American Regions Mathematics League Contests 2015–2020

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Preface: American Regions Math League

My first visit to ARML was in 2002. A coach in our local (upstate New York) mathematics league noticed that I enjoyed problem solving and that I enjoyed seeing teenagers solve problems, and she suggested I get on a bus for a weekend at Penn State. I cheerfully agreed, and I spent two days training for competition with the Upstate New York Math Team (two of whose students were IMO participants!). It was mind-blowingly awesome to watch the students solve problems in Eisenhower Auditorium and to see a national champion crowned.

I spent most of the day during ARML2002 as a grader in the Power Question grading room. I saw some of the most brilliant people in the mathematics community reading excellent mathematics, talking about excellent mathematics, making jokes with mathematical punch lines, and the like. I was hooked from the very beginning. I have been to every ARML since. ARML brings the mathematical community together for face-to-face competition and so much more.

ARML was founded in 1976 as the Atlantic Region Mathematics League. It was an outgrowth of NYSML, the New York State Mathematics League, founded in 1973, and designed to serve as a competition for the best of the teams in the math leagues in New York. Both were the joint vision of Alfred Kalfus and Steve Adrian. Alfred served as the first President of ARML and Steve, along with Joe Quartararo (from 1976 to 1982), did the organizational work, site preparations, and publicity. Marty Badoian as Vice President and Eric Walstein also played crucial roles in ARML's early days. The impetus for ARML came when Marty Badoian brought a Massachusetts team to NYSML in 1974 and 1975 and they did so well that an expanded competition seemed desirable. ARML was conceived as an interstate competition covering the eastern seaboard, formed through the joint action of the New York State Mathematics League, the New England Association of Mathematics Leagues, and leagues from New Jersey, Pennsylvania, Maryland, and Virginia. But ARML is a great contest and it soon attracted math students and math teachers from all over the country. By 1984 ARML was renamed the American Regions Mathematics League. It takes place simultaneously at four sites: Penn State University in State College, Pennsylvania; the University of Iowa in Iowa City, Iowa; the University of Georgia in Athens, Georgia; and the University of Nevada at Las Vegas. (Beginning after 2020, the southern site will be at the University of Alabama at Huntsville.) ARML brings together some 2200 of the finest young mathematicians in the United States and Canada. In the past, teams from Russia have competed and in the last few years, teams from China, Taiwan, Macau, Vietnam, South Korea, the Philippines, Brazil, and Colombia have been taking part. Taiwanese educators even created a similar competition for schools in Taiwan called TRML.

ARML is a competition between regions. A region may be as large as a state; there are teams from Texas, Minnesota, and Georgia. A region may be half a state; for example, eastern and western Massachusetts field teams. A region may be a county, such as Nassau county in New York. A region may be a city such as Chicago or New York, or a region may even be a high school. Each team consists of 15 students and a region may send more than one team.

The contest consists of 6 parts. First is the Team Round. A team's 15 students are in one classroom. They receive 10 problems to solve as a group in 20 minutes. They cannot use calculators (though they were able to use calculators on the Team Round for about a decade ending with the 2008 competition). Typically, problems are divided up so that every problem is being worked on by at least one student, answers are posted, and hurried consultations take place if there is disagreement. Next comes the Power Question. Teams are given 60 minutes to solve a series of in-depth questions on one topic and/or prove a number of theorems on that topic. The teams' papers are graded by a hardworking group of teachers tucked away in a corner of the auditorium. Following the Power Question, the teams come together in a large auditorium or arena for the Individual Round. Here, starting in 2009, students individually solve 10 problems, given in pairs, with 10 minutes for each pair. Only the answer is scored; no supporting work is considered. Neither calculators nor collaboration are allowed. Initial problems are easier, but an easier problem is generally paired with a more difficult one. We try to write problems so that 80–90% of the students can solve the first problem and less than 5% can solve the last one. The next round is the Relay Round. The teams of 15 are divided up into 5 groups of 3. Each first person in a group gets the same problem, each second person gets the same problem, and each third person gets the same problem. The first person's problem has all the information necessary to solve it, but the second person's problem requires the first person's answer, and the third person needs the second person's answer. None of the three knows the other's problems. A well-written relay problem enables the second and third team members to do considerable work while waiting. They may even be able to discover a candidate pool of likely answers to their problems.

Those four rounds are the only rounds that count for the Team and Individual competition. Scoring is as follows: each correct answer on the Team Round earns 5 points. Sometimes a team gets all 50 points, but 40 or 45 has typically been the top score and the average for all the teams has been between 20 and 25. The Power Question is worth 50 points; often one or more teams earn a perfect score. Each Individual problem is worth 1 point per contestant, meaning that the team can earn as many as $10 \cdot 15 = 150$ points on this round. Generally, 100 is a great score. On the Relay Round, only the third person's answer is scored. If the group of 3 gets the problem correct within 3 minutes they earn 5 points, and a correct answer within 6 minutes earns 3 points. Thus, each relay is worth a maximum of $5 \cdot 5 = 25$ points; a score of 14 is quite good. The winning team's score for the entire contest usually lies between 225 and 250 points.

The Super Relay has followed the Relay Round since 1996. It is just for fun and is a relay race for the whole team involving 15 problems. The first team that gets the correct answer to the last problems is the winner, earning a round of applause and bragging rights 'til next year. To keep the Super Relay from getting bogged down, we usually slip one or two problems into the relay that can be solved without requiring the previous person's answer. Students pass answers from both ends into position 8. That student's problems requires two answers in order to be solved.

The Tiebreaker Round follows the relays. Until 2006, ARML gave only 3 top individual prizes, forcing a playoff to take place among the top scorers. With increased sponsorship, ARML was able to increase the number of awards to 20. The Tiebreaker is a dramatic moment. At each site the contenders come to the front of the auditorium, and they receive the problem at the same time that it is flashed on an overhead screen. Each student is timed and the winners are determined by who gets the correct answer most quickly. Three tiebreakers are now given. A site will give tiebreakers until it either has 20 correct answers or has exhausted the problems and students. Afterward times are compared with the other sites.

ARML is quite an undertaking. It brings together great students, and it is one of the few mathematics contests that involves exciting travel, meeting lots of other students, and renewing friendships established during summer programs. It relies on the work of a large number of dedicated adults who do all the organizing and spend hours finding or writing problems to use in practice sessions. It is a contest that promotes and demands creativity and imagination. The students who take part in ARML are very experienced problem solvers, quick and insightful. Those of us who write problems for ARML respect the abilities of our participants and, consequently, we spend hours developing problems that spring from high school mathematics, yet are out of the ordinary, problems that can't be attacked in rote fashion. We don't just take a theorem and write a problem that employs it. We ask questions, we imagine situations we'd never thought of before, we try out this idea and that possibility, we run into dead-ends, and we occasionally stumble across a really neat idea and that's the one that makes it into an ARML competition. There are many different topics on each year's ARML, but more important, there is a wide variety of ways of thinking.

This publication makes available the problem sets and solutions for the 2015 to 2020 ARML contests. Six earlier publications contain ARML problems from its beginning in 1976 to 2014 and NYSML problems from its beginning in 1973 to 1992. The first was *NYSML-ARML Contests 1973–1982*, published in 1983 by Mu Alpha Theta. The second was *NYSML-ARML Contests 1983–1988* by Gilbert Kessler and Lawrence Zimmerman, published by the National Council of Teachers of Mathematics in 1989. The third was *ARML-NYSML Contests 1989–1994* by Lawrence Zimmerman and Gilbert Kessler, published by MathPro Press in 1995. The fourth was *American Regions Math League & ARML Power Contests* by Don Barry and Thomas Kilkelly, published by ARML in 2003. The fifth was *American Regions Math League & Power & Local Contests* by Don Barry, Paul Dreyer, and Thomas Kilkelly, published by ARML in 2009. The sixth was *American Regions Math League & Power & Local Contests* by Paul J. Karafiol, Thomas Kilkelly, Micah Fogel, and Paul Dreyer, published by ARML in 2015.

ARML has an executive board that organizes each contest. Alfred Kalfus was president from 1976 until 1989 when Mark Saul of Bronxville High School, NY took over. Mark retired in 2001 and Tim Sanders, director of the Great Plains Mathematics League, took over. In June of 2004, J. Bryan Sullivan became president. In June of 2013, Paul Dreyer became president. Current board members include Steve Adrian, Don Barry, Linda Berman, Barbara Chao, Steve Condie, Michael Curry, Paul Dreyer, Bowen Kerins, Quan Lam, Jason Mutford, George Reuter, Matthew Schneider, and Anchala Sobrin.

I became a co-author of ARML in 2014. I was (and still am) the head author for NYSML, and I had contributed some problems for ARML Local in the past. It has been a challenge to construct contests like NYSML and ARML. How do we challenge students who come to these contests from a varied set of backgrounds? How do we write problems that are accessible to the broad cross-section of students who will take the contest while making sure that very few students can solve the toughest problems? ARML wants to make sure that every student is challenged but also that every student finds some success. We continue to endeavor to write contests that allow all students to showcase their skills while giving them some questions that will push their mathematical creativity to new heights.

ARML comes together because of the hard work of a dedicated group of problem writers and editors. Since I became head author in 2017, ARML has worked to expand the committee, and because of this expansion, we have been able to provide problems for ARML and ARML Local beginning in 2020. The problem committee currently consists of Evan Chen, Thinula De Silva, Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Silas Johnson, Winston Luo, Jason Mutford, Andy Niedermaier, Graham Rosby, Tristan Shin, April Verser, and of course myself. Problems are proposed in the late summer. Drafts are sent to the committee in October. Problems are proposed for specific slots in the contest. Then solutions are written and edited and rewritten and reedited. Problems are recast to be crystal clear for students and coaches alike. Then, Chris Jeuell heads up the final editing process in the March before the contest takes place. Chris makes sure every 'i' is dotted and every 't' is crossed. I am grateful for his watchcare over the final product, and I am grateful for the care and concern that every author takes to ensure an excellent contest for all involved.

There are so many people to thank for the way they have encouraged me personally. Richard Kalman, in 2002, took the time to recognize the new guy in the ARML Power Question grading room and developed a relationship with me that eventually led to me working with him at Math Olympiads for Elementary Schools. He encouraged me to become a problem-poser and not just a problem-solver. Leo Schneider, who wrote NYSML from 2000 to 2010, took time with me (the brand-new Vice President at the time) to encourage me as a problem-poser. It is because of Leo that I wrote NYSML's short-lived seasonal competitions, and it is because of Leo that I felt confident in heading up NYSML's author committee after Leo passed away.

It was Michael Curry who wanted to write a math contest for local leagues to use in 2011, and he called me to ask if I would be the author. I pushed back, unsure of what I could offer such a venture. He pushed back, confident that I could do the job. He saw potential I didn't see, and he continues to do so. Through the last decade or two, my wife, Nicole, and my children (and now my grandchildren!) have supported my crazy ideas and my time spent down rabbit holes of mathematical thought. They don't always understand how I gain such satisfaction from these problem-posing pursuits, but they know that I get to do excellent things with excellent people in excellent places, and that matters to them. It has been a joy to be able on occasion to take them to some of those places and let them see what I do. I live a truly charmed life. I am grateful to everyone who has encouraged me to do something I didn't think was possible. Hopefully, there are people like that in your life, spurring you to greatness and helping you to be better at the things you enjoy. Even more hopefully, you are that kind of person to other people, encouraging them to see beyond their perceived limitations so they can fly higher than they thought was possible.

George R. Reuter, Jr.

As a young person doing ARML in the spring of 1987, I would never have thought that, 20 years later, I would be *writing* the contest—or that, ten years after that, I'd be finishing up a stint as Head Author of what I still maintain is the most challenging and mathematically rich contest accessible to a wide range of students nationally and internationally. How time flies!

There's tremendous joy in creating a math contest. From the outside, there's a thrill to watching students struggle with, and then discuss animatedly, mathematical ideas and structures provoked by a problem you wrote. Anyone at ARML sees that. As the inheritor of that proud tradition, I take little credit for the wonderful problems, deep questions, and exciting competition—that belongs to my fantastic team of writers. And there's an exuberance most people never see, the back-and-forth of the author's meeting, which for one weekend every year brought me the experience of doing terrific mathematics with good friends—kind of like the best summer math programs, only for adults and compressed into a weekend.

I was fortunate enough to experience a third sort of joy. By the time I left, the writing team included several former students, from my classes and my own role coaching the Chicago ARML "Developmental" team. There's a special satisfaction I found in coaching someone in solving one set of contest problems, only to have them bring back problems more creative than my own—and often, ones I myself had difficulty solving.

I'm delighted that George and the team have kept producing contests that continue to provide that special mix of creativity and challenge, and I wish them the best. I would be remiss if I didn't thank the many people who brought me along on this journey. My teachers: Arnold Ross, David Kelly, and Paul J. Sally, Jr., each of whom taught me more mathematics and more about doing mathematics than I had possibly expected, and Steve Viktora, who nurtured my talent even when I wasn't aware of it. My friends and collaborators, particularly Don Barry, whom I met at Phillips Academy and who handed the head author role to me on his retirement, Richard Rukin, the Chicago All-Star Math Team coaching squad, and Doug O'Roark, who has been my good friend and closest collaborator for over twenty years. And of course my family, Ari, Jonah, and Helen, who put up with weekends away, "just-one-more-minutes", and other inconveniences so I could contribute to the contest, and Allison, who held down the fort so many times.

Those who know me understand that it's only fitting I would retire as head writer in 2017. Here's to many times seventeen more great years of math!

Paul J. Karafiol
Head Author, 2009–2017

Preface: ARML Local

ARML Local was created in 2008 to offer schools an opportunity to either practice for ARML or to take part in a contest similar to ARML without having to travel. ARML Local has participants from over 20 states as well as Canada, China, Egypt, Hong Kong, Malaysia, the Philippines, South Korea, Sri Lanka, and Taiwan.

An ARML Local team consists of up to 6 students. Each contest site administrator receives the contest via email, prints out the contest deck, administers the test, corrects the problems, and enters the team's results online. Prizes are typically given to the top two teams and the top individual scorer, although for the 2020 contest the prizes were expanded thanks to sponsorship from the D. E. Shaw group and Star League.

ARML Local consists of three rounds:

> The <u>Team Round</u> consists of 15 questions. The team works together for 45 minutes to solve the problems.
>
> The <u>Individual Round</u> consists of 10 questions given in pairs with 10 minutes allowed for each pair. A tiebreaker question is also given to top scoring individuals at each site in order to help determine an overall individual winner.
>
> The <u>Relay Rounds</u> depart from the approach used at ARML to some degree by taking advantage of the fact that teams consist of 6 members. First of all, there are three rounds not two. Secondly, the first relay is for three teams of two, the second is for two teams of three, and the third is for one team of all six students. The times allotted for the relays are 6, 8, and 10 minutes respectively.
>
> The one exception to this format occurred in 2020 when ARML Local was administered entirely online, and we switched to a variant of the collaborative relay format developed by mathleague.org. In that case, relay teams received different sets of questions and were able to work together on the questions. The times alloted for the relays were 3, 4, and 5 minutes, respectively, to submit answers to all of the questions for credit (as opposed to the standard relay, where only the final answer is considered for credit).

More information about the contest is available at `https://www.arml.com/arml_local/`.

Over the period of 2015 to 2020, ARML Local has been coordinated by Paul Dreyer, with problem submissions by David Altizio, Evan Chen, Michael Curry, Thinula De Silva, Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Silas Johnson, Paul J. Karafiol, Jason Mutford, George Reuter, Tristan Shin, Michael Tang, and April Verser, with editing and reviewing assistance from Chris Jeuell, Oleg Kryzhanovsky, Andy Niedermaier, and George Reuter. We want to thank all of these people, as well as the coaches and site coordinators worldwide for making ARML Local possible.

Paul Dreyer, Ph.D.

Preface: ARML Power Contest

In 1994, under the leadership of its president, Mark Saul, ARML introduced the ARML Power Contest. Modeled on the Power Question which involves, as Mark noted, "cooperative effort in exploring a problem situation through the solution of chains of related problems", ARML began to offer a competition by mail. In November and February, each participating team receives a set of problems based on a major theme. Some of the problems require a numerical answer, some require justification, while others require that a proof be written. The mathematics of the problems has been geared so that students in an honors class, a math club, or on a math team can have a unique, challenging, yet mostly successful problem solving and mathematics writing experience. There is no limit to the size of the team, but the time for solving the problem set is limited to 45 minutes. Currently, the team's solutions are mailed to the author where they are graded by the author and some of his colleagues. There are 40 points per contest; the team score is the sum of the scores of the two contests. A team may represent a school, a math club, a state, or even an entire country. The contest must be taken by all members of the team at the same time and place. In the beginning the contest only had a few teams in competition; the contest has grown over time to include teams from almost every state as well as several Canada-based teams. In recent years there have also been competitors from Colombia, Spain, Egypt, and a large contingent of teams from China. An agreement with the Mu Alpha Theta mathematics honor society to grant a few of its chapters free entry to the contest has spurred even more interest as teams return after their free year to keep their students engaged in the challenging and fun mathematics.

The Power Question and the Power Contest are both designed to simulate actual mathematical research activity. This is not easy. As Mark noted,

> "Power Contest problems are difficult to write. They must provide meaningful problem situations both for the novice and veteran mathletes. They must attract schools with strong traditions in mathematics competitions, yet offer experiences for students new to such events. Thus, they must build mathematically significant results out of mathematically trivial materials."

The idea for a Power Contest question usually starts off quite small and simple, such as a special case of a theorem the author has just read in a journal, or an extension of another problem. The idea then grows and the parts take shape, leading students through simple calculations, explanation, and examples to deeper, more powerful results. The goal is to make the mathematics inviting to all participating teams yet also challenge and discriminate amongst the top mathematical talent in the competition, while keeping it assessable by the graders. Creating the problems is as much a challenge for the author as solving them is to the students!

For the majority of its existence, through the 2012–2013 competition, Thomas Kilkelly was the main author, coordinator, and grader for the ARML Power Contest. In developing the problems he has been assisted at times by Andy Niedermaier, a former ARML contestant from Minnesota, and by Ivaylo Kertezov from Sofia, Bulgaria. Many others have contributed ideas, and helped proofread and grade the contest. Starting with the 2013–2014 competition, stewardship of the contest passed to

Micah Fogel of the Illinois Mathematics and Science Academy. His goals are to maintain the quality of the contest while modernizing the administration by creating electronic registration, distribution, and scoring systems. Please visit the contest on the Web at `https://www.arml.com/power/`.

The current Power Contest author wishes to thank the ARML Board for entrusting him with the work, Thomas Kilkelly for making the contest what it is today, the Illinois Mathematics and Science Academy for supporting the work, and the students and coaches who provide wonderful feedback and encouragement.

Micah Fogel

Sponsor and Supporters of ARML

Sponsor: The D. E. Shaw group (https://www.deshaw.com/)

The D. E. Shaw group is a global investment and technology development firm. It employs a substantial number of top-notch computer scientists and mathematicians, many of whom joined the firm directly from leading university math and science programs. A number of them are former ARML competitors. Given the nature and challenges of its work, the D. E. Shaw group has a particular interest in promoting excellence in mathematics. It is especially proud of its years-long association with ARML and of its current sponsorship of this national competition.

