangement” is $1/2$ raised to the number of terms in the sum. The “number of arrangements” is how many distinct orders the terms in the sum can be arranged—be careful, because nominally the second row should have 5 arrangements, but the 20 must not be the last term because the first 4 terms being 26 already exceed 100 so is covered by the first row. The “probability of case” is the product of the probability per arrangement times the number of arrangements. The expected value is the product of the sum and the probability of case. The “Total” row gives for each meaningful column the sum, over the cases, of the values in that column; “—” is used where a sum is not useful.

Case	Sum	Probability per Arrangement	Number of Arrangements	Probability of Case	Expected Value
$26 + 26 + 26 + 26$	104	$1/16$	1	$1/16$	$104/16$
$20 + 26 + 26 + 26 + 26$	124	$1/32$	4	$2/16$	$248/16$
$20 + 20 + 26 + 26 + 26$	118	$1/32$	10	$5/16$	$590/16$
$20 + 20 + 20 + 26 + 26$	112	$1/32$	10	$5/16$	$560/16$
$20 + 20 + 20 + 20 + 26$	106	$1/32$	5	$5/32$	$265/16$
$20 + 20 + 20 + 20 + 20$	100	$1/32$	1	$1/32$	$50/16$
Total	—	—	—	1	$1817/16$

The desired answer is in the cell at the lower right corner of the table: $\frac{1817}{16}$.

Team 10

For all positive integers m and n , $\gcd(m, n) \cdot \text{lcm}(m, n) = mn$, so it is reasonable to ask if the same concept works upon bringing in a third positive integer l : $\gcd(l, m, n) \cdot \text{lcm}(l, m, n) = lmn$? No, that typically does not work—let’s see why not, and what is the most general case that will work for three input values. Let’s focus on an arbitrary prime factor p , and say that l has x factors of p (that is, p^x divides l but p^{x+1} does not), m has y factors of p , and n has z factors of p , with x , y , and z being nonnegative integers. Then $\gcd(m, n)$ has a factor $p^{\min(y, z)}$ and $\text{lcm}(m, n)$ has a factor $p^{\max(y, z)}$, so the product of the gcd and lcm has $\min(y, z) + \max(y, z) = y + z$ factors of p , which is the number of factors of p in mn —this always works when two values are being processed. Going to three values, $\gcd(l, m, n)$ has a factor $p^{\min(x, y, z)}$ and $\text{lcm}(l, m, n)$ has a factor $p^{\max(x, y, z)}$, so the product of the gcd and lcm has $\min(x, y, z) + \max(x, y, z)$ factors of p , while the product lmn has $x + y + z$ factors of p —for the two sums to be equal, the middle value $x + y + z - (\min(x, y, z) + \max(x, y, z))$ must be 0, which then requires the minimum value to be 0; in other words, at most one of x , y , and z can be nonzero, so p can divide at most one of l , m , and n —then, and only then, the formula given by the problem can work. Thus, for the given problem with the three values 30, a , and b , no two of those values can have a common divisor greater than 1, thus $\gcd(30, a) = \gcd(30, b) = \gcd(a, b) = 1$, which is referred to mathematically as “30, a , and b are mutually relatively prime”. Because $30 = 2 \times 3 \times 5$, a and b must not have any of 2, 3, or 5 as factors. Now, $10 \leq a \leq 99$ and $10 \leq b \leq 99$ are given as restrictions. Thus, a and b are required to be [1] distinct 2-digit primes or [2] the product of primes with the product having two digits. In case [1], there are 21 two-digit primes (25 primes under 100 minus 4 primes under 10), so a can be any one of those 21 primes and b can be any of the 20 remaining primes, so $21 \times 20 = 420$ choices. Case [2] must involve the square of a one-digit prime (other than 2, 3, or 5—meaning 7) or the product of a one-digit prime and a small two-digit prime, leaving only $7^2 = 49$, $7 \times 11 = 77$, and $7 \times 13 = 91$ for one of a and b , with one of the remaining two-digit primes for the other, thus another $2 \times (21 + 20 + 20) = 122$ choices. Therefore, the total count of choices is $420 + 122 = 542$.