Supporters:

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Penguin Group	https://www.penguin.com/services-academic/
Princeton University Press	https://press.princeton.edu/
Star League	https://starleague.us/
WIRIS	http://www.wiris.com/en/mathtype
Wolfram Research	https://www.wolfram.com/

Acknowledgements: Problem Writers and Reviewers

Problem Writers and Reviewers for ARML 2015–2019

2015: **Writers:** Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Winston Luo, Andy Niedermaier, George Reuter, Andy Soffer, and Eric Wepsic

Reviewer: Chris Jeuell

2016: **Writers:** Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Winston Luo, Andy Niedermaier, George Reuter, Graham Rosby, Andy Soffer, and Eric Wepsic

Reviewers: Benji Fisher, Chris Jeuell, and Eric Wepsic

2017: **Writers:** Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Silas Johnson, Paul J. Karafiol, Winston Luo, Andy Niedermaier, George Reuter, Graham Rosby, Andy Soffer, and Eric Wepsic

Reviewers: Don Barry, Benji Fisher, Micah Fogel, and Chris Jeuell

2018: **Writers:** Evan Chen, Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Silas Johnson, Paul J. Karafiol, Winston Luo, Jason Mutford, Andy Niedermaier, George Reuter, Graham Rosby, and Eric Wepsic

Reviewers: Don Barry, Micah Fogel, Chris Jeuell, and Eric Wepsic

2019: **Writers:** Evan Chen, Paul Dreyer, Edward Early, Zuming Feng, Zachary Franco, Chris Jeuell, Silas Johnson, Winston Luo, Jason Mutford, Andy Niedermaier, George Reuter, Graham Rosby, and Eric Wepsic

Reviewers: Don Barry, Micah Fogel, Chris Jeuell, and Eric Wepsic

Problem Writer for the ARML Power Contest

Feb. 2015–Feb. 2020: Micah Fogel

Problem Writers and Reviewers for ARML Local 2015–2020

2015: **Writers:** Paul Dreyer and George Reuter
Reviewer: Andy Niedermaier

2016: **Writers:** Evan Chen, Michael Curry, and Paul Dreyer
Reviewers: Chris Jeuell, Andy Niedermaier, and George Reuter

2017: **Writers:** Evan Chen and Paul Dreyer
Reviewers: Oleg Kryzhanovsky and Andy Niedermaier

2018: **Writers:** David Altizio, Paul Dreyer, Edward Early, Zachary Franco, Chris Jeuell, Paul J. Karafiol, Jason Mutford, and George Reuter
Reviewer: Andy Niedermaier

2019: **Writers:** David Altizio, Thinula De Silva, Paul Dreyer, Chris Jeuell, and Michael Tang
Reviewer: Chris Jeuell

2020: **Writers:** Evan Chen, Thinula De Silva, Paul Dreyer, Zuming Feng, Zachary Franco, Chris Jeuell, Silas Johnson, George Reuter, Tristan Shin, and April Verser
Reviewers: Matthew Babbitt, Chris Jeuell, and Oleg Kryzhanovsky

Prize Winners

ARML Prize Winners 2015–2020

2015 Team Competition

Division A

1. SFBA/NorCal A1 211
2. Phillips Exeter (PEARL) A1 .. 200
3. Southern California A1 188

Division B

1. Wild Wild West B1 122
2. Missouri B1 86
3. Thomas Jefferson B1 83

The D. E. Shaw Group Individual High Scorers for ARML 2015

1. Brice Huang .... West Windsor-Plainsboro (Perfect Score)
2. Michael Tang .... Minnesota
3. Alex Song .... Phillips Exeter
4. Nicholas Sun .... Phillips Exeter
5. Shyam Narayanan .... Kansas
6. Daniel Zhu .... Montgomery
7. Colin Tang .... Washington
8. Jason Lee .... Montgomery County
9. Swapnil Garg .... SFBA/NorCal
10. Rachel Zhang .... Missouri A1
11. Gideon Leeper .... New York City A1
12. Allen Liu .... Upstate NY
13. Sam Korsky .... Chicago
14. Maya Sankar .... SFBA
15. Siye Zhu .... Phillips Academy
16. Alexander Katz .... Bergen County
17. Kevin Zhao .... South Carolina
18. Frank Han .... Minnesota
19. Ben Wei .... Waterloo
20. David Stoner .... South Carolina

2016 Team Competition

Division A

1. SFBA/NorCal A1 210
2. TJHSST A1 202
3. PEARL A1 193

Division B

1. Fairfax B1 153
2. UTS Incircle B1 113
3. WWP B1 108

The D. E. Shaw Group Individual High Scorers for ARML 2016

1.	Daniel Kim	Bergen County Academies
2.	Kaan Dokmeci	Connecticut
3.	Allen Liu	Upstate New York
4.	Kevin Liu	Indiana
5.	Eric Gan	Texas
6.	Edwin Peng	SFBA/NorCal
7.	Yuan Yao	PEARL
8.	Jason Lee	Montgomery
9.	James Lin	Alternate 1 (PEARL)
10.	Luke Robitaille	Texas
11.	Joshua Lee	TJHSST
12.	Hongyi Chen	Colorado
13.	Ben Epstein	Eastern Mass
14.	Anton Cao	SFBA/NorCal
15.	Wanlin Li	Nassau
16.	Eshaan Nichani	San Diego
17.	Brian Gu	Washington
18.	Qi Qi	PEARL
19.	Wanqi Zhu	Chicago
20.	Ellen Li	Buchholz

2017 Team Competition

Division A

SFBA/Norcal A1 245
Thomas Jefferson High School for Science and Technology A1 240
Chicago A1 237

Division B

Ontario Math Circles B1 209
Georgia B1 197
Ohio B1 195

The D. E. Shaw Group Individual High Scorers for ARML 2017

1.	Brian Reinhart	Florida (Perfect Score)
2.	Nathan Ramesh	Eastern Massachusetts (Perfect Score)
3.	Kevin Liu	Indiana (Perfect Score)
4.	Akshaj Kadaveru	TJHSST (Perfect Score)
5.	Peter Zhu	Ohio A1 (Perfect Score)
6.	Michael Tang	Minnesota
7.	Andrew Gu	Western Pennsylvania
8.	Jeffrey Huang	Texas
9.	Allen Chen	Chicago
10.	Steven Ma	Connecticut
11.	Guanpeng Xu	Phillips Academy
12.	Saaketh Vedantam	Florida
13.	Andrew Lin	SFBA/NorCal
14.	Graham O'Donnell	Florida
15.	Michelle Shen	Indiana
16.	Ashwin Agnihotri	Bergen Co. Academies
17.	William Zhao	Ontario Math Circles
18.	Frank Lu	Minnesota
19.	Serina Hu	New York City
20.	Dylan Liu	Texas

2018 Team Competition

Division A

Thomas Jefferson High School for Science and Technology A1	234
SFBA/Norcal A1	223
Lehigh Valley A1	207

Division B

Buchholz B1	176
Fairfax JV2 B2	154
Colorado B1	152

The D. E. Shaw Group Individual High Scorers for ARML 2018

1.	Luke Robitaille	Texas
1.	Joshua Lee	Thomas Jefferson
1.	Nicky Sun	Chicago
2.	Kevin Liu	Indiana
3.	Kevin Ren,	San Diego
4.	Daniel Xia	Lehigh Valley
5.	Brian Huang	New York City
6.	Michael Li	Olympiads School
7.	Brian Reinhart	Florida
8.	Frank Han	Minnesota
9.	Anders Olsen	Oregon
10.	Joey Heerens	Indiana
11.	Allen Chen	Chicago
12.	Andrew Gu	Western PA
12.	Howard Halim	University of Toronto Schools
12.	Nathan Ramesh	Eastern Massachusetts
12.	Victor Rong	Ontario Math Circles
12.	William Sun	Thomas Jefferson
13.	Jeffery Li	SFBA/NorCal
14.	Tristan Shin	San Diego

2019 Team Competition

Division A

1.	Thomas Jefferson High School for Science and Technology A1	245
2.	Texas A1	233
3.	SFBA/Norcal A1	231

Division B

1.	Georgia B1	192
2.	Ohio B1	186
3.	Central Jersey B1	178

The D. E. Shaw Group Individual High Scorers for ARML 2019

1.	David Chen	Thomas Jefferson
2.	Luke Robitaille	Texas
3.	William Wang	West Windsor-Plainsboro
4.	Sam Wang	Fairfax, Virginia
5.	Austen Mazenko	Colorado
6.	Daniel Xia	Lehigh Valley
7.	Kevin Cong	Phillips Exeter Academy
8.	Jeremy Zhou	Phillips Academy
9.	Andrew Cai	Texas
10.	Eric Gan	Texas
11.	Haydn Gwyn	Montgomery County
12.	Sam Delatore	Ohio
13.	James Xiu	Michigan
14.	Ben Kang	Thomas Jefferson
15.	Frank Wang	Central Jersey
16.	Christopher Wu	Texas
17.	Ajit Kadaveru	Thomas Jefferson
18.	William Zhao	Ontario
19.	Rajat Mittal	Chicago
20.	Zipeng (Leo) Lin	Fairfax, Virginia

Jing Ding from China (IRML) earned 9 points and participated in the tiebreaker. His score on the tiebreaker was equivalent to a 12th place finish.

ARML Power Contest Prize Winners

2014–2015

1.	Montgomery Blair High School	Silver Spring, MD
2.	Thomas Jefferson High School	Alexandria, VA
3.	Lynbrook High School	San Jose, CA

2015–2016

1.	Colorado Math Circle	Boulder, CO
2.	Long Beach Math Circle	Long Beach, CA
3.	Montgomery Blair High School	Silver Spring, MD

2016–2017

1.	Long Beach Math Circle	Long Beach, CA
2.	Thomas Jefferson High School	Alexandria, VA
2.	Western Washington ARML	Redmond, WA

2017–2018

1.	Thomas Jefferson High School	Alexandria, VA
1.	Lynbrook High School	San Jose, CA
3.	San Diego Math Circle	San Diego, CA

2018–2019

1.	Thomas Jefferson High School	Alexandria, VA
2.	Bergen County Academies	Hackensack, NJ
2.	San Diego Math Circle	San Diego, CA
2.	AlphaStar Academy ARMLMC 10	Santa Clara, CA
2.	Lynbrook High School	San Jose, CA
2.	Long Beach Math Circle	Long Beach, CA

2019–2020

1.	Thomas Jefferson High School	Alexandria, VA
2.	Bergen County Academies	Hackensack, NJ
3.	AlphaStar Academy	Santa Clara, CA

ARML Local Prize Winners

2015 Competition

Top-Scoring Teams

1. Star League A-Star Air
2. Thomas Jefferson A

Individual Winner

Franklyn Wang Thomas Jefferson A

2016 Competition

Top-Scoring Teams

1. SFBA-NorCal / A-Star: prAyer, Reading, and Meditation (praise the Lord)
2. SFBA-NorCal / A-Star: Make up a name for us, Moor

Individual Winner

Arav Karighattam SFBA-NorCal / A-Star: Andrew Lin is not pr0

2017 Competition

Top-Scoring Teams

1. SFBA-NorCal / A-Star: Cubberley A1
2. SFBA-NorCal / A-Star: Judicially Activist Kids Addicted to Super Smash

Individual Winner

Ishika Shah SFBA-NorCal / A-Star

2018 Competition

Top-Scoring Teams

1. SFBA-NorCal / A-Star: Question Mark!
2. SFBA-NorCal / A-Star: Cant Think

Individual Winner

Junhee Lee Iowa City West Purple Moo Cows

2019 Competition

Top-Scoring Teams

1. SFBA-NorCal / A-Star: BetaStar Academy Air
2. BCA GASSS

Individual Winner

Howard Halim	University of Toronto Schools 1

2020 Competition

Top-Scoring Teams

1. Texas A
2. Lehigh Valley Magnetic
3. SFBA/NorCal Hibiscus
4. Tin Man
5. SFBA/NorCal Magenta
6. YEA
7. SoCal ARML Team E
8. All Stars A2
9. San Diego: Throwing

Individual Winners

Adithya Balachandran	Lehigh Valley Magnetic
Luke Robitaille	Texas A
Alex Xu	Michigan All Stars - Troy
Neel Basu	SoCal ARML Team E
Andrew Cai	Texas A
Samrit Grover	1111111111
Brian Liu	Lehigh Valley Magnetic
Vidur Jasuja	Tin Man
Douglas Chen	Engineer's Induction
William Chen	SFBA/NorCal Magenta
Eric Shen	YEA
Won (Joshua) Jang	SoCal ARML Team E

ARML Awards

The *Alfred Kalfus Founder's Award* is given each year for long-term service to ARML. Al Kalfus was the founder and first president of both NYSML and ARML. He was devoted to his students and dedicated to excellence. Some measure of the man can be found in a preface that he wrote to the first book of ARML problems covering the 1976 to 1982 contests: "Meanwhile, enjoy the problems given in these pages. Remember, they were posed not to stump you, but to provoke your mathematical ingenuity, to guide you to clever tricks and new methods of attack, to open new facets of topics to the uninitiated, and to offer you the unique joy that only the solving of a truly challenging problem can bring." This award was first given in 1985.

The *Samuel L. Greitzer Coach Award* is given each year for outstanding service to a regional team. For many years Dr. Samuel Greitzer, a professor at Rutgers University, coached the United States IMO team. He also served as the editor for Arbelos, a small journal designed for high school students that was full of challenging mathematics problems and investigations. The award was first given in 1987.

The *Harry and Ruth Ruderman Award* is awarded at ARML to the winning team of the ARML Power Contest. Harry Ruderman influenced and inspired generations of students and teachers. He was a prolific writer, problem poser, and lecturer. For over a decade, he served as a contributor and reviewer of ARML questions as well as being the head judge. It was first awarded in 1996.

Part I

ARML Contests

2015 Contest

Contents

2015 Team Problems

T-1. Compute the greatest integer x such that $\left\lfloor\sqrt{\left\lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\right\rfloor}\right\rfloor = 2$.

T-2. Compute the number of ordered pairs of integers (x, y) such that $\frac{1}{x} + \frac{540}{xy} = 2$.

T-3. A positive integer has the *Kelly Property* if it contains a zero in its base-17 representation. Compute the number of positive integers less than 1000 (base 10) that have the Kelly Property.

T-4. Compute the smallest positive integer n such that $n+i$, $(n+i)^2$, and $(n+i)^3$ are the vertices of a triangle in the complex plane whose area is greater than 2015.

T-5. A loop is made by connecting rods of lengths $1, 2, \ldots, 90$ in that order. (The rod of length 90 is connected to the rods of lengths 89 and 1.) The loop is laid in the shape of an equilateral triangle of perimeter 4095. Rods cannot be bent or broken. Compute the sum of the lengths of the shortest rods on each side of the triangle. (For example, the loop with rods 1, 2, …, 9 can be arranged into an equilateral triangle because $4+5+6 = 7+8 = 9+1+2+3$.)

T-6. Four spheres S_1, S_2, S_3, and S_4 are mutually externally tangent and are tangent to a plane, on the same side of the plane. Let S_1 and S_2 have radius r and let S_3 and S_4 have radius s. Given that $r > s$, compute r/s.

T-7. In $\triangle ABC$, point D is on $\overline{AB}$ and point E is on $\overline{AC}$. The measures of the nine angles in triangles ADE, BCD, and CDE can be arranged to form an arithmetic sequence. Compute the greatest possible degree measure for $\angle A$.

T-8. Let $a_1 = a_2 = a_3 = 1$. For $n > 3$, let a_n be the number of real numbers x such that

$$x^4 - 2a_{n-1}x^2 + a_{n-2}a_{n-3} = 0.$$

Compute the sum $a_1 + a_2 + a_3 + \cdots + a_{1000}$.

T-9. For any real number k, let region R_k consist of all points (x, y) such that $x \geq 0$, $y \geq 0$, and $\lfloor x+y \rfloor + \{x\} \leq k$, where $\{u\}$ denotes the fractional part of u. Compute the value of k for which the area of R_k is equal to 100.

T-10. Let $ABCD$ be a parallelogram with m$\angle A > 90°$. Point E lies on $\overrightarrow{DA}$ so that $\overline{BE} \perp \overline{AD}$. The circumcircles of $\triangle ABC$ and $\triangle CDE$ intersect at points F and C. Given that $AD = 35$, $DC = 48$, and $CF = 50$, compute AC.

2015 Team Answers

T-1. 6560

T-2. 15

T-3. 106

T-4. 9

T-5. 72

T-6. $2 + \sqrt{3}$

T-7. 84

T-8. 2329

T-9. $29 - \sqrt{221}$

T-10. $5\sqrt{23}$, $5\sqrt{79}$

2015 Team Solutions

T-1. Ignore the floor functions and replace 2 with 3. Then x would be $((3^2)^2)^2 = 6561$. If $x < 6561$, then $\sqrt{x} < 81$, so $\lfloor\sqrt{x}\rfloor \leq 80$, and $\sqrt{\lfloor\sqrt{x}\rfloor} < 9$, so $\lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\rfloor \leq 8$, and $\lfloor\sqrt{\lfloor\sqrt{\lfloor\sqrt{x}\rfloor}\rfloor}\rfloor \leq 2$. Thus the greatest possible integral value of x is $6561 - 1 = \mathbf{6560}$.

T-2. Multiply through by xy and rearrange terms to obtain $540 = 2xy - y = (2x-1)\cdot y$. Because $2x - 1$ is odd, the question reduces to counting the number of odd divisors (positive and negative) of 540. Because $540 = 2^2 \cdot 3^3 \cdot 5$, there are $(3+1)(1+1) = 8$ positive odd divisors, hence 540 has 16 odd integer divisors and 16 potential integral values for x. However, the divisor -1 corresponds to $x = 0$, which is extraneous to the original equation. Hence there are **15** solutions, given in the tables below.

$\boldsymbol{2x-1}$	-135	-45	-27	-15	-9	-5	-3
$\boldsymbol{x}$	-67	-22	-13	-7	-4	-2	-1
$\boldsymbol{y}$	-4	-12	-20	-36	-60	-108	-180

$\boldsymbol{2x-1}$	1	3	5	9	15	27	45	135
$\boldsymbol{x}$	1	2	3	5	8	14	23	68
$\boldsymbol{y}$	540	180	108	60	36	20	12	4

T-3. Compute the base-17 representation of 1000:

$$1000 = 3 \cdot 17^2 + 7 \cdot 17^1 + 14 \cdot 17^0.$$

Let y_p denote the number of integers less than 1000 with the Kelly Property whose 17^2-digit is p. If $p = 1$ or $p = 2$, then so long as one of the two rightmost digits is zero, the number will have the Kelly Property. There are $17^2 - 16^2$ such numbers because 16^2 of the 17^2 options have non-zero values in both of the two rightmost digits. Thus $y_1 = y_2 = 17^2 - 16^2 = 33$. For $p = 0$, the only possibility for a number with the Kelly Property is for its 1s digit to be zero. There are 16 such numbers, because the 17s digit can be any non-zero value. Thus $y_0 = 16$. Finally consider the case $p = 3$. If the 17s digit is zero, then any value for the 1s digit will yield a number with the Kelly Property, contributing another 17 numbers with the Kelly Property. If the 17s digit is a number 1 through 7 inclusive, then to have the Kelly Property, the number must have a zero in its 1s digit, contributing another 7 numbers with the Kelly Property. Thus $y_3 = 17 + 7 = 24$.

The number of positive integers less than 1000 which have the Kelly Property is therefore

$$y_0 + y_1 + y_2 + y_3 = 16 + 33 + 33 + 24 = \mathbf{106}.$$

T-4. The complex number $(n+i)^2$ can be broken into real and imaginary parts as $n^2+2ni+i^2 = (n^2-1)+2ni$. The complex number $(n+i)^3$ can be broken into real and imaginary parts as $n^3+3n^2i+3ni^2+i^3 = (n^2-3n)+(3n^2-1)i$. Therefore the triangle has the same area in the complex plane as the triangle in the Cartesian plane with coordinates $(n,1)$, $(n^2-1,2n)$, and $(n^3-3n,3n^2-1)$. The shoelace formula gives that the area of this triangle is the absolute value of

$$\begin{aligned}
&\frac{1}{2}\begin{vmatrix} n & 1 \\ n^2-1 & 2n \end{vmatrix} + \frac{1}{2}\begin{vmatrix} n^2-1 & 2n \\ n^3-3n & 3n^2-1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} n^3-3n & 3n^2-1 \\ n & 1 \end{vmatrix} \\
&= \frac{1}{2}(2n^2+3n^4-n^2-3n^2+1+n^3-3n-n^2+1-2n^4+6n^2-3n^3+n) \\
&= \frac{1}{2}(n^4-2n^3+3n^2-2n+2).
\end{aligned}$$

Therefore the problem reduces to finding the smallest positive n such that

$$f(n) = \left|n^4-2n^3+3n^2-2n+2\right| > 4030.$$

Notice that the dominant term is n^4, and so $n=8$ gives $n^4 = 4096$, which is a good approximation to 4030. However, substituting $n=8$ gives $f(n) = |4096-2\cdot 512+3\cdot 64-2\cdot 8+2| = 3250$, which is too small. Trying $n=9$ gives $f(n) = |6561-2\cdot 729+3\cdot 81-2\cdot 9+2| = 5330 > 4030$, so the answer is $n = \mathbf{9}$.

Alternate Solution: Let $z = (n+i)$, and let K_1 be the area of the triangle whose vertices are 0, z, and z^2, let K_2 be the area of the triangle whose vertices are 0, z^2, and z^3, and let K_3 be the area of the triangle whose vertices are 0, z, and z^3. Then if K is the area of the triangle with vertices z, z^2, and z^3, $K = K_1+K_2-K_3$.

First compute K_1. Let θ be the angle that $n+i$ makes with the x-axis. By DeMoivre's Theorem, the angle between the ray from 0 through z and the ray from 0 through z^2 is also θ, so using the triangle area formula $\frac{1}{2}ab\sin C$ yields

$$K_1 = \frac{1}{2}\cdot|n+i|\cdot\left|(n+i)^2\right|\cdot\sin\theta.$$

Because absolute values are multiplicative, the expression on the right-hand side simplifies to

$$K_1 = \frac{1}{2}\cdot|n+i|^3\cdot\sin\theta,$$

and because $|n+i| = \sqrt{n^2+1}$,

$$K_1 = \frac{1}{2}\cdot(n^2+1)^{3/2}\sin\theta.$$

Now $\sin\theta = \frac{1}{\sqrt{n^2+1}}$, so

$$K_1 = \frac{n^2+1}{2}.$$

Similarly, $K_2 = \frac{(n^2+1)^2}{2}$.

Computing K_3 is not much more involved. The angle between z and z^3 is 2θ. Using the trigonometric identity $\sin 2x = 2 \sin x \cos x$ and the observation that $\cos\theta = \frac{n}{\sqrt{n^2+1}}$ yields

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= 2\left(\frac{1}{\sqrt{n^2+1}}\right)\left(\frac{n}{\sqrt{n^2+1}}\right) \\
&= \frac{2n}{n^2+1}.
\end{aligned}$$

Thus

$$K_3 = \frac{1}{2}\cdot(n^2+1)^2\cdot\frac{2n}{n^2+1} = n(n^2+1).$$

Hence

$$\begin{aligned}
K &= K_1 + K_2 - K_3 \\
&= \frac{n^2+1}{2} + \frac{(n^2+1)^2}{2} - n(n^2+1) \\
&= (n^2+1)\left(\frac{1}{2} + \frac{n^2+1}{2} - n\right) \\
&= (n^2+1)\left(\frac{n^2-2n+2}{2}\right) \\
&= \frac{1}{2}\left(n^2+1\right)\left((n-1)^2+1\right).
\end{aligned}$$

To compute the smallest positive integral value of n such that $K > 2015$, first multiply by 2 to obtain $(n^2+1)\left((n-1)^2+1\right) > 4030$. The two factors are approximately n^2 and $(n-1)^2$, so look for n such that $n^4 > 4030$. Because $8^4 = 2^{12} = 4096$, try $n = 8$ to obtain a product of $(64+1)(49+1) = 3250$ which is too small; $n = 9$ yields a product of $(81+1)(64+1) = 5330$. Thus $n = \mathbf{9}$.

T-5. The side length of the triangle is $\frac{1}{3}\cdot\frac{91\cdot 90}{2} = 91\cdot 15 = 3\cdot 5\cdot 7\cdot 13 = 1365$. Call this value S. Start walking around the triangle at the length-1 rod, going in the direction away from the length-90 rod. Let the last rods of each of the three sides have lengths a, b, and c, with $a < b < c$. (In the 9-rod example, $(a, b, c) = (3, 6, 8)$). Thus the three sides are composed of rods $a+1, \ldots, b$; $b+1, \ldots c$; and $c+1, \ldots, 90, 1, \ldots, a$.

Focus on the side that ends in the length-c rod. It has $c-b$ rods in total, and $S = (c-b)(\frac{b+c+1}{2})$. If $c-b$ is odd, then it divides S. If $c-b$ is even, then it divides $2S$. If $c-b \le 15$, then $\frac{S}{c-b}$, that is, the average length of the rods on that side, will be at least 91, which is too big.

If $c-b = 21$, then $\frac{S}{c-b} = 65$, so this side must have rods $55, \ldots, 75$. That would imply $b = 54$. Does there exist an a such that $S = (54-a)(\frac{a+54+1}{2})$? As it turns out, $a = 15$ is a solution. Thus $(a, b, c) = (15, 54, 75)$ produces an equilateral triangle.

If $c-b = 26$, then $\frac{S}{c-b} = 52.5$, so this side must have rods $40, \ldots, 65$. But note that this means the side *after* this must contain rods 66 through 90, as well as the length-1 rod and possibly more after that. The total length of those rods is at least $78 \cdot 25 + 1 > S$, which is too large.

Thus for all values of $c - b$ that divide $2S$ *except* 21, there will not exist an equilateral triangle satisfying the conditions of the problem. Thus the only possible value for the sum of the lengths of the shortest rods of the three sides is $1 + 16 + 55 = \mathbf{72}$.

Remark: This problem admits several interesting generalizations, three of which are discussed below.

No squares. If $(A, B, C) = (2a + 1, 2b + 1, 2c + 1)$, then it must be the case that $8S = B^2 - A^2 = C^2 - B^2$. That is, A^2, B^2, C^2 must form an arithmetic progression. In general, to create a regular k-gon with the n-loop (rods of lengths $1, 2, \ldots, n$), then it is necessary to be able to create arithmetic progressions of k odd perfect squares. It so happens that there are no such progressions for $k \geq 4$, a result shown by Euler. Therefore it is impossible to make the loop into a square (or larger k-gon) for *any* integer n.

Equilateral triangles are hard to find. The values of n less than 1000 for which the n-loop can be made into an equilateral triangle are 9, 90, 125, 153, 189, 440, 819, and 989. In the $n = 125$ case, there are two different ways to create an equilateral triangle! Can you determine whether any other value n has this property?

There are infinitely many such equilateral triangles, though. One can use Diophantine equations to find an infinite family of values of n that allow for equilateral triangles.

T-6. Let the centers of S_1 and S_2 be at $(r, r, 0)$ and $(-r, r, 0)$, so that they are tangent to the plane $y = 0$ at points $(r, 0, 0)$ and $(-r, 0, 0)$. Then the centers of S_3 and S_4 lie on the plane $x = 0$, at $(0, s, -s)$ and $(0, s, s)$. Because all of the spheres are mutually tangent, the distance between the center of S_1 and the center of S_4 is $r + s$, so the 3-dimensional Pythagorean Theorem gives

$$(r + s)^2 = r^2 + (r - s)^2 + s^2$$

which simplifies to

$$r^2 - 4rs + s^2 = 0.$$

Solving this as a quadratic in r gives

$$r = (2 \pm \sqrt{3})s.$$

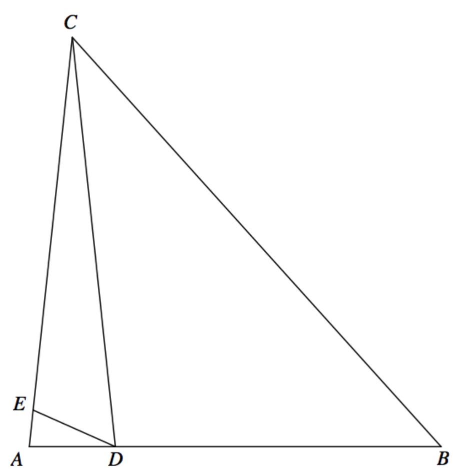
Because $r > s$, it follows that $\frac{r}{s} = \mathbf{2} + \sqrt{\mathbf{3}}$.

Alternate Solution: Let ω be the given plane, and let the centers of the four spheres be P_1, P_2, P_3, and P_4 respectively. Because S_1 and S_2 have equal radii and S_3 and S_4 have equal radii, note that $\overline{P_1P_2}$ and $\overline{P_3P_4}$ are parallel to ω. Note also that $P_1P_2 = 2r$, $P_3P_4 = 2s$, and $P_1P_3 = P_1P_4 = P_2P_3 = P_2P_4 = r + s$. Let R be the midpoint of $\overline{P_3P_4}$, and let Q denote the point in the plane parallel to ω containing P_3 and P_4, such that $\overleftrightarrow{QP_1} \perp \omega$. Then $\triangle QRS_3$ is a right triangle with legs of length r and s, and hypotenuse $QS_3 = \sqrt{r^2 + s^2}$. But $\triangle QP_1P_3$ is also a right triangle, with hypotenuse $\overline{P_1P_3}$. Because $QP_1 = r - s$, the Pythagorean Theorem yields $(r-s)^2 + r^2 + s^2 = (r+s)^2$. Hence $r^2 + s^2 = 4rs$. Divide by s^2 to obtain the equation $\frac{r^2}{s^2} + 1 = 4 \cdot \frac{r}{s}$; substitute $u = \frac{r}{s}$ to obtain the quadratic equation $u^2 - 4u + 1 = 0$. Thus $u = 2 \pm \sqrt{3}$. It is given that $r > s$, hence $\frac{r}{s} = \mathbf{2 + \sqrt{3}}$.

T-7. The average value of the angles is $60°$, so the nine terms must be $60° \pm nd$, where d is the common difference and $0 \le n \le 4$. Thus $d < 15°$. Looking at the supplementary angles at E, one angle is $60° + kd$, and the other is $60° + \ell d$. Let $m = k + \ell$. Then $d = \frac{60°}{m}$. Combining these facts yields only three possible values for m, namely, $m \in \{5, 6, 7\}$.

If $d = 10°$, then the supplementary angles must be $80°$ and $100°$, in some order, and so $\mathrm{m}\angle A \le 90°$. But then $\mathrm{m}\angle ADE = 10°$, which is impossible. One can successfully fill the diagram with $\mathrm{m}\angle A = 70°$, which is the maximum for $d = 10°$.

If $d = 12°$, then the supplementary angles can be $\{72°, 108°\}$ or $\{84°, 96°\}$. Note that $\mathrm{m}\angle A$ cannot be $108°$, so try $96°$. Then $\mathrm{m}\angle AED = 72°$ and $\mathrm{m}\angle ADE = 12°$. But $\mathrm{m}\angle CDE + \mathrm{m}\angle BDC = 168°$, and the two largest unused angles are $84°$ and $60°$, so this case is impossible. One can successfully fill the diagram with $\mathrm{m}\angle A = 84°$, as shown below. That is the maximum for this case.



$\triangle BCD$ has angles with measures $(48°, 36°, 96°)$.
$\triangle ADE$ has angles with measures $(84°, 24°, 72°)$.
$\triangle CDE$ has angles with measures $(12°, 60°, 108°)$.

If $d = \frac{60}{7}^\circ$, then the largest angles, $\frac{660}{7}^\circ$ and $\frac{600}{7}^\circ$, must be located at E. The next-largest angle, $\frac{540}{7}^\circ$, is less than 84°. Hence the largest possible angle is $\mathbf{84^\circ}$.

T-8. Consider the quartic equation $x^4 - 2px^2 + q = 0$, where p and q are nonnegative. This equation can be rewritten as $(x^2 - p)^2 = p^2 - q$. Split into cases to determine the number of distinct real roots:

- If $p^2 - q < 0$, there are **0** real roots.
- If $p^2 - q = 0$ **and** $p > 0$, there are **2** real roots, at $x = \pm\sqrt{p}$.
- If $p^2 - q = 0$ **and** $p = 0$, then there is **1** real root, at $x = 0$. (Note that this is just the case $x^4 = 0$.)
- If $p^2 - q > 0$ **and** $q > 0$, then there are **4** real roots, at $x = \pm\sqrt{p \pm \sqrt{p^2 - q}}$.
- If $p^2 - q > 0$ **and** $q = 0$, then there are **3** real roots, at $x = 0$ and $x = \pm\sqrt{2p}$.

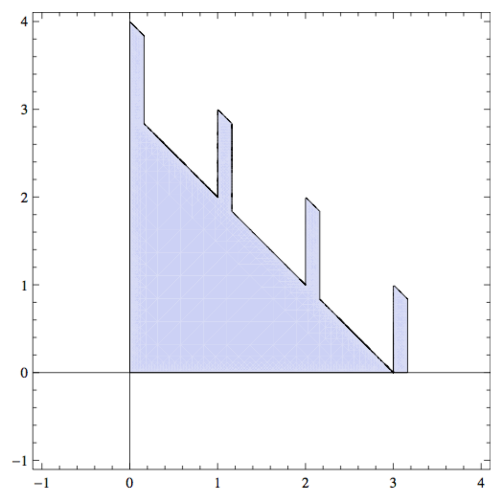
To determine a_n, iteratively apply the above rules, with $p = a_{n-1}$ and $q = a_{n-2}a_{n-3}$. Because each term a_n depends only on the three previous values a_{n-1}, a_{n-2}, and a_{n-3}, it suffices to find a group of three consecutive terms that occurs twice.

n	$p = a_{n-1}$	$q = a_{n-2}a_{n-3}$	$p^2 - q$	a_n
4	1	1	0	2
5	2	1	3	4
6	4	2	12	4
7	4	8	8	4
8	4	16	0	2
9	2	16	-12	0
10	0	8	-8	0
11	0	0	0	1
12	1	0	1	3
13	3	0	9	3
14	3	3	6	4
15	4	9	7	4
16	4	12	4	4

The 3-term subsequence $4, 4, 4$ occurs starting at a_5 and again at a_{14}. Thus the sequence has the following 9-term period: $4, 4, 4, 2, 0, 0, 1, 3, 3$. The sum of these terms is 21, so the sum $a_1 + a_2 + \cdots + a_{1000}$ is

$$\begin{aligned}\sum_{n=1}^{1000} a_n &= \left(\sum_{n=1}^{4} a_n\right) + \left(\sum_{n=5}^{994} a_n\right) + \left(\sum_{n=995}^{1000} a_n\right)\\ &= (1+1+1+2) + 110 \cdot 21 + (4+4+4+2+0+0)\\ &= \mathbf{2329}.\end{aligned}$$

T-9. Let $N = \lfloor k \rfloor$ and let $r = \{k\}$. The region R_k is the triangle $x+y < N$, augmented by vertical "slices" of the region $N \leq x+y < N+1$. There are N slices that are parallelograms of height 1 and width r, as well as one slice that is a trapezoid of area $r - \frac{r^2}{2}$. The region for $k = \sqrt{10}$ is shown below.



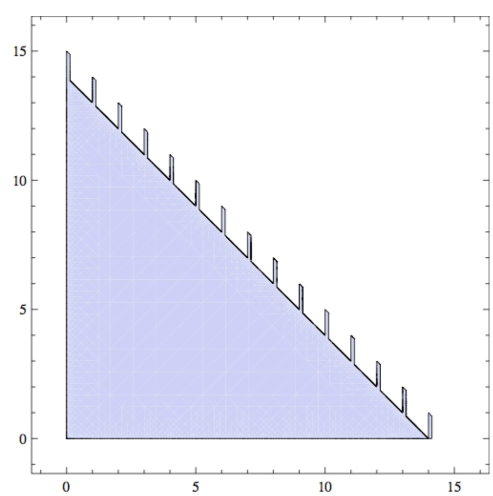
In all, the area is $\frac{N^2-r^2}{2} + (N+1)r$. The goal is to compute k such that

$$\frac{N^2 - r^2}{2} + (N+1)r = 100.$$

Replace r with zero and solve for N to obtain an estimated value for N. This substitution yields $\frac{N^2}{2} = 100$, and so N is approximately $\sqrt{200}$ which is slightly larger than 14. Trying $N = 14$, the equation becomes

$$\frac{14^2 - r^2}{2} + 15r = 100.$$

Simplifying this equation yields $r^2 - 30r + 4 = 0$, and so $r = 15 - \sqrt{221}$. Because $14^2 < 221 < 15^2$, this value of r satisfies $0 < r < 1$. Hence, $k = N + r = \mathbf{29 - \sqrt{221}}$.
The region corresponding to this value of k is shown below.



T-10. Let N lie on $\overline{BC}$ so that $BEAN$ is a rectangle with diagonal $\overline{BA}$. Then $EN = AB = 48$. Trapezoid $ENCD$ is isosceles, and hence a cyclic quadrilateral, so N lies on the circumcircle of $\triangle CDE$. Let G be the intersection of $\overline{BA}$ and $\overline{EN}$. Then $BG = GA = EG = GN = 24$.

Recall that the power of a point P with respect to a circle O is the product of the distance $PX \cdot PY$, where P, X, and Y are collinear, and X and Y are on O; The power of a point P is invariant for any such chord $\overline{XY}$. Because $\overline{BA}$ is a chord of the circumcircle of $\triangle ABC$ and $\overline{EN}$ is a chord of the circumcircle of $\triangle CDE$, the power of point G with respect to both circumcircles is $24 \cdot 24$.

Now recall that given two intersecting circles, the locus of points with the same power in each circle is the line connecting the two points of intersection of the circles, so G lies on $\overline{CF}$.

Applying the power of a point theorem again to chord CF gives $GC \cdot CF = 24 \cdot 24 = 496$, and $GC + CF = 50$, so CG and GF are 18 and 32, in some order.

Apply Stewart's Theorem to triangle ABC to obtain

$$AC^2 \cdot BG + BC^2 \cdot GA = CG^2 \cdot AB + AG \cdot GB \cdot AB,$$

or

$$AC^2 \cdot 24 + 35^2 \cdot 24 = CG^2 \cdot 48 + 24 \cdot 24 \cdot 48$$

which gives

$$AC = \sqrt{2CG^2 - 73}.$$

The two values for CG then yield $AC = \mathbf{5\sqrt{23}}$ or $AC = \mathbf{5\sqrt{79}}$.

2015 Individual Problems

I-1. The perimeter of regular heptagon $HEPTGON$ is 2015 more than the perimeter of square $ARML$. Let $x = HE - AR$. Compute the greatest possible integer value of x.

I-2. Let n be the greatest integer such that n^n is a divisor of $27^{27^{27}}$. Compute the number of positive divisors of n.

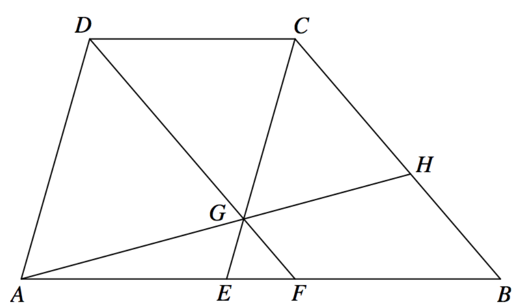
I-3. A rectangular box has integer edge lengths. The sum of the numerical values of its volume, its surface area, and its twelve edge lengths is 2015. Compute the length of the box's interior diagonal.

I-4. Compute the smallest positive integer n such that

$$\sum_{k=0}^{n} \log_2\left(1 + \frac{1}{2^{2^k}}\right) \geq 1 + \log_2\left(\frac{2014}{2015}\right).$$

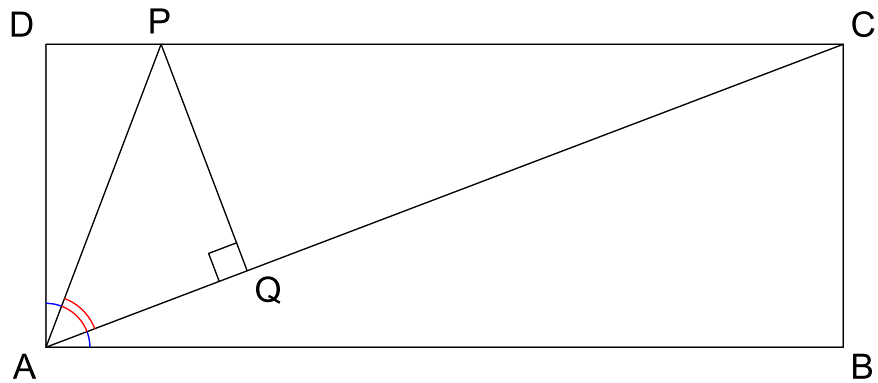
I-5. Let f be a function such that for all x, $f(x) = f(x+1) + f(x-1)$. Given that $f(20) = 15$ and $20 = f(15)$, compute $f(20152015)$.

I-6. In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, $AB = 14$ and $CD = 6$. Points E and F lie on $\overline{AB}$ so that $\overline{AD} \parallel \overline{CE}$ and $\overline{BC} \parallel \overline{DF}$. Segments $\overline{DF}$ and $\overline{CE}$ intersect at G, and $\overrightarrow{AG}$ intersects $\overline{BC}$ at H. Compute $\dfrac{[CGH]}{[ABCD]}$.



I-7. Let f be the function defined by $f(x) = x^3 - 49x^2 + 623x - 2015$, and let $g(x) = f(x+5)$. Compute the sum of the roots of g.

I-8. In rectangle $WASH$, point E lies on $\overline{SH}$ so that $\angle AWS \cong \angle HWE$. Point D lies on $\overline{WS}$ so that $\overline{ED} \perp \overline{WS}$. Given that $[WASH] = 100$ and $[SED] = 32$, compute $\sin \angle SWE$.



I-9. An (a, r, m, ℓ)-*trapezoid* is a trapezoid with bases of length a and r, and other sides of length m and ℓ. Compute the number of positive integer values of ℓ such that there exists a $(20, 5, 15, \ell)$-trapezoid.

I-10. Six people of different heights are getting in line to buy donuts. Compute the number of ways they can arrange themselves in line so that no three consecutive people are in increasing order of height, from front to back.

2015 Individual Answers

I-1. 287

I-2. 79

I-3. $5\sqrt{19}$

I-4. 3

I-5. -5

I-6. $\dfrac{27}{160}$

I-7. 34

I-8. $\dfrac{4+6\sqrt{6}}{25}$

I-9. 29

I-10. 349

2015 Individual Solutions

I-1. Let $a = HE$ and $b = AR$ so that $x = a - b$ and $7a - 4b = 2015$. Substituting $a = x + b$,

$$\begin{aligned} 7(x+b) - 4b &= 2015 \\ 7x + 3b &= 2015 \\ x &= \frac{2015 - 3b}{7} = 287 + \frac{6}{7} - \frac{3b}{7} < 288. \end{aligned}$$

The greatest possible integer x is thus **287**, which occurs when $a = 289$ and $b = 2$. Coincidentally, both a and b are also integers.

I-2. Because n must be a power of 3, set $n = 3^a$ for some positive integer a. Then $3^{a \cdot 3^a}$ must be a divisor of $27^{27^{27}} = 3^{3 \cdot (3^3)^{27}} = 3^{3^{82}}$. That is, $a \cdot 3^a \leq 3^{82}$. Trying decreasing integers starting with $a = 82$, one quickly arrives at $a = 78$ as the largest integral value that satisfies the inequality. Thus $n = 3^{78}$ is the largest integer for which n^n is a divisor of $27^{27^{27}}$, and 3^{78} has **79** positive divisors.

I-3. Let a, b, and c denote the length, width, and height of the box. With

$$4(a+b+c) + 2(ab+ac+bc) + abc = 2015,$$

it follows that

$$\begin{aligned} a = \frac{2015 - 4b - 4c - 2bc}{bc + 2b + 2c + 4} &= \frac{2015 - 2b(c+2) - 4c}{(b+2)(c+2)} \\ &= \frac{2023 - 2b(c+2) - 4(c+2)}{(b+2)(c+2)} \\ &= \frac{2023}{(b+2)(c+2)} - 2, \end{aligned}$$

hence $b + 2$ and $c + 2$ must both divide $2023 = 7 \cdot 17^2$. If $b = c = 15$, then $a = 7 - 2 = 5$. If one of b or c is 5 and the other is 15, then $a = 17 - 2 = 15$, thus in any case, the length of the interior diagonal is $\sqrt{5^2 + 15^2 + 15^2} = \mathbf{5\sqrt{19}}$.

I-4. It is simpler to replace $1 + \frac{1}{2^{2^k}}$ with $1 + x^{2^k}$, and later evaluate at $x = \frac{1}{2}$. Then the left-hand side of the inequality is

$$\sum_{k=0}^{n} \log_2 \left(1 + x^{2^k}\right) = \log_2 \prod_{k=0}^{n} \left(1 + x^{2^k}\right).$$

Consider the product $\prod_{k=0}^{n}(1 + x^{2^k})$. For $n = 0$, this product is $1 + x$. For $n = 1$, this product is $1 + x + x^2 + x^3$. For $n = 2$, this product is $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$. It is natural to conjecture that the product simplifies to $\sum_{i=0}^{2^{n+1}-1} x^i$. This can be proven by induction on n as follows. The base case has already been checked. If $\prod_{k=0}^{n}(1+x^{2^k}) = \sum_{i=0}^{2^{n+1}-1} x^i$, then by induction,

$$\begin{aligned}
\prod_{k=0}^{n+1}(1+x^{2^k}) &= \left[\prod_{k=0}^{n}(1+x^{2^k})\right](1+x^{2^{n+1}}) \\
&= \left(\sum_{i=0}^{2^{n+1}-1} x^i\right)(1+x^{2^{n+1}}) \\
&= \sum_{i=0}^{2^{n+1}-1} x^i + \sum_{i=2^{n+1}}^{2^{n+2}-1} x^i \\
&= \sum_{i=0}^{2^{n+2}-1} x^i.
\end{aligned}$$

Thus the product evaluates to a finite geometric series, and $\prod_{k=0}^{n}(1+x^{2^k}) = \frac{1-x^{2^{n+1}}}{1-x}$. Set $x = \frac{1}{2}$, and take the base-2 logarithm of both sides:

$$\begin{aligned}
\sum_{k=0}^{n} \log_2\left(1+\frac{1}{2^{2^k}}\right) &= \log_2\left(1-\frac{1}{2^{2^{n+1}}}\right) - \log_2\left(1-\frac{1}{2}\right) \\
&= \log_2\left(1-\frac{1}{2^{2^{n+1}}}\right) + 1.
\end{aligned}$$

Rephrased in this way, the problem asks for the smallest positive integer n such that

$$\log_2\left(1-\frac{1}{2^{2^{n+1}}}\right) + 1 \geq 1 + \log_2\left(1-\frac{1}{2015}\right).$$

Because $\log_2$ is an increasing function, it preserves inequalities. It therefore suffices to compute the smallest positive integer n such that

$$1-\frac{1}{2^{2^{n+1}}} \geq 1-\frac{1}{2015}.$$

Rearrange this inequality to obtain $2^{2^{n+1}} \geq 2015$. If $n=2$, then the left-hand side is $2^8 = 256 < 2015$, but if $n=3$, the left-hand side is $2^{16} = 65536 > 2015$, so the smallest positive integer n satisfying the criterion is **3**.

I-5. Use the given equation to expand $f(x+1)+f(x+2)$.

$$f(x+1)+f(x+2) = f(x)+f(x+2)+f(x+1)+f(x+3).$$

Cancel $f(x+1)+f(x+2)$ from both sides:

$$0 = f(x)+f(x+3).$$

Thus if $f(1) = a$, $f(2) = b$, and $f(3) = c$, the sequence $f(1), f(2), f(3), \ldots$ is given by $a, b, c, -a, -b, -c, a, b, c, \ldots$. This sequence is periodic with period six. Thus $b = f(2) = f(20) = 15$, and $c = f(3) = f(15) = 20$. Because $20152015 \equiv 1 \bmod 6$, it follows that $f(20152015) = f(1) = a$. Lastly, recall that $f(2) = f(1) + f(3)$, so $f(20152015) = a = 15 - 20 = \mathbf{-5}$.

I-6. Let h be the altitude of trapezoid $ABCD$, so that $[ABCD] = 10h$. Because $AECD$ and $FBCD$ are both parallelograms, $AE = FB = 6$, so $EF = 2$. Therefore $[CEB] = 4h$.

Triangles EGF and CGD are similar, so $\frac{CG}{EG} = \frac{DG}{FG} = \frac{CD}{EF} = 3$. This means that $\frac{CG}{CE} = \frac{3}{4}$.

Triangles AGF and AHB are also similar, so $\frac{HB}{GF} = \frac{AB}{AF} = \frac{14}{8}$. Because $\frac{DG}{FG} = 3$, $\frac{GF}{DF} = \frac{1}{4}$. Because $FBCD$ is a parallelogram, $DF = CB$, so $\frac{GF}{CB} = \frac{1}{4}$. Therefore $\frac{HB}{CB} = \frac{HB}{GF} \cdot \frac{GF}{CB} = \frac{14}{8} \cdot \frac{1}{4} = \frac{7}{16}$. Thus $\frac{CH}{CB} = \frac{9}{16}$.

Let $\alpha = \mathrm{m}\angle ECB$. Then $[CEB] = \frac{1}{2} CE \cdot CB \cdot \sin\alpha$ and $[CGH] = \frac{1}{2} CG \cdot CH \cdot \sin\alpha$, so $[CGH] = [CEB] \cdot \frac{CG}{CE} \cdot \frac{CH}{CB}$. Therefore, by the above results, $[CGH] = 4h \cdot \frac{3}{4} \cdot \frac{9}{16} = \frac{27}{16}h$. Because $[ABCD] = 10h$, $\frac{[CGH]}{[ABCD]} = \mathbf{\frac{27}{160}}$.

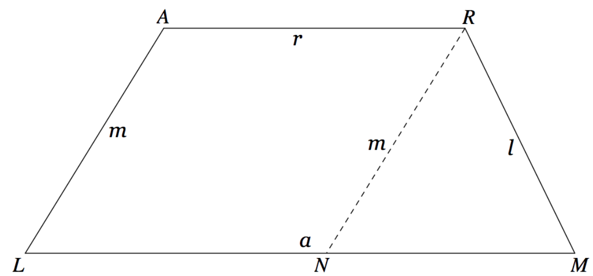
I-7. The sum of the roots of $f(x)$ is 49 by Vieta's formula. If r is a root of f, then $r - 5$ must be a root of g and conversely, because $g(r-5) = f((r-5)+5) = f(r)$. (If all three roots are real, this result can be seen graphically, because the graph of $g(x)$ is a translation of the graph of $f(x)$ by 5 units to the left.) Thus the sum of the roots of $g(x)$ is $49 - 3 \cdot 5 = \mathbf{34}$.

I-8. Let $\theta = \mathrm{m}\angle HWE$ and $a = HW$. Then $HE = a\tan\theta$, $WA = a\cot\theta$, and $\sin\angle SWE = \sin(90^\circ - 2\theta) = \cos 2\theta$. Because $[WSA] = \frac{1}{2}[WASH]$,

$$\frac{32}{50} = \frac{[SED]}{[WSA]} = \frac{SE^2}{WS^2} = \frac{(a\cot\theta - a\tan\theta)^2}{a^2 + (a\cot\theta)^2} = \frac{(\cot\theta - \tan\theta)^2}{1 + (\cot\theta)^2} = \frac{(\cos^2\theta - \sin^2\theta)^2}{\cos^2\theta} = \frac{\cos^2 2\theta}{\cos^2\theta}.$$

Cross-multiplying, $50\cos^2 2\theta = 32\cos^2\theta = 16 + 16\cos 2\theta$, resulting in a quadratic in $\cos 2\theta$ with solutions $\cos 2\theta = \frac{4 \pm 6\sqrt{6}}{25}$. Because $\theta < 45^\circ$, the negative solution is rejected, so the answer is $\mathbf{\frac{4+6\sqrt{6}}{25}}$.

I-9. Let $ARML$ be an (a, r, m, ℓ)-trapezoid, with $AR = r$, $RM = \ell$, $ML = a$, and $LA = m$, so that $\overline{AR} \parallel \overline{ML}$, and $a \geq r$. Then place N on $\overline{ML}$ so that $\overline{NR} \parallel \overline{LA}$, as shown in the diagram below.



Because $ARNL$ is a parallelogram, $RN = m$ and $LN = r$, so $NM = a - r$. By the triangle inequality, $m + \ell > a - r \Rightarrow \ell > a - r - m$, $m + a - r > \ell$, and $a - r + \ell > m \Rightarrow \ell > m + r - a$. In the case where $a = 20$, $r = 5$, and $m = 15$, these three inequalities yield $\ell > 0$ (twice) and $30 > \ell$. Because ℓ is an integer, there are **29** possibilities for ℓ.

Alternate Solution: Using the diagram above, consider the extreme (degenerate) cases. With a $0°$ angle at L, $a = \ell + m + r$, making $\ell = 20 - 15 - 5 = 0$. With a $180°$ angle at L, $a + m = r + \ell$, making $\ell = 20 + 15 - 5 = 30$. Thus $0 < \ell < 30$ for a total of **29** possible integer values.

I-10. Let A be the event that the first three people are in increasing height order, let B be the event that the second, third, and fourth people are in increasing height order, let C be the event that the third, fourth, and fifth people are in increasing height order, and let D be the event that the last three people are in increasing height order. Letting $|x|$ denote the number of orderings that result in event x occurring, then $|A \cup B \cup C \cup D|$ is the number of orderings where there are three consecutive people in increasing height order. The desired answer results by subtracting this number from $6! = 720$, the total number of orderings of six individuals.

Proceed by using the principle of inclusion-exclusion:

$$\begin{aligned}
|A \cup B \cup C \cup D| \quad = \quad & (|A| + |B| + |C| + |D|) \\
& - (|A \cap B| + |B \cap C| + |C \cap D| + |A \cap C| + |B \cap D| + |A \cap D|) \\
& + (|A \cap B \cap C| + |B \cap C \cap D| + |A \cap B \cap D| + |A \cap C \cap D|) \\
& - (|A \cap B \cap C \cap D|).
\end{aligned}$$

Observe that $|A| = \binom{6}{3} \cdot 3!$, because there are $\binom{6}{3}$ ways to choose which 3 people to put in increasing height order in the first three spots, and then there are 3! ways to arrange the remaining 3 people in any order. Similarly, $|B| = |C| = |D| = \binom{6}{3} \cdot 3!$.

Next, observe that the event $A \cap B$ really means that persons 1 through 4 are all in increasing height order, because event A says that persons 1 through 3 are in order, and event B says that persons 2 through 4 are in order, and both of these hold exactly when persons 1 through 4 are all in increasing height order. There are $\binom{6}{4}$ ways to select which four people to place in these spots, one way to order them, and 2! ways to order the remaining two people. So $|A \cap B| = \binom{6}{4} \cdot 2!$. Identical arguments hold for $B \cap C$ and $C \cap D$, so $|B \cap C| = |C \cap D| = \binom{6}{4} \cdot 2!$.

Next, observe that $|A \cap C|$ and $|B \cap D|$ count the number of ways to arrange the first 5 people in order and the number of ways to arrange the last 5 people in order, respectively. Therefore following an argument similar to above, $|A \cap C| = |B \cap D| = \binom{6}{5}$.

The event $A \cap D$ is the event that the first three people are arranged in order, and the last three people are arranged in order. There are $\binom{6}{3}$ ways to select which three people are in the first three spots, and then only one way to arrange them, and only one way to arrange the remaining three people, so $|A \cap D| = \binom{6}{3}$.

The events $A \cap B \cap C$ is identical to the event $A \cap C$, because both events describe the event that the first 5 people in line are in order. Similarly, the event $B \cap C \cap D$ is identical to the event $B \cap D$. These were calculated above, so $|A \cap B \cap C| = |B \cap C \cap D| = \binom{6}{5}$.

The events $A \cap B \cap D$, $A \cap C \cap D$, and $A \cap B \cap C \cap D$ are all identical to the event that all 6 people are in increasing height order, so $|A \cap B \cap D| = |A \cap C \cap D| = |A \cap B \cap C \cap D| = 1$.

Combine these computations to obtain:

$$|A \cup B \cup C \cup D| = 4 \cdot \binom{6}{3} \cdot 3! - 3 \cdot \binom{6}{4} \cdot 2! - 2 \cdot \binom{6}{5} - \binom{6}{3} + 2 \cdot \binom{6}{5} + 2 \cdot 1 - 1 = 371.$$

Therefore the number of orderings where no three consecutive people are in increasing height order is $720 - 371 = \mathbf{349}$.

Power Question 2015: The Power of Riffles

This power question concerns sequences of positive integers, which will be written as $\langle a_1, a_2, \ldots \rangle$, or simply $\langle a_n \rangle$. In all cases, the first term of the sequence will have index 1, that is, no term will be denoted a_0.

Throughout this power event, the word "sequence" is equivalent to "positive integer sequence". Sequences may not contain non-positive or non-integer values!

A sequence such as $\langle 71, 54, 37, 20, 3, 20, 3, 20, 3, 20, 3, 20, 3, \ldots \rangle$ is called *periodic*; in this case, the period of the sequence is 2, and the periodicity begins at the fourth term. Formally, a sequence will be called periodic if there exists a positive integer p and a positive integer s such that $a_n = a_{n+p}$ for all $n \geq s$; the *period of the sequence* is the least such value of p, and the *beginning of periodicity of the sequence* is the least such value of s.

The following are four examples of sequences:

1. A *constant sequence* such as $\langle 2, 2, 2, \ldots \rangle$ is the sequence all of whose terms are 2. The sequence might also be written as $a_n \equiv 2$.
2. The powers of two are given by $a_n = 2^{n-1}$, and begin $\langle 1, 2, 4, \ldots \rangle$.
3. The Fibonacci sequence will be denoted by $\langle F_n \rangle$; they are defined by $F_1 = F_2 = 1$ and the rule $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.
4. The *modulo sequence for* n, $\langle 1, 2, 3, \ldots, n, 1, 2, 3, \ldots, n, 1, 2, \ldots \rangle$, that is, the periodic sequence satisfying the conditions (i) that the sequence has period n, (ii) that the periodicity begins with the first term, and (iii) that the first n terms are $1, 2, 3, \ldots, n$.

As the examples suggest, a sequence can be defined by an explicit or recursive formula, and need not have any easily expressible algebraic formula.

For a given sequence $\langle a_n \rangle$, the *riffle* of $\langle a_n \rangle$, denoted $\langle a'_n \rangle$, is given by setting $a'_1 = a_1$, and for $n \geq 2$:

$$a'_n = \begin{cases} a'_{n-1} - a_n & \text{if } a_n < a'_{n-1} \\ a'_{n-1} + a_n & \text{otherwise.} \end{cases}$$

Note that each term in $\langle a'_n \rangle$ is a positive integer, and so $\langle a'_n \rangle$ is indeed a sequence.

1. Compute a'_{10} for each sequence below. [5 pts]

   a. $a_n = n$

   b. $a_n = n + 1$

   c. $a_n = n^2$

   d. $a_n = 2^{n-1}$

   e. $a_n = F_n$, the Fibonacci sequence

2. Compute a'_{2015} for each sequence below. [3 pts]

   a. $a_n = n$

   b. $a_n = 2^{n-1}$

   c. the modulo sequence for $n = 5$

3. Determine the smallest n for which $a'_n = 2015$ for each sequence below, or show that no such n exists. [7 pts]

   a. $a_n = n$

   b. $a_n = F_n$

   c. the modulo sequence for $n = 1000$

   d. $a_n = 3^n - 2^n$

4. a. Compute examples of two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ with equal periods such that $\langle a'_n \rangle$ and $\langle b'_n \rangle$ have different periods, or show that no such pair of sequences exists. [2 pts]

   b. Show that $\langle a_n \rangle$ is periodic if and only if $\langle a'_n \rangle$ is periodic. [3 pts]

5. Suppose that $\langle a_n \rangle$ has period p and that $\langle a'_n \rangle$ has period q. [6 pts]

   a. Show that $p \leq q \leq p \cdot M$, where M is the maximum value of $\langle a'_n \rangle$.

   b. Determine whether p must be a divisor of q.

6. A sequence $\langle a_n \rangle$ is *invertible* if there exists at least one sequence $\langle b_n \rangle$ for which $\langle b'_n \rangle = \langle a_n \rangle$; in that case, the sequence $\langle b_n \rangle$ is an *inverse* of $\langle a_n \rangle$. Determine whether the following sequences $\langle a_n \rangle$ are invertible. [4 pts]

   a. $a_n = n$

   b. $a_n = \binom{n+1}{2}$

   c. $a_n = 2^n$

   d. $a_n = n^m$, where $m > 1$ is a fixed integer

7. a. Compute an example of a sequence with period 17 that is invertible, and an example of a sequence with period 17 that is not invertible. [2 pts]

   b. Suppose that $\langle a_n \rangle$ is periodic with period 2 beginning at $n = 1$, and that $a_2 > 2a_1$. Show that $\langle a_n \rangle$ is invertible. [2 pts]

   c. Let $\langle a_n \rangle$ be a strictly increasing sequence. That is, $a_n < a_{n+1}$ for all n. Suppose further that $\langle a_n \rangle$ is invertible. Prove that $a_n \geq 2^{n-1}$ for all n. [2 pts]

d. Determine the set of real numbers S for which the following statement is true: $\langle a_n \rangle$ is invertible if and only if $\frac{a_{n+1}}{a_n} \notin S$ for all $n \geq 1$. [3 pts]

8. Suppose $\langle a_n \rangle$ is invertible. Let $\langle a_n \rangle^{-1}$ denote the inverse of $\langle a_n \rangle$. More generally, if $k \geq 1$ and $\langle a_n \rangle^{-k}$ is invertible, denote its inverse by $\langle a_n \rangle^{-(k+1)}$. (It may be helpful to define $\langle a_n \rangle^0 = \langle a_n \rangle$.) Define $\langle a_n \rangle$ to be *k-invertible* if the sequences

$$\langle a_n \rangle, \quad \langle a_n \rangle^{-1}, \quad \ldots, \quad \langle a_n \rangle^{-(k-1)}$$

are all invertible.

a. Determine whether there exists a set S for which the following statement is true: $\langle a_n \rangle$ is 2-invertible if and only if $\frac{a_{n+1}}{a_n} \notin S$ for all $n \geq 1$. [4 pts]

b. Determine whether there exists a sequence $\langle a_n \rangle$ that is 2015-invertible but *not* 2016-invertible. [3 pts]

c. Determine whether there exists a sequence $\langle a_n \rangle$ that is k-invertible for all $k \geq 1$. (Such a sequence will be called *infinitely invertible*, or ∞-*invertible*.) [4 pts]

Solutions to 2015 Power Question

1. The first 10 terms of a'_n for each sequence are given below.

   a. $a_n = n$: $\langle a'_n \rangle = \langle 1, 3, 6, 2, 7, 1, 8, 16, 7, \mathbf{17}, \ldots \rangle$

   b. $a_n = n + 1$: $\langle a'_n \rangle = \langle 2, 5, 1, 6, 12, 5, 13, 4, 14, \mathbf{3}, \ldots \rangle$

   c. $a_n = n^2$: $\langle a'_n \rangle = \langle 1, 5, 14, 30, 5, 41, 90, 26, 107, \mathbf{7}, \ldots \rangle$

   d. $a_n = 2^{n-1}$: $\langle a'_n \rangle = \langle 1, 3, 7, 15, 31, 63, 127, 255, 511, \mathbf{1023}, \ldots \rangle$

   e. $a_n = F_n$: $\langle a'_n \rangle = \langle 1, 2, 4, 1, 6, 14, 1, 22, 56, \mathbf{1}, \ldots \rangle$

2. a. Although no proof is required for this problem, it is useful to determine which values of n satisfy $a'_n = 1$. If $a'_k = 1$, the next three terms of the sequence will be $k+2, 2k+4, k+1$—note the two consecutive increases—after which $\langle a'_n \rangle$ increases and decreases alternately. Thereafter, if a'_m and a'_{m+2} follow consecutive decreases, then $a'_{m+2} = a'_m - 1$, and similarly $a'_{m+2} = a'_m + 1$ if a'_m and a'_{m+2} follow consecutive increases. Thus, if $a'_{k+3} = k + 1$, the next value of m such that $a'_m = 1$ is $m = k + 3 + 2k = 3k + 3$.

   Using this relation yields $a'_n = 1$ for $n = 1, 6, 21, 66, 201, 606, 1821$. Subsequently, $a'_{1824} = 1822$, and $a'_{1825} = 3647$. As 2015 is 190 terms later, $a'_{2015} = 3647 + \frac{1}{2} \cdot 190 = \mathbf{3742}$.

   b. Use induction to show that $a'_n = 2^n - 1$. The base case is $a'_1 = a_1 = 1 = 2^1 - 1$. Assume then that $a'_n = 2^n - 1$. Then because $a_{n+1} = 2^{(n+1)-1} = 2^n > a'_n$, it follows that $a'_{n+1} = a'_n + a_n = 2 \cdot 2^n - 1 = 2^{n+1} - 1$. Hence $a'_{2015} = \mathbf{2^{2014}}$.

   c. Note that $\langle a_n \rangle$ is periodic, and therefore bounded. A reasonable conjecture is that $\langle a'_n \rangle$ is periodic. Although no proof is required for this problem, it is useful to provide one now. First, though, it must be shown that if all the terms of a sequence $\langle a_n \rangle$ are at most M, then all the terms of $\langle a'_n \rangle$ are at most $2M$:

   Proof: Assume towards a contradiction that there exists at least one term in $\langle a'_n \rangle$ that exceeds $2M$. Let k be the smallest index satisfying $a'_k > 2M$. Then $a'_{k-1} \le 2M$, and so $\langle a'_n \rangle$ has an increase from a'_{k-1} to a'_k. As this is an increase, it must be the case that $a_k \ge a'_{k-1}$. Then $2a_k \ge a'_{k-1} + a_k = a'_k > 2M$, which implies $a_k > M$, a contradiction. $\square$

   Note that in general, the $2M$ bound cannot be improved. For example, the constant sequence $\langle 1, 1, 1, \ldots \rangle$ has a riffle $\langle 1, 2, 1, 2, \ldots \rangle$. Now it can be shown that if $\langle a_n \rangle$ is periodic, then $\langle a'_n \rangle$ is periodic as well.

   Proof: If $\langle a_n \rangle$ is periodic, then it is bounded. Let its period be p and its maximum value be M. For any fixed offset r, the subsequence $\langle a'_{p+r}, a'_{2p+r}, a'_{3p+r}, \ldots \rangle$ must repeat a value in its first $2M + 1$ terms. Suppose the first instance of a repeated value in this subsequence is $a'_{kp+r} = a'_{lp+r}$. Because $a_{kp+r+c} = a_{lp+r+c}$ for all positive integers c, it follows by induction that $a'_{kp+r+c} = a'_{lp+r+c}$. Therefore $\langle a'_n \rangle$ is periodic, and its period is at most $(2M+1)p$. $\square$

Because $\langle a'_n \rangle$ is periodic with period 5, it is enough to find k, l, and r such that $a'_{5k+r} = a'_{5l+r}$. The first 20 terms of $\langle a'_n \rangle$ are given below.

1	3	6	2	7	6	4	1	5	10
9	7	4	8	3	2	4	1	5	10

The first instance of such a repeated pair of terms is $a'_7 = a'_{17} = 4$. Thus $\langle a'_n \rangle$ is periodic with period 10. From here it is straightforward to get $a'_{2015} = a'_{15} = \mathbf{3}$.

3. a. From the solution to 2a, recall that if $k > 1$ and if $a'_k = 1$, then the next term to equal 1 will be a'_{3k+3}. Furthermore, it was shown that the terms $a'_{k+1}, a'_{k+2}, \ldots, a'_{3k+3}$ consist of alternating increases and decreases, with the terms coming after the increases or decreases themselves increasing or respectively decreasing by 1. That is, $a'_{k+2}, a'_{k+4}, \ldots, a'_{3k+2} = 2k+4, 2k+5, \ldots, 3k+4$; and $a'_{k+1}, a'_{k+3}, \ldots, a'_{3k+3} = k+2, k+1, \ldots, 1$.

For each set of terms $a_k, a_{k+1}, \ldots, a_{3k+3}$, consider the range of values covered by the respective increasing and decreasing subsequences:

k	Up	Down
1	$6, 7$	$3, 2, 1$
6	$16, 17, \ldots, 22$	$8, 7, \ldots, 1$
21	$46, 47, \ldots, 67$	$23, 22, \ldots, 1$
66	$136, 137, \ldots, 202$	$68, 67, \ldots, 1$
201	$406, 407, \ldots, 607$	$203, 202, \ldots, 1$
606	$1216, 1217, \ldots, 1822$	$608, 607, \ldots, 1$
1821	$3646, 3647, \ldots, 5467$	$1823, 1822, \ldots, 1$
5466	$10936, 10937, \ldots, 16402$	$5468, 5467, \ldots, 1$

It is clear then that the first time 2015 occurs in $\langle a'_n \rangle$ will be in the decreasing subsequence beginning at $a'_{5467} = 5468$. Because $5468 - 2015 = 3453$, it will take $2 \cdot 3453 = 6906$ more terms to reach 2015. That is, $2015 = a'_{5467+6906} = \boldsymbol{a'_{12373}}$.

b. In this case, $a'_n = 2015$ has no solution. Note that $\langle a'_n \rangle$ begins with $1, 2, 4, 1$. That is, $a'_4 = 1$.

Now suppose $a'_n = 1$, for some index $n > 2$. Then $a_{n+1} = F_{n+1} > 1$ and $a_{n+2} = F_{n+2} \geq F_{n+1} + 1$. Therefore $a'_{n+1} = F_{n+1} + 1$, $a'_{n+2} = F_{n+1} + 1 + F_{n+2} = F_{n+3} + 1$, and $a'_{n+3} = (F_{n+3} + 1) - F_{n+3} = 1$. That is, every third term of $\langle a'_n \rangle$ will equal 1, and the intermediate terms will each be 1 greater than a Fibonacci number.

This result applies to $\langle a'_n \rangle$ from $n = 4$ onward. Because 2014 is not a Fibonacci number, 2015 does not appear in $\langle a'_n \rangle$.

c. It can be shown by contradiction that 2015 does not appear in $\langle a'_n \rangle$. If $a'_n = 2015$, then because $a_n \leq 1000$, a'_{n-1} must be smaller than a'_n. That is, $a'_{n-1} = a'_n - a_n = 2015 - a_n \geq 1015$. But because $a'_{n-1} \geq 1015$ and $a_n \leq 1000$, it follows that $a'_n = a'_{n-1} - a_n < 1015$. Thus 2015 does not appear in $\langle a'_n \rangle$.

d. The number 2015 does not appear in $\langle a'_n \rangle$. First, it will be shown that $\langle a'_n \rangle$ has no decreases, or equivalently, $a_n > a'_{n-1}$ for all n. Proceed by induction. In the base case, $a_2 = 5 > 1 = a'_1$. Then assume $a'_k = a_1 + \cdots + a_k$ for some $k \geq 1$. Note that $3^1 + 3^2 + \cdots + 3^k = \frac{1}{2}(3^{k+1} - 3)$ and $2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 2$, so $a'_k = \frac{3^{k+1}}{2} - 2^{k+1} + \frac{1}{2}$. Then $a_{k+1} - a'_k = \frac{3^{k+1}-1}{2} > 0$, so $a'_{k+1} = a'_k + a_{k+1}$, which is another increase. This completes the inductive step.

Because $\langle a'_n \rangle$ is strictly increasing, it is necessary only to compute terms until reaching one that is at least 2015. The first several terms of $\langle a'_n \rangle$ are $1, 6, 25, 90, 301, 966, 3025$. Hence 2015 does not occur in the sequence.

4. a. There are many such examples. Consider $\langle a_n \rangle = \langle 1, 2, 1, 2, \ldots \rangle$ and $\langle b_n \rangle = \langle 1, 3, 1, 3, \ldots \rangle$: their respective riffles, with periods 4 and 6, are shown below.

$$\langle a'_n \rangle = \langle 1, 3, \mathbf{2}, 4, 3, 1, \mathbf{2}, 4, 3, 1, \ldots \rangle$$
$$\langle b'_n \rangle = \langle \mathbf{1}, 4, 3, 6, 5, 2, \mathbf{1}, 4, 3, 6, \ldots \rangle$$

b. The forward direction of the proof was shown earlier as a part of the solution to 2c, so now consider the reverse direction of the proof.

Suppose that $\langle a'_n \rangle$ is periodic with period p. Let r be an integer such that $a'_s = a'_{s+p}$ for all $s \geq r$. Because $a_n = |a'_n - a'_{n-1}|$ for all $n > 1$, it follows that $a_{s+p} = |a'_{s+p} - a'_{s+p-1}| = |a'_{s+2p} - a'_{s+2p-1}| = a_{s+2p}$ for all $s \geq r$. This establishes the periodicity of $\langle a_n \rangle$, and completes the proof. $\square$

5. a. By assumption, the sequence $\langle a_n \rangle$ has period p, the riffle $\langle a'_n \rangle$ has period q, and $\max_n a'_n = M$. Let the periodicity of $\langle a'_n \rangle$ begin at r, that is, for all $k \geq r$, $a'_k = a'_{k+q}$. Then

$$a_{k+1} = \left| a'_{k+1} - a'_k \right| = \left| a'_{k+1+q} - a'_{k+q} \right| = a_{k+1+q}$$

for all such k, and so $\langle a_n \rangle$ is periodic with period at most q. That is, $p \leq q$.

Let s be an integer for which $a_k = a_{k+p}$ for all $k \geq s$. Consider the $M+1$ values $a'_s, a'_{s+p}, \ldots, a'_{s+Mp}$. These are $M+1$ terms, all of which are values between 1 and M. Hence there must exist integers u and v such that $0 \leq u < v \leq M$ and $a'_{s+up} = a'_{s+vp}$. It can be shown via induction that $\langle a'_n \rangle$ is periodic from a'_{s+up} onwards. Assume $a'_{s+up+z} = a'_{s+vp+z}$ for some $z \geq 0$. (The case $z = 0$ has already been established.) Because $a'_{s+up+z} = a'_{s+vp+z}$ and $a_{s+up+z+1} = a_{s+vp+z+1}$, it follows that $a'_{s+up+z+1} = a'_{s+vp+z+1}$, concluding the inductive step. Thus $\langle a'_n \rangle$ is periodic with period at most $(v-u)p \leq Mp$.

b. Show that q is a multiple of p by contradiction. If $p = 1$, then q must be a multiple of p. So assume that $p > 1$. In the proof that follows, the essential idea is that if q is *not* a multiple of p, then $\langle a_n \rangle$ must be periodic with a period of $c = \gcd(p, q)$.

Assume towards a contradiction that q is not a multiple of p. Let $c = \gcd(p, q)$, and let $d = \frac{p}{c}$. Note that $c < p$, and so $d > 1$.

Let r be an integer for which $a_{r+k} = a_{r+k+p}$ for all $k \geq 0$. Then for *at least one* integer i with $1 \leq i \leq c$, the terms $a_{r+i}, a_{r+i+c}, \ldots, a_{r+i+(d-1)c}$ must contain *at least two* different values. (If this weren't the case, then $\langle a_n \rangle$ would have period at most c, which is a contradiction.)

So let x, y be integers, with $0 \leq x < y < d$ for which $a_{r+i+xc} \neq a_{r+i+yc}$. Because $c = \gcd(p, q)$, $\frac{p}{c}$ and $\frac{q}{c}$ are relatively prime. Hence there exists an integer $m > 0$ such that $m \cdot \frac{q}{c} \equiv 1 \bmod \frac{p}{c}$. Thus $m(y-x)q \equiv (y-x)c \bmod p$. Then

$$\begin{aligned} a_{r+i+xc} &= |a'_{r+i+xc} - a'_{r+i+xc-1}| \\ &= |a'_{r+i+xc+m(y-x)q} - a'_{r+i+xc+m(y-x)q-1}| \\ &= a_{r+i+xc+m(y-x)q}. \end{aligned}$$

But $r+i+xc+m(y-x)q$ is no less than r and is congruent to $r+i+yc \bmod p$, so

$$a_{r+i+xc+m(y-x)q} = a_{r+i+yc} \neq a_{r+i+xc}.$$

This result contradicts the existence of two distinct values within one of the $\frac{p}{c}$ subsequences of $\langle a_n \rangle$, which would imply that $\langle a_n \rangle$ is periodic with period at most c.

6. a. The sequence is not invertible. Note that if $\langle b_n \rangle$ is an inverse of $\langle a_n \rangle$, then $b_3 = |a_3 - a_2| = 1$. But then it follows that $a_3 = b'_3 = b'_2 - b_3 = 2 - 1 = 1 \neq a_3$.

b. The sequence is not invertible. The first few terms of $\langle a_n \rangle$ are $1, 3, 6, 10, 15$. So if $\langle b_n \rangle$ is an inverse of $\langle a_n \rangle$, it follows that $b_4 = |a_4 - a_3| = 4$. But then $a_4 = b'_4 = b'_3 - b_4 = 6 - 4 = 2$.

c. The sequence is invertible, with inverse $\langle b_n \rangle$, where $b_1 = 1$, and $b_n = 2^{n-2}$ for $n > 1$. It is straightforward to show by induction that $\langle b'_n \rangle = \langle a_n \rangle$.

d. This sequence is not invertible for any value of m. As with the earlier examples of sequences that were not invertible, the key is to find a pair of terms a_n, a_{n+1} for which $\frac{a_{n+1}}{a_n} < 2$. Because $\sqrt[m]{2} > 1$, there exists an integer k for which $(\frac{k+1}{k})^m < 2$. So suppose there exists a sequence $\langle b_n \rangle$ for which $b'_1, b'_2, \ldots, b'_k = a_1, a_2, \ldots, a_k$. Because $(\frac{k+1}{k})^m < 2$, it follows that $a_{k+1} < 2a_k$. Therefore $b_{k+1} = |a_{k+1} - a_k| = a_{k+1} - a_k$, which is less than $a_k = b'_k$. Thus $b'_{k+1} = b'_k - b_{k+1} = 2a_k - a_{k+1}$, which is less than a_{k+1}.

7. a. The periodic sequence $\langle 17, 16, 15, \ldots, 1, 17, 16, \ldots \rangle$ is invertible, whereas the periodic sequence $\langle 1, 2, 3, \ldots, 17, 1, 2, \ldots \rangle$ is not invertible.

b. Let $\langle b_n \rangle$ be defined as follows: $b_1 = a_1$, and for $n > 1$, $b_n = a_2 - a_1$. Then $b_2 > b_1$, and it follows that $b'_1 = b_1 = a_1$, $b'_2 = b_1 + b_2 = a_2$, and $b'_3 = b'_2 - b_3 = a_1$. Because $b'_1 = b'_3$ and $\langle b_n \rangle$ is constant for $n \geq 2$, $\langle b'_n \rangle$ is periodic from $n \geq 2$ onwards. Thus $\langle b_n \rangle$ is an inverse of $\langle a_n \rangle$.

c. Proceed by induction on n. For the base case, $a_1 \geq 1 = 2^{1-1}$. Suppose that $\langle b'_n \rangle = \langle a_n \rangle$, and assume that $a_n \geq 2^{n-1}$. Then $a_{n+1} \geq a_n$ implies that $b_{n+1} \geq a_n = 2^{n-1}$, so $a_{n+1} = a_n + b_{n+1} \geq 2^{n-1} + 2^{n-1} = 2^n$.

d. Given a sequence $\langle a_n \rangle$ of positive integers, define the *ruffle* sequence $\langle \hat{a}_n \rangle$ as follows: let $\hat{a}_1 = a_1$, and for $n > 1$ let $\hat{a}_n = |a_n - a_{n-1}|$.

Claim: If $\langle a_n \rangle$ is invertible, it can have only one inverse, namely $\langle \hat{a}_n \rangle$.

Proof of Claim: Suppose there exists a sequence $\langle b_n \rangle$ for which $\langle b'_n \rangle = \langle a_n \rangle$. Then $b_1 = a_1$, and for $n > 1$, $b_n = |a_n - a_{n-1}|$. That is, $\langle b_n \rangle = \langle \hat{a}_n \rangle$. □

Note the restricted nature of the claim: it does *not* guarantee that $\langle (\hat{a})'_n \rangle = \langle a_n \rangle$. For example, the ruffle of $\langle 1, 2, 3, 4, \ldots \rangle$ is the constant sequence $\langle 1, 1, 1, 1, \ldots \rangle$, but the riffle of that constant sequence is the period-2 sequence $\langle 1, 2, 1, 2, \ldots \rangle$. The reason the claim appears to generate an extraneous inverse to $\langle 1, 2, 1, 2, \ldots \rangle$ is that this sequence doesn't have an inverse, which was one of the hypotheses of the claim!

Now consider the question at hand:

Stronger Claim: A sequence $\langle a_n \rangle$ is invertible if and only if $\frac{a_{n+1}}{a_n} \notin [1, 2)$ for all $n \geq 1$.

Proof of Stronger Claim: Suppose $\langle a_n \rangle$ is a positive integer sequence such that $\frac{a_{n+1}}{a_n} \notin [1, 2)$ for all $n \geq 1$. Then $\hat{a}_n > 0$ for all $n \geq 1$. It is true that $\hat{a}_1 = (\hat{a})'_1 = a_1$. Proceed by induction to show that the ruffle sequence $\langle \hat{a}_n \rangle$ is the inverse of $\langle a_n \rangle$:

Suppose that $(\hat{a})'_k = a_k$ for some $k \geq 1$. Then $\hat{a}_{k+1} = |a_{k+1} - a_k|$. If $a_{k+1} < a_k$, then $\hat{a}_{k+1} = a_k - a_{k+1} < a_k = (\hat{a})'_k$, and so $(\hat{a})'_{k+1} = (\hat{a})'_k - \hat{a}_{k+1} = a_k - (a_k - a_{k+1}) = a_{k+1}$. If $a_{k+1} > a_k$, then $\hat{a}_{k+1} = a_{k+1} - a_k \geq a_k = (\hat{a})'_k$, and so $(\hat{a})'_{k+1} = (\hat{a})'_k + \hat{a}_{k+1} = a_k + (a_{k+1} - a_k) = a_{k+1}$. This completes the inductive step. (And the earlier claim entails that there are no other inverses of $\langle a_n \rangle$.)

To show that the condition on S is necessary, use proof by contradiction. Suppose k is the smallest integer for which $\frac{a_{k+1}}{a_k} \in [1, 2)$. By the earlier induction, $(\hat{a})'_k = a_k$. Then $\hat{a}_{k+1} = a_{k+1} - a_k < a_k = (\hat{a})'_k$, and so $(\hat{a})'_{k+1} = (\hat{a})'_k - \hat{a}_{k+1} = a_k - (a_{k+1} - a_k) = 2a_k - a_{k+1}$, which is smaller than a_k and therefore cannot equal a_{k+1}. □

8. a. No such set S exists. For a given sequence $\langle a_n \rangle^k$, let R_a^k be the set of values $\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \ldots\}$. Note that earlier it was shown that $\langle a_n \rangle^0$ is invertible if and only if $R_a^0 \cap [1, 2) = \varnothing$.

Let $\langle b_n \rangle^0$ be the periodic sequence $\langle 1, 3, 1, 3, 1, 3, \ldots \rangle$. Then $R_b^0 = \{\frac{1}{3}, 3\}$. This sequence is invertible, and $\langle b_n \rangle^{-1} = \langle \hat{b}_n \rangle = \langle 1, 2, 2, 2, 2, 2, \ldots \rangle$. Clearly, $\langle b_n \rangle^{-1}$ is not invertible, because it has repeated values.

Now let $\langle c_n \rangle^0$ be the sequence whose first few terms are $1, 3, 9, 30, 10$, and whose subsequent terms are $c_n = 10^{n-4}$ for $n \neq 6$. That is, the first eight terms are 1, 3, 9, 30, 10, 100, 1000, 10000. Then $R_c^0 = \{3, \frac{10}{3}, \frac{1}{3}, 10\}$, and it follows that $R_c^0 \cap [1, 2) = \varnothing$. Thus $\langle c_n \rangle^0$ is invertible. The first several terms of $\langle c_n \rangle^{-1}$ are $2, 6, 21, 20, 90, 900, 9000$. $R_c^{-1} = \{3, \frac{7}{3}, \frac{20}{21}, \frac{9}{2}, 10\}$. It follows that $R_c^{-1} \cap [1, 2) = \varnothing$, so $\langle c_n \rangle^0$ is indeed 2-invertible.

On the other hand, $R_b^0 \subset R_c^0$. In order for the original claim to be true, either $\frac{1}{3}$ or 3 must be an element of S. But both these values are in R_c^0, and $\langle c_n \rangle^0$ is 2-invertible. Therefore the claim cannot be true, as no such set S satisfies the condition.

b. Such a sequence exists. For an integer $k \geq 1$ and sequence $\langle a_n \rangle$, let $\langle a_n \rangle^k$ denote the kth riffle of $\langle a_n \rangle$. That is, $\langle a_n \rangle^{k+1}$ is the riffle of $\langle a_n \rangle^k$. (And as before, let $\langle a_n \rangle^0 = \langle a_n \rangle$.)

Consider the kth riffle of the sequence $a_n = 2^{n-1}$. Note that $\langle a_n \rangle$ is invertible, because $\frac{a_{n+1}}{a_n} = 2$ for all n. However, because the first two terms of $\langle a_n \rangle^{-1}$ are $1, 1$, $\langle a_n \rangle$ is not 2-invertible.

Claim: Let $\langle b_n \rangle$ be the kth riffle of $\langle a_n \rangle$, where $k \geq 1$. Then $\langle b_n \rangle$ is strictly increasing, with $\frac{b_{n+1}}{b_n} > 2$ for all n.

Proof of Claim: The claim is true for $k = 1$, as $\langle a_n \rangle^1 = \langle 1, 3, 7, 15, 31, \ldots \rangle$, i.e., the sequence $2^n - 1$. So assume the claim is true for some positive integer k, and let $\langle c_n \rangle = \langle b'_n \rangle = \langle a_n \rangle^{k+1}$. First show that $\langle c_n \rangle$ is increasing. Proceed by induction. For the base case, $c_1 = b_1$, and because $b_2 > b_1$, it follows that $c_2 = b_1 + b_2$. Assume (further) now that $\langle c_n \rangle$ is increasing for $c_1, c_2, \ldots, c_j$. Note that $b_j < \frac{b_{j+1}}{2}, b_{j-1} < \frac{b_{j+1}}{4}, \ldots, b_1 < \frac{b_{j+1}}{2^j}$, and so $c_j = b_1 + b_2 + \cdots + b_j < b_{j+1}(2 - \frac{1}{2^j})$. Then because $\frac{b_{j+1}}{b_j} > 2$, $\langle c_n \rangle$ increases at c_{j+1}. This completes the (inner) inductive step, demonstrating that $\langle c_n \rangle$ is increasing. $\square$

It remains to complete the outer induction; that is, that the ratios of consecutive terms are decreasing but are always larger than 2. It can be shown that they are all larger than 2, by noting that

$$\begin{aligned}
\frac{c_{n+1}}{c_n} &= \frac{b_1 + b_2 + \cdots + b_n + b_{n+1}}{b_1 + b_2 + \cdots + b_n} \\
&= 1 + \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} \\
&> 1 + \frac{b_{n+1}}{\frac{b_{n+1}}{2^n} + \frac{b_{n+1}}{2^{n-1}} + \cdots + \frac{b_{n+1}}{2^1}} \\
&> 1 + \frac{1}{1 - \frac{1}{2^n}} \\
&> 2.
\end{aligned}$$

This argument completes the proof.

(**Note:** It can furthermore be shown that $\frac{c_2}{c_1} > \frac{c_3}{c_2} > \frac{c_4}{c_3} > \cdots > 2$.)

It has thus been shown that for any positive integer k, $\langle a_n \rangle^k$ is invertible. (Its ratios are all larger than 2.) Thus $\langle a_n \rangle^{2014}$ can be inverted 2014 times to get $\langle a_n \rangle$, and one more time to get $1, 1, 2, 4, 8, \ldots$, but *cannot* be inverted once more (which would be a 2016th inversion).

This solution can easily be amended to find sequences that can be inverted k times but not $k + 1$ times.

c. No such sequence $\langle a_n \rangle$ exists. Proceed by contradiction: if a sequence $\langle a_n \rangle$ is invertible, its unique inverse is the ruffle sequence, $\langle \hat{a}_n \rangle$. So assume that $\langle a_n \rangle$ is infinitely invertible. Let a_n^{-k} denote the nth term of $\langle a_n \rangle^{-k}$, which is well-defined. Then $a_1^{-k} = a_1$ for all $k \geq 1$, and $a_n^{-k} = |a_n^{-k+1} - a_{n-1}^{-k+1}|$ for all $n > 1, k \geq 1$.

Consider the sequence $a_2, a_2^{-1}, a_2^{-2}, \ldots$. The differences between consecutive terms are all just a_1. There must exist an integer s_2 such that $a_2^{-k} < a_1$ for all $k \geq s_2$. For example, if $a_1, a_2 = 4, 17$, then $a_2, a_2^{-1}, a_2^{-2}, \ldots = 17, 13, 9, 5, 1, 3, 1, 3, 1, 3, \ldots$. That is, after some point, all the values are less than a_1.

Now consider the sequence $a_3, a_3^{-1}, a_3^{-2}, \ldots$. The differences between consecutive terms are the terms of the previous sequence. Note that at some point, all the differences are at most $a_1 - 1$, after which, there must be some $s_3 \geq 1$ for which $a_3^{-s_3} < a_1 - 1$. Then all subsequent terms are at most $a_1 - 2$.

This process can be continued to show that for any $r > 1$, there exists some integer $s_r > s_{r-1}$ for which $a_r, a_r^{-1}, a_r^{-2}, \ldots$ eventually achieves a value less than $a_1 - (r-2)$. Then all subsequent terms are at most $a_1 - (r-2)$. But this cannot continue indefinitely, as eventually, some term must equal zero, which is not allowed. Hence there does not exist an infinitely invertible sequence.

Authors' Note: The authors initially stumbled upon this topic in writing individual questions; in fact, problem 2a was "almost" an individual question. Many other questions are suggested, such as the following:

1. Does there exist an integer $N > 1$ such that the first N terms of both $\langle a_n \rangle$ and $\langle a'_n \rangle$ are permutations of the set of integers $\{1, 2, \ldots, N\}$? If so, what are all such integers?
2. For a specific sequence $\langle a_n \rangle$, is it possible to determine which positive integers never occur in its riffle $\langle a'_n \rangle$, or which occur only finitely many times in its riffle, or which occur infinitely many times in its riffle? For example, what about the sequence $a_n = n$?

Either of these questions, or extensions of the previous questions, would make a promising topic for a mathematics research project!

2015 Relay Problems

R1-1. The average of six distinct real numbers is 275. The average of the four least numbers is 200. The average of the four greatest numbers is 340. Compute the average of the middle two numbers.

R1-2. Let $T = \text{TNYWR}$. A rectangle has area $\dfrac{T}{4} + \dfrac{1}{8}$ and a diagonal of length $\dfrac{T}{2}$. Compute the perimeter of the rectangle.

R1-3. Let $T = \text{TNYWR}$, and let k be a real number such that the region above the x-axis satisfying the inequalities $|2x| + |y| \leq 2k$ and $|x| + |y| \geq k$ has area T. Compute k.

R2-1. Given that $1 - r + r^2 - r^3 + \cdots = s$ and $1 + r^2 + r^4 + r^6 + \cdots = 4s$, compute s.

R2-2. Let $T = \text{TNYWR}$. Two swimming pools Y and P are surrounded by rectangular walkways of width w feet. The sides of Y are 4 feet longer than the sides of P, and the area of the walkway surrounding Y is $128T$ square feet greater than the area of the walkway surrounding P. Compute w.

R2-3. Let $T = \text{TNYWR}$, and let p and q be real numbers with $p < q$. Given that $p^2 - 5p = q^2 - 5q = T$, compute $\dfrac{1}{p} - \dfrac{1}{q}$.

2015 Relay Answers

R1-1. 255

R1-2. 256

R1-3. 16

R2-1. $\dfrac{4}{7}$

R2-2. $\dfrac{32}{7}$

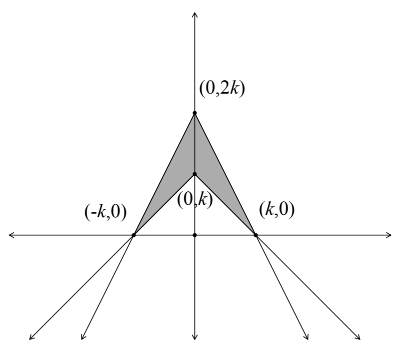
R2-3. $\dfrac{\sqrt{2121}}{32}$

2015 Relay Solutions

R1-1. Let the numbers be $a < b < c < d < e < f$. Then $\frac{a+b+c+d}{4} = 200 \Rightarrow a+b+c+d = 800$, $\frac{c+d+e+f}{4} = 340 \Rightarrow c+d+e+f = 1360$, and $\frac{a+b+c+d+e+f}{6} = 275 \Rightarrow a+b+c+d+e+f = 1650$. From the first two equations it follows that $a+b+2c+2d+e+f = 2160$, so $c+d = 510$, and the average is $\frac{510}{2} = \mathbf{255}$.

R1-2. Let ℓ and w be the length and width of the rectangle, respectively. Then it is given that $\sqrt{\ell^2+w^2} = \frac{T}{2}$, hence $\ell^2 + w^2 = \frac{T^2}{4}$. The area is $\ell w = \frac{T}{4} + \frac{1}{8} = \frac{2T+1}{8}$, so $2\ell w = \frac{2T+1}{4}$ and $(\ell+w)^2 = \ell^2 + 2\ell w + w^2 = \frac{T^2+2T+1}{4}$. Therefore $\ell + w = \left|\frac{T+1}{2}\right|$, and the perimeter is $2\ell + 2w = |T+1|$. With $T = 255$, the perimeter is $\mathbf{256}$.

R1-3. The region is the concave kite whose vertices are $(\pm k, 0)$, $(0, k)$, and $(0, 2k)$, as shown below.



The area of the kite is twice the area of the portion contained in the first quadrant, which is a triangle with base k and height k. So the area of the triangle is $\frac{k^2}{2}$ and the area of the entire kite is k^2. With $T = 256$, $k = \sqrt{256} = \mathbf{16}$.

R2-1. Use the formula for the sum of an infinite geometric series twice to get $\frac{1}{1+r} = s$ and $\frac{1}{1-r^2} = 4s$. Divide the first equation by the second to yield $\frac{1-r^2}{1+r} = \frac{s}{4s}$, or $1 - r = \frac{1}{4}$. Therefore $r = \frac{3}{4}$, and $s = \frac{1}{1+\frac{3}{4}} = \mathbf{\frac{4}{7}}$.

R2-2. Let pool P have length x feet and width y feet. Pool P is surrounded by a walkway of width w feet, so the total rectangular shape has length $x+2w$ feet and width $y+2w$ feet. Thus the area of the walkway itself is $(x+2w)(y+2w) - xy = xy + 2xw + 2yw + 4w^2 - xy = 2(x+y)w + 4w^2$. Substitute $x+4$ for x and $y+4$ for y in this expression to obtain the area of the walkway surrounding pool Y: $2((x+4)+(y+4))w + 4w^2$. The difference between these areas is

$$2((x+4)+(y+4))w + 4w^2 - 2(x+y)w - 4w^2 = 16w.$$

Equate this quantity with $128T$ to give $16w = 128T$, or $w = 8T$. The correct value of T is $\frac{4}{7}$, so $w = \mathbf{\frac{32}{7}}$.

R2-3. Both p and q satisfy the equation $x^2 - 5x - T = 0$, so they must be the roots of the polynomial function $f(x) = x^2 - 5x - T$. Vieta's formulas thus yield $p + q = 5$ and $pq = -T$. Therefore $(p-q)^2 = (p+q)^2 - 4pq = 25 + 4T$. Because $p < q$, $p - q = -\sqrt{25+4T}$. Therefore

$$\frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq} = \frac{\sqrt{25+4T}}{T}.$$

The correct value of T is $\frac{32}{7}$, so

$$\frac{1}{q} - \frac{1}{p} = \frac{\sqrt{25 + 4 \cdot \frac{32}{7}}}{\frac{32}{7}} = \frac{\sqrt{\frac{303}{7}}}{\frac{32}{7}} = \boldsymbol{\frac{\sqrt{2121}}{32}}.$$

2015 Tiebreaker Problems

TB-1. Compute the ordered pair of integers (a, b) that minimizes $\left|a^4 + a^b - 2015\right|$.

TB-2. Compute the number of ways in which the numbers $1, 2, \ldots, 8$ can be filled into a 3×3 array, leaving the center unoccupied, so that if a is anywhere to the left of b and in the same row, or if a is anywhere above b and in the same column, then $a < b$.

TB-3. Let $S = \{15, 24, 30, 40, 50, 60, 80\}$. When one element, k, is removed from S, the product of the remaining elements is a perfect cube. Compute k.

2015 Tiebreaker Answers

TB-1. $(-7, 3)$

TB-2. 18

TB-3. 60

2015 Tiebreaker Solutions

TB-1. Let $f(a,b) = a^4 + a^b$. Note that for fixed values of b, $f(a,b)$ is an increasing function of a when a is positive (this claim is true even if b is negative, because in that case, $(a+1)^4 - a^4 > 1 > a^b - (a+1)^b$). Further, $f(7,b) = 2401 + 7^b > 2015$. Note that if $a = 7$, then decreasing b will always decrease the value of $f(a,b)$, but will always satisfy $f(7,b) > 7^4 > 2015$. In other words, the answer cannot be of the form $(7,b)$, because $(7,b-1)$ would make $|f(a,b) - 2015|$ smaller. Thus it suffices to only consider values of a less than 7. For each positive integer $a = 1,\ldots,6$, compute the value of b for which $f(a,b)$ is nearest to 2015 in absolute value. These values are shown in the table below.

a	b	$f(a,b)$	$\lvert f(a,b) - 2015\rvert$
1	any value	2	2013
2	11	2064	49
3	7	2268	253
4	5	1280	735
5	4	1250	765
6	3	1512	503

Thus the ordered pair (a,b) minimizing $|f(a,b) - 2015|$ is either $(2,11)$, or has $a < 0$. Moreover, if $a < 0$, but b is even, then the ordered pair (a,b) produces the same value as the ordered pair $(-a,b)$, and the previous work also applies. Thus either $(2,11)$ is the correct ordered pair, or $a < 0$ and b is odd. Therefore the term a^b is negative. If $b > 4$, then $a^4 + a^b < 0$, and no negative value of a will cause $f(a,b)$ to differ from 2015 by less than 49, so it must be that $b < 4$. If $b < 0$, then $\left|a^b\right| \le 1$, and $|f(a,b) - a^4| \le 1$. By the triangle inequality,

$$|f(a,b) - 2015| \ge \left|a^4 - 2015\right| - 1.$$

As a must be an integer, this quantity is optimized at $a = -7$, yielding $|f(a,b) - 2015| \ge 385$, which is larger than the value of 49 produced by $(2,11)$. It follows then, that either $b = 1$ or $b = 3$, and that $a < 0$.

Note that both $f(a,1)$ and $f(a,3)$ increase as a is *decreased* (provided that $a \le -1$). Thus it suffices to compute successively increasing values of $f(a,1)$ and $f(a,3)$ until they become larger than 2015.

$$\begin{array}{rr}
f(-1,1) = 0 & f(-1,3) = 0 \\
f(-2,1) = 14 & f(-2,3) = 8 \\
f(-3,1) = 78 & f(-3,3) = 54 \\
f(-4,1) = 252 & f(-4,3) = 192 \\
f(-5,1) = 620 & f(-5,3) = 500 \\
f(-6,1) = 1290 & f(-6,3) = 1080 \\
f(-7,1) = 2394 & f(-7,3) = 2058
\end{array}$$

The value nearest to 2015 is 2058. Moreover, $2058 < 2064$, so the correct ordered pair is $\mathbf{(-7, 3)}$.

TB-2. The 1 must appear in the top-left cell of the array. Similarly, the 8 must appear in the bottom-right cell of the array. This leaves six cells remaining which are split into two groups of three: the three cells nearest the bottom-left corner of the array, and the three cells nearest the top-right corner of the array. Given any three numbers from $\{2, 3, \ldots, 7\}$, there is exactly one way to put them into one of these groups to satisfy the conditions, disregarding any relationships between the two groups. There are therefore $\binom{6}{3} = 20$ such ways to put numbers into this array. However, it cannot be that the largest number in one group is smaller than the smallest number in another group, otherwise either the middle column or the middle row would fail the condition. There are two such possibilities which must be subtracted, yielding a total of **18** possibilities.

TB-3. Factoring the given numbers, the set S is

$$\{3 \cdot 5, 2^3 \cdot 3, 2 \cdot 3 \cdot 5, 2^3 \cdot 5, 2 \cdot 5^2, 2^2 \cdot 3 \cdot 5, 2^4 \cdot 5\}.$$

The product of all of these is $2^{14} \cdot 3^4 \cdot 5^7$. If the product is to be a perfect cube, all exponents must be multiples of 3, thus there is an excess of $2^2 \cdot 3 \cdot 5 = \mathbf{60}$.

2015 Super Relay Problems

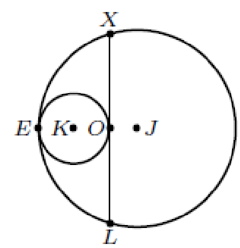
1. Triangle TAR has a right angle at R, with $RT = 3$ and $AT = 5$. Triangle LMN is similar to triangle TAR, and $MN = 8$. Compute $AR \cdot ML$.

2. Let T = TNYWR. Compute the unique fixed point of the function f defined by $f(x) = T \cdot (x - T + 1)$. [Note: A *fixed point* of a function $g(x)$ is a value z such that $g(z) = z$.]

3. Let T = TNYWR. A dessert tray containing T chocolate-covered strawberries has been ordered for an ARML coaches meeting. Marlys eats some of the strawberries, leaving 24 strawberries for the other attendees to consume. Then Marlys will have eaten $K\%$ of the strawberries. Compute the value of K.

4. Let T = TNYWR, and let $a = \log \frac{T}{8}$. Given that $\log K = 3 - 2a$, compute the value of K.

5. Let T = TNYWR. In the diagram to the right, circle K has area $T\pi$ and is internally tangent to circle J at point E. Chord $\overline{XL}$ is tangent to circle K at point O. Given that the product of the radii of circles J and K is $4T - 20$, compute XL.



6. Let T = TNYWR. Mary places 20 green marbles into a hat, Lydia places 15 purple marbles into the hat, and Joshua places T blue marbles into the hat. If a marble is randomly selected from the hat, the probability that it will be purple is $\frac{8}{K}$. Compute K.

7. Let T = TNYWR, and let K be the smallest prime factor of T. Suppose that a large cube has edge length $\sqrt{3K}$. A smaller cube has edge length $\sqrt{K-1}$ and is placed on a face $\mathcal{F}$ of the larger cube so that a face of the smaller cube completely rests on $\mathcal{F}$. Compute the surface area of the resulting figure.

8. Let A_1 be the number you will receive from position 7 and let A_2 be the number you will receive from position 9. Let a_1 be the **smallest** prime factor of A_1 and let a_2 be the **largest** prime factor of A_2. Consider the complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Given that $|z_2|^2 = 2|z_1|^2$ and that b_1 and b_2 are positive integers where $b_1 + b_2$ is as small as possible, compute the value of $|z_2|^2$.

9. Let $T = \text{TNYWR}$. An ARML coach wants to choose a three-person relay team (without regard to order). If the coach had $T+1$ students to choose from, she could form N more relay teams than if she only had T students to choose from. Compute $\lceil\sqrt{2N}\rceil$.

10. Let $T = \text{TNYWR}$. Given that $\sin(T^\circ) = \cos(10^\circ + K^\circ)$, compute the smallest positive value of K.

11. Let $T = \text{TNYWR}$. Alex, Bailey, and Casey are making candied apples. The recipe calls for 1 bag of apples, 2 cups of butter, and 3 cups of caramel, and yields 8 candied apples. Each person has one of the three ingredients to make one or more batches of apples, but alas, the amounts of their ingredients are not proportional, so there will be leftovers of at least one ingredient. Alex has $\frac{T}{8}$ bags of apples, Bailey has 40 cups of butter, and Casey has $T - 17$ cups of caramel. Compute the number of candied apples they can make.

12. Let $T = \text{TNYWR}$. Circle O has center (T, T) and radius T. Two lines are tangent to circle O at points (x_1, y_1) and (x_2, y_2) and each line has slope -1. Compute $\dfrac{x_1x_2 + y_1y_2}{T}$.

13. Let $T = \text{TNYWR}$. Compute the value of K for which

$$\sqrt{T} = \frac{T}{20}\sqrt[4]{K+60}\,.$$

14. Let $T = \text{TNYWR}$. The union of the intervals $[4, T]$ and $[T - 40, 17]$ is a closed interval of length L. Compute L.

15. Let $A + R = 17, R + M = 1$, and $M + L = 24$. Compute $A + L$.

2015 Super Relay Answers

1. 40
2. 40
3. 40
4. 40
5. 40
6. 40
7. 40
8. 26
9. 40
10. 40
11. 40
12. 40
13. 40
14. 40
15. 40

2015 Super Relay Solutions

1. By the Pythagorean Theorem, $AR = 4$. By similar triangles, $ML = AT \cdot \frac{MN}{AR} = 5 \cdot \frac{8}{4} = 10$. Thus $AR \cdot ML = 4 \cdot 10 = \mathbf{40}$.

2. Let $x = T \cdot (x - T + 1)$. Then $x(T-1) = T^2 - T = T(T-1)$. Thus either $T = 1$ (in which case, x is not uniquely determined) or $x = T$. With $T = 40$, it follows that $x = \mathbf{40}$.

3. The given information implies that $\frac{T-24}{T} = \frac{K}{100}$. With $T = 40$, $K = 100 \cdot \frac{16}{40} = \mathbf{40}$.

4. Note that $\log K = \log 1000 - \log(\frac{T}{8})^2 = \log \frac{64000}{T^2}$. Thus $K = \frac{64000}{T^2}$, and with $T = 40$, $K = \mathbf{40}$.

5. Let R and r be the radii of the larger and smaller circles, respectively. Then $JO = R - 2r$, and $XO^2 = JX^2 - JO^2 = R^2 - (R-2r)^2 = 4Rr - 4r^2$. The given information implies that $r^2 = T$, so $XO^2 = 4(4T-20) - 4T = 12T - 80$. Thus $XL = 2\sqrt{12T-80}$. With $T = 40$, $XL = 2\sqrt{480-80} = \mathbf{40}$.

   Alternate Solution: Instead of using the Pythagorean Theorem, extend $\overrightarrow{EO}$ to intersect circle J at point D. Then by the Power of a Point Theorem applied to O, $XO^2 = DO \cdot OE = (2R-2r)(2r) = 4Rr - 4r^2$. As above, this equation is equivalent to $XO^2 = 12T - 80$, hence $XL = \mathbf{40}$.

6. The given information implies that $\frac{8}{K} = \frac{15}{35+T}$, so $K = \frac{280+8T}{15}$. With $T = 40$, $K = \frac{600}{15} = \mathbf{40}$.

7. The larger cube has a surface area of $6(3K)$, and the smaller cube has a surface area of $6(K-1)$. When the smaller cube is placed atop the larger cube, the surface area increases by $4K-4$, so that the total surface area of the resulting solid is $18K + 4K - 4 = 22K - 4$. With $T = 40, K = 2$, hence the answer is $22 \cdot 2 - 4 = \mathbf{40}$.

8. The given information implies that $a_2^2 + b_2^2 = 2(a_1^2 + b_1^2)$. With $A_1 = A_2 = 40$, it follows that $a_1 = 2$ and $a_2 = 5$. Thus $25 + b_2^2 = 8 + 2b_1^2$, so $2b_1^2 - b_2^2 = 17$ $(*)$. By inspection, $b_1 = 3, b_2 = 1$ is a solution, and it is easy to check that all ordered pairs of positive integers (b_1, b_2) that yield smaller values of $b_1 + b_2$ do not satisfy $(*)$. Thus $|z_2|^2 = 5^2 + 1^2 = \mathbf{26}$.

   Author's Note: The year 2015 marked ARML's fortieth anniversary, which we felt compelled to commemorate in a special Super Relay "XL". Unlike every other answer on this Super Relay, the final answer is 26, not 40. This problem is dedicated to ARML's then head author, Paul J. Karafiol, whose favorite number is 17—and sure enough, $26_{17} = 40_{10}$.

9. It follows that $N = \binom{T+1}{3} - \binom{T}{3} = \frac{(T+1)T(T-1)}{6} - \frac{T(T-1)(T-2)}{6} = \left(\frac{T(T-1)}{6}\right) \cdot (T+1-T+2) = \frac{T(T-1)}{2}$. Thus $2N = T(T-1)$, and $T - 1 < \sqrt{2N} < T$, so $\left\lceil \sqrt{2N} \right\rceil = T$. With $T = 40$, $\left\lceil \sqrt{2N} \right\rceil = \mathbf{40}$.

10. Note that $\sin(T^\circ) = \cos(90 - T^\circ)$. Assuming that $0 \leq T \leq 80$, the smallest value of K will occur when $90 - T = 10 + K$, in which case, $K = 80 - T$. With $T = 40$, $K = \mathbf{40}$.

11. The ingredients are in the ratio $1 : 2 : 3$. So, for example, if $40 \geq 2 \cdot \frac{T}{8}$ and $T - 17 \geq 3 \cdot \frac{T}{8}$, then $8 \cdot \frac{T}{8} = T$ candied apples can be made. Similarly, if $\frac{T}{8} \geq \frac{40}{2}$ and $T - 17 \geq \frac{3}{2} \cdot 40$, then $\frac{40}{2} \cdot 8 = 160$ candied apples can be made. Finally, if $\frac{T}{8} \geq \frac{1}{3}(T - 17)$ and $20 \geq \frac{2}{3}(T - 17)$, then $\left(\frac{T-17}{3}\right) \cdot 8$ candied apples can be made. With $T = 40$, only the first pair of inequalities is satisfied, resulting in **40** candied apples.

12. Because the circle is centered at (T, T), it is symmetric about the line $y = x$. Because the tangent lines have a slope of -1, the slopes of the segments joining the points of tangency to the center must equal 1. Thus $x_1 = y_1$ and $x_2 = y_2$. The solutions of the equations $(x - T)^2 + (y - T)^2 = T^2$ and $y = x$ are $(x, y) = \left(T \pm \frac{T}{\sqrt{2}}, T \pm \frac{T}{\sqrt{2}}\right)$. Thus $x_1x_2 = y_1y_2 = \dfrac{T^2}{2}$ and $\dfrac{x_1x_2 + y_1y_2}{T} = \dfrac{\frac{T^2}{2} + \frac{T^2}{2}}{T} = T$. With $T = 40$, the answer is **40**.

13. Raise each side to the fourth power to obtain $T^2 = \frac{T^4}{20^4}(K + 60)$. Thus $K = \frac{20^4}{T^2} - 60$. With $T = 40$, it follows that $K = \frac{160000}{1600} - 60 = 100 - 60 = \mathbf{40}$.

14. Because it is given that the union of the two intervals is itself an interval, it follows that $L = \max(T, 17) - \min(4, T - 40)$. With $T = 40$, conclude that $L = 40 - 0 = \mathbf{40}$.

15. Note that $A + L = (A + R) + (M + L) - (R + M) = 17 + 24 - 1 = \mathbf{40}$.

Puzzles for the 40th ARML Contest

In 2015, ARML celebrated its 40th contest with a 40-themed puzzle hunt. At 4:40 PM EST on June 1st, 2015, eleven encrypted PDF files were made available to our competitors on our website, along with the password to decrypt the first puzzle. Not entirely like an ARML relay, each puzzle solution provided the password to decrypt the next puzzle. Upon solving the 10th puzzle and opening the final file, students received a link that enabled them to submit all of their answers, and the first student from each ARML region that did so received a $100 Amazon gift card.

Initially, we had intended to give out 40 gift cards, in line with the theme. In the end, we received 225 correct submissions from 44 different ARML regions, so we decided to give out $25 gift cards to the first submission from the last four regions as well. The list of all correct submissions and prize winners, along with links to (unencrypted) PDF files of the puzzles and answers presented here, are at `https://www.arml.com/puzzlehunt`. The fastest responses came from two students from the SFBA/NorCal region, Victor Chen and Aaron Lin, who each completed all ten puzzles in under two hours.

The puzzles were developed by Paul Dreyer (Ways to 40, Weighs to 40, Ways to 40, 4T, 4, and T, Ways to 4 Seas – Sum Battleships, Ways to XL, Ways to 40 – Shikaku Edition, and Ways to 4 T's), Christian Steinruecken (Ways to 40 – Minesweeper Edition), and Kenny Young (Ways Around 40 and Ways to 4-D). The puzzles were playtested by Dana Young, Kenny Young, and Philip Loh.

The puzzles are presented first, followed by their answers. Note that puzzles 1, 2, 4, 8, and 10 have multiple pages. Also please note that some puzzles and answers are in color. To minimize printing costs, they are presented here in black-and-white but can be retrieved at the website mentioned above. Enjoy!

Puzzle 1: Ways to 40

	Third Op'n	Second Operation							
	+ - / x /	+		-		X		/	
First Operation +		R	S	T	L	T	N	B	A
		2	5	O	U	H	A	E	R
-		E	A	P	A	1	3	S	S
		0	4	U	L	E	A	T	E
X		W	T	0	7	I	S	T	N
		4	5	N	K	N	O	E	A
/		M	S	T	U	M	A	A	L
		I	5	R	E	T	H	M	1

There are many ways to get to 40. On the next page there are 12 rows of four numbers. Enter one of the four arithmetic operations (+,-,x,/) into each blank square such that the resulting expression, when calculated from left to right, equals 40.

Standard order of operations does *not* apply, so 3 + 2 x 9 - 5 = 40 (3 plus 2 is 5, 5 times 9 is 45, 45 minus 5 is 40).

Each triplet of operations in order corresponds to a code letter or number in the table to the left (e.g. (+,x,-) is N). The twelve letters and numbers in order form a code phrase that is the password to the next puzzle (no spaces, all upper case letters or numbers).

Puzzle 1: Ways to 40

											Code
1		3		5		5	=	4	0		
1		5		9		8	=	4	0		
2		2		7		5	=	4	0		
2		3		7		2	=	4	0		
3		9		3		4	=	4	0		
4		1		2		4	=	4	0		
4		5		2		4	=	4	0		
4		5		5		5	=	4	0		
5		3		4		5	=	4	0		
6		4		8		8	=	4	0		
7		1		7		2	=	4	0		
8		6		9		1	=	4	0		

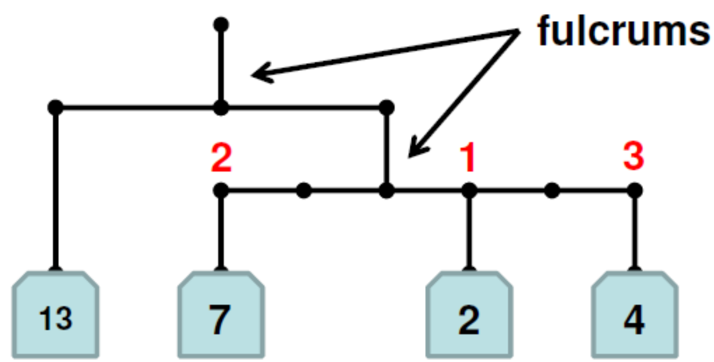
Puzzle 2: Weighs to 40

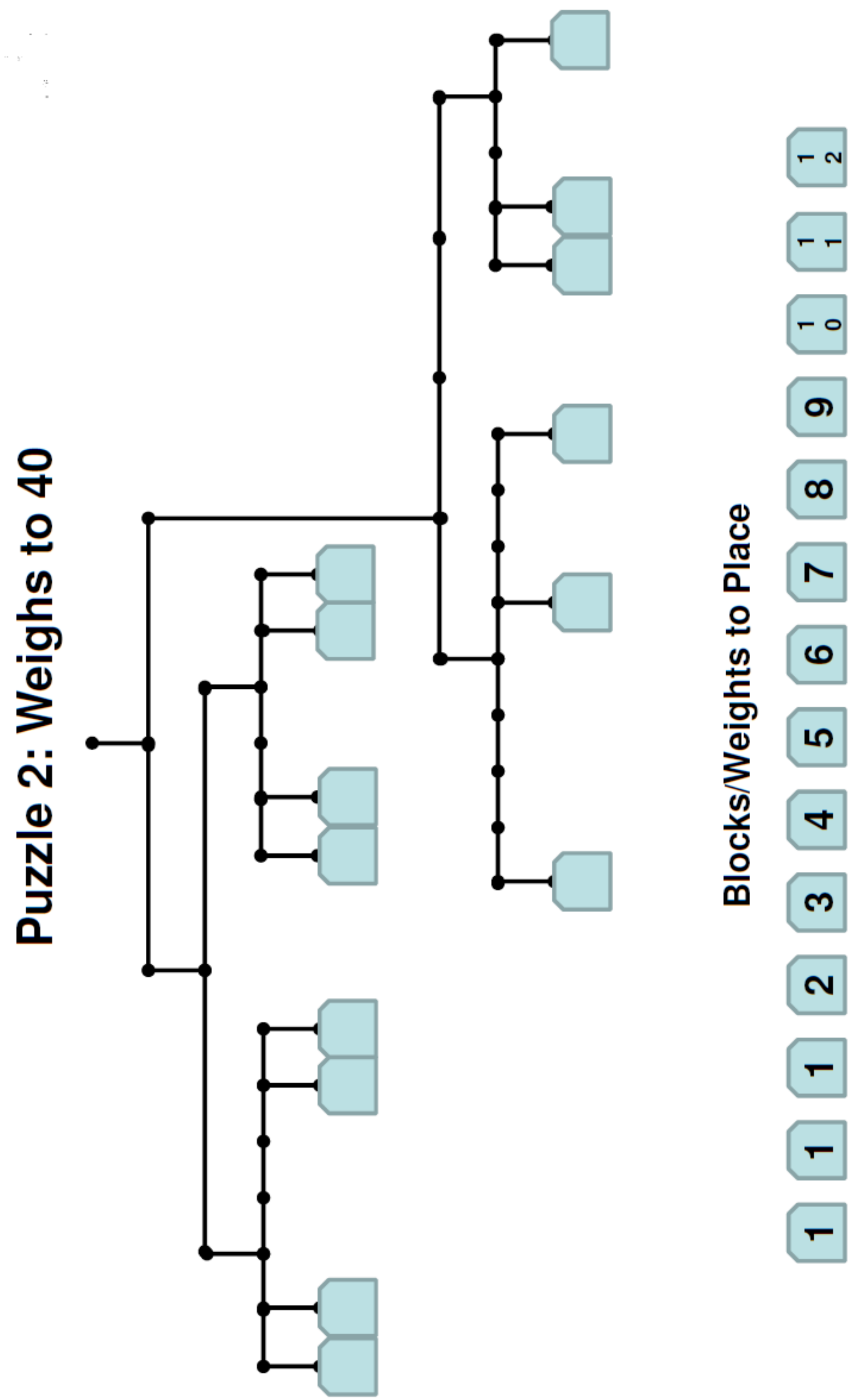
For the puzzle on the next page, assign weights to the hanging blocks to make all of the rods horizontal (horizontal rods indicate there is are equal amounts of torque on both sides of the fulcrum/pivot point).

The *torque* of each block on a fulcrum is its weight times its distance from the fulcrum. The lower rod in the example below balances as 2 x 7 = 1 x 2 + 3 x 4.

The total weight of any balanced rod equals the sum of all the hanging blocks distributed underneath it. In the example, the 3 weights total 13 and are equal in torque to the block of weight of 13 on the left side of the upper fulcrum.

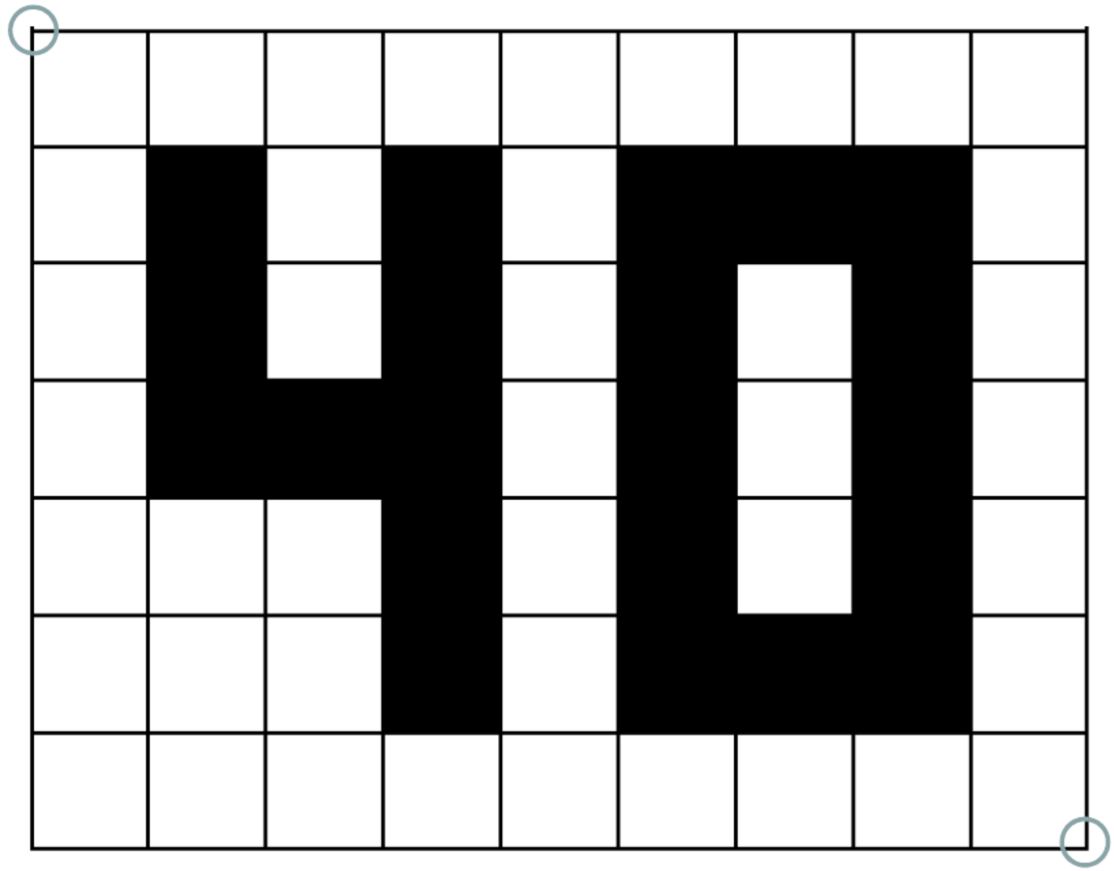
The password to the next puzzle is the weights of the blocks in the balance on the next page in the first row, left to right, then the second row, left to right (14 characters, no spaces). Use hexadecimal notation for weights 10 (A), 11 (B) and 12 (C). In the example below, the password would be D724.




Puzzle 2: Weighs to 40
Blocks/Weights to Place
1
1
1
2
3
4
5
6
7
8
9
10
11
12

Puzzle 3: Ways Around 40

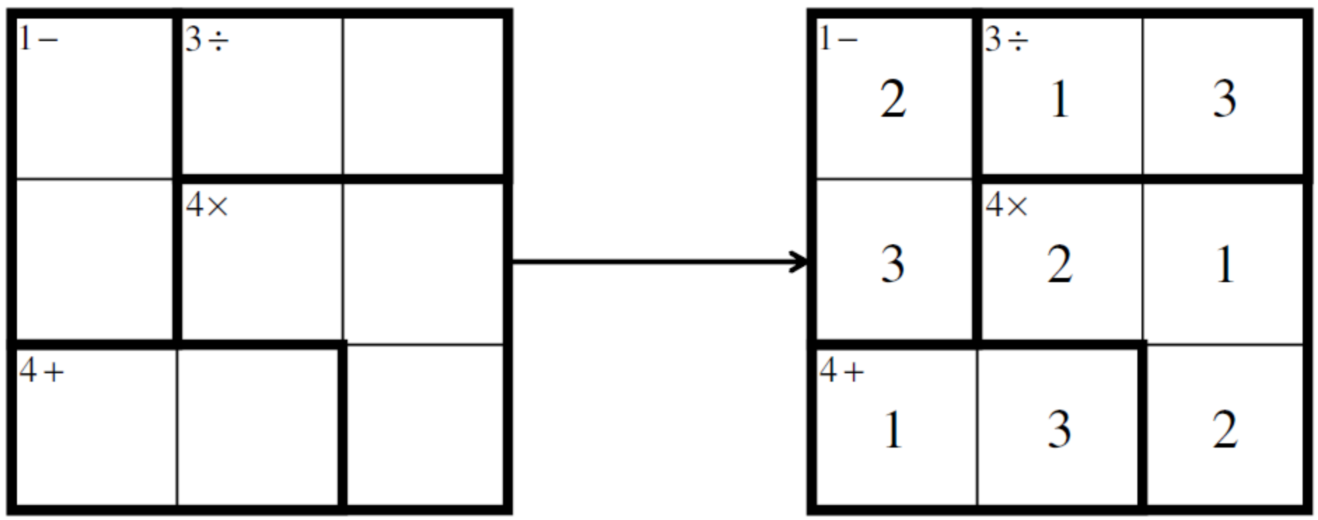
The password to the next puzzle is the number of routes exactly 16 segments long from the upper left corner of the grid to the lower right corner of the grid that do *not* intersect the interior of the black boxes (segments along the boundary of the "40" are allowed).



Puzzle 4: Ways to 40, 4T, 4, and T

In a KenKen puzzle, the numbers 1 to *n* are entered into an *n* x *n* square grid such that:

- Each number appears in every row and column exactly once.
- In each "cage", the regions outlined in heavier lines, there is a target number and an arithmetic operation. The numbers entered into the cage must combine in any order to produce the target number via the arithmetic operation.



See http://www.kenkenpuzzle.com/ for more information on these puzzles.

Puzzle 4: Ways to 40, 4T, 4, and T

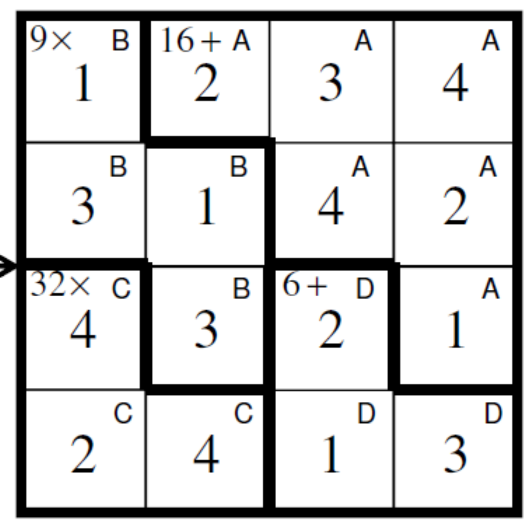
In a *reverse* KenKen puzzle, the numbers 1 to *n* are already placed in an *n* by *n* grid such that each number appears in every row and column exactly once. What is missing is the outlines of the cages. The puzzle is to determine the outlines of the cages given the target numbers and the operations. Each target number/operation pair is associated with a letter. The password to the next puzzle has ten characters, the letter associated to each cage (in alphabetical order) followed by the total number of cells in the cage(s) with that label. Use hexadecimal notation for numbers larger than 9. In the example, the password would be A6B4C3D3.

Note that on the puzzle on the next page, some of the target numbers contain a *T*. This number must also be determined in order to solve the puzzle. In addition, no cage may consist of a single cell, no two cages with the same arithmetic operation can contain the same number of cells, and no two cages with the same target number can share an edge.

A = 16 +
B = 9×
C = 32×
D = 6 +

1	2	3	4
3	1	4	2
4	3	2	1
2	4	1	3

Example



Puzzle 4: Ways to 40, 4T, 4, and T

Reminders:
No cage may consist of a single cell

No two cages with the same arithmetic operation can contain the same number of cells

No two cages with the same target number can share an edge

A = T+ (1 cage)

B = T× (1 cage)

C = 4+ (1 cage)

D = 40x (3 cages)

E = 4T+ (1 cage)

4	5	2	1	3
3	2	1	5	4
2	1	4	3	5
5	4	3	2	1
1	3	5	4	2

Note: There are two solutions to the puzzle, both of which result in the same password

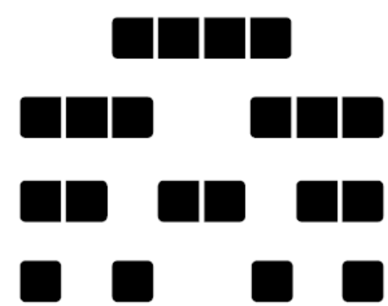
Puzzle 5: Ways to 4 Seas – Sum Battleships

Locate the fleet of ships in the grid. Each ship piece occupies a single cell. Ships can be rotated 90 degrees, but do not touch each other, not even diagonally. Each number to the right or below the grid is equal to the sum of the numbers in the cells containing ship pieces in that row or column. To get the password for the next puzzle, for each row from top to bottom, write the letter corresponding to the column in the cell where the left-most ship piece in that row appears. In the example, the password would be AACACAC.

A	B	C	D	E	F	G	
5	5	2	2	2	2	5	13
1	2	3	1	2	2	6	
2	3	4	3	2	3	4	14
3	4	5	2	3	2	3	3
4	5	6	2	1	2	2	
5	6	7	2	3	1	3	
6	7	8	1	2	1	1	
14		27	5	5	5		

R	O	T	D	F	S	E	I	M	
3	2	7	4	5	2	2	4	3	5
2	7	2	3	2	7	4	3	3	16
3	3	8	4	5	1	3	6	4	6
4	6	2	4	2	8	5	2	4	6
4	2	3	2	2	2	3	3	3	3
3	2	7	7	3	8	8	6	4	12
3	2	2	4	4	1	4	4	6	11
3	3	9	4	2	8	3	3	5	6
6	4	8	9	5	3	4	3	5	9
5	9	9	13	9	0	11	14	4	

Ships to Place



Puzzle 6: Ways to XL

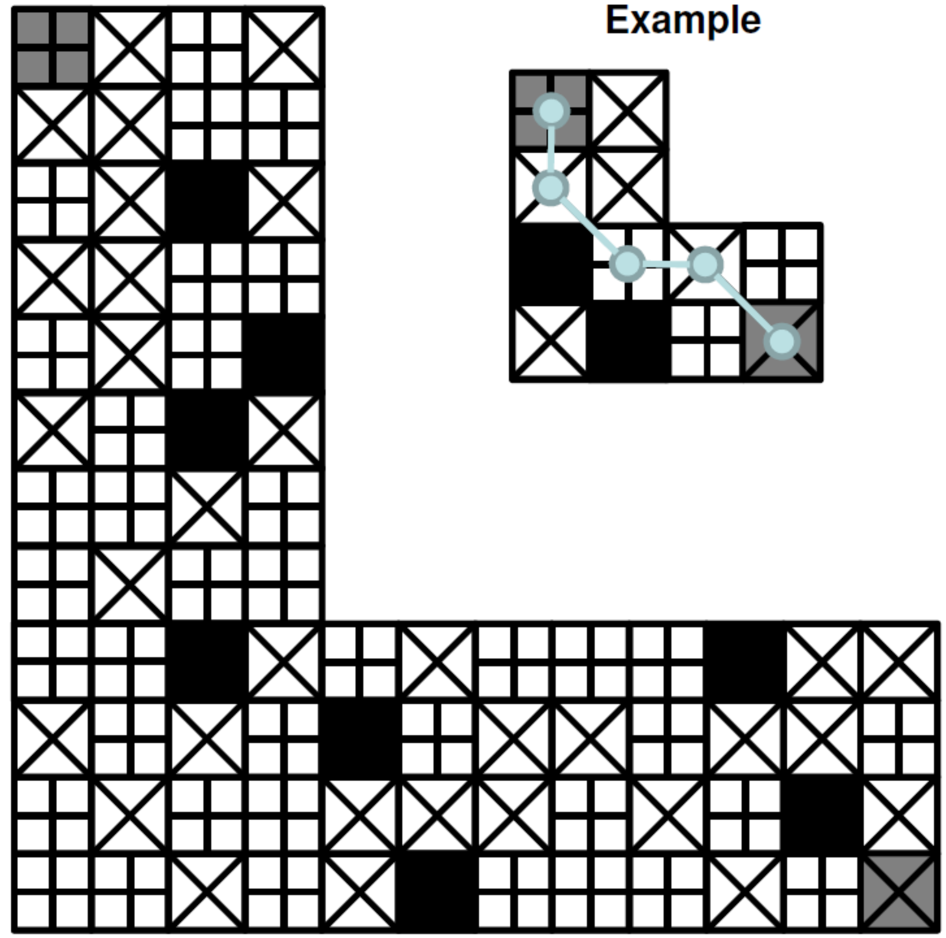
In the grid to the right, find the ***shortest*** path from the top left square to the bottom right square, subject to the following rules:

The path can only leave a square horizontally or vertically

The path can only leave a square diagonally

The path cannot enter a square at all.

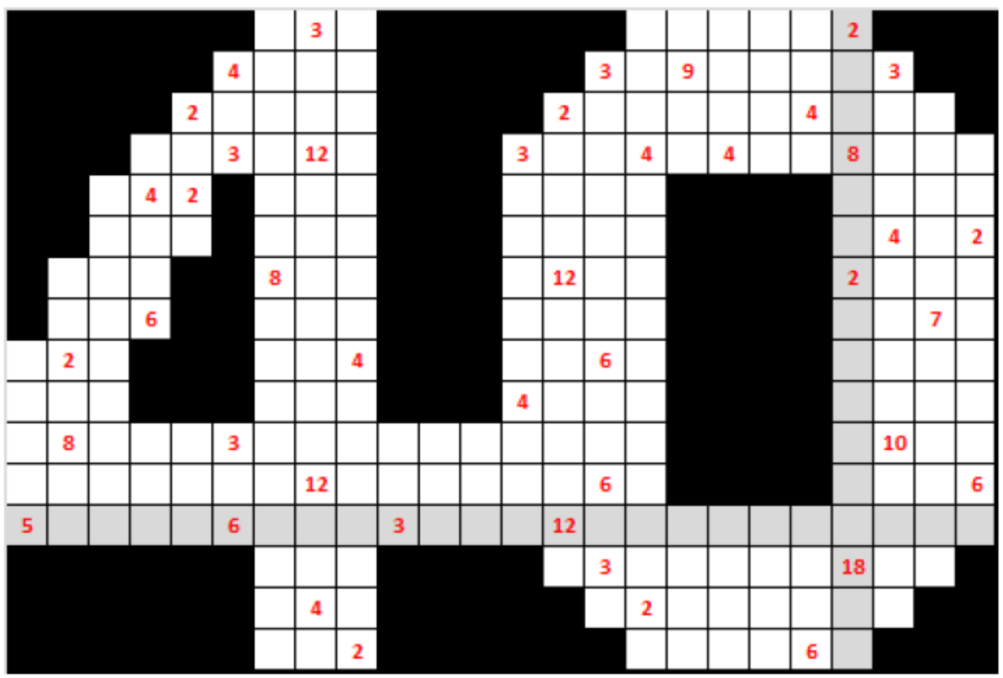
The password to the next puzzle is a 12 digit number with the k^{th} digit equal to the number of squares in the shortest path in row k, going from top to bottom. In the example the password would be 1121.



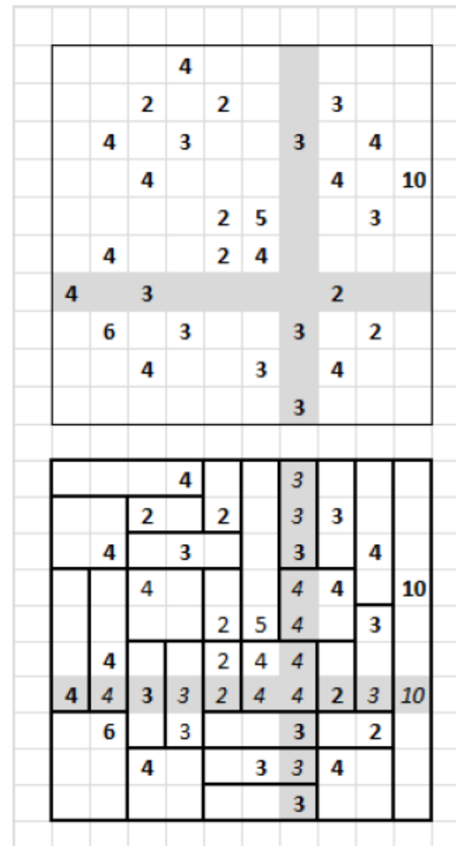
Example

Puzzle 7: Ways to 40 – Shikaku Edition

Partition the non-black cells in the grid into rectangles such that each rectangle contains exactly one cell containing a number and the number is equal to the number of cells in the rectangle. To get the password to the next puzzle, in each cell in the fourth row from the bottom and the fourth column from the right, write the number of the rectangle containing it. The password to the next puzzle is the sum of these numbers. In the example to the right, the password would be 69.



Example

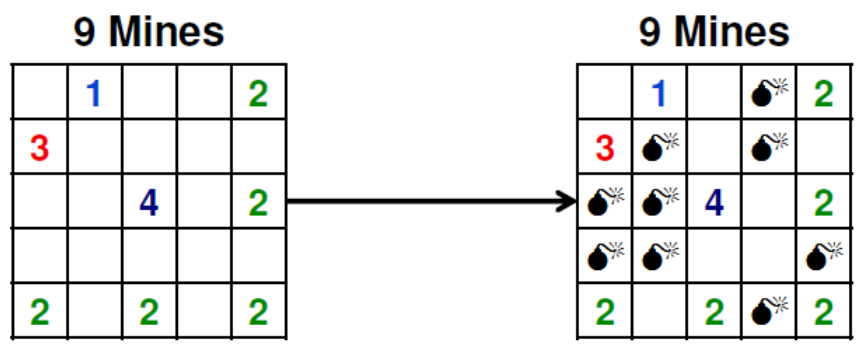


More information about Shikaku puzzles: http://www.nikoli.com/en/puzzles/shikaku/

Puzzle 8: Ways to 40 – Minesweeper Edition

In a Minesweeper puzzle, the goal is to locate all of the mines in a grid of squares. The total number of mines in the grid is given. Some of the squares in the grid contain a mine. Numbers within the grid give the number of mines in the squares surrounding the number (left, right, above, below, and diagonally). A mine will never be in a square containing a number, nor in a black grid square.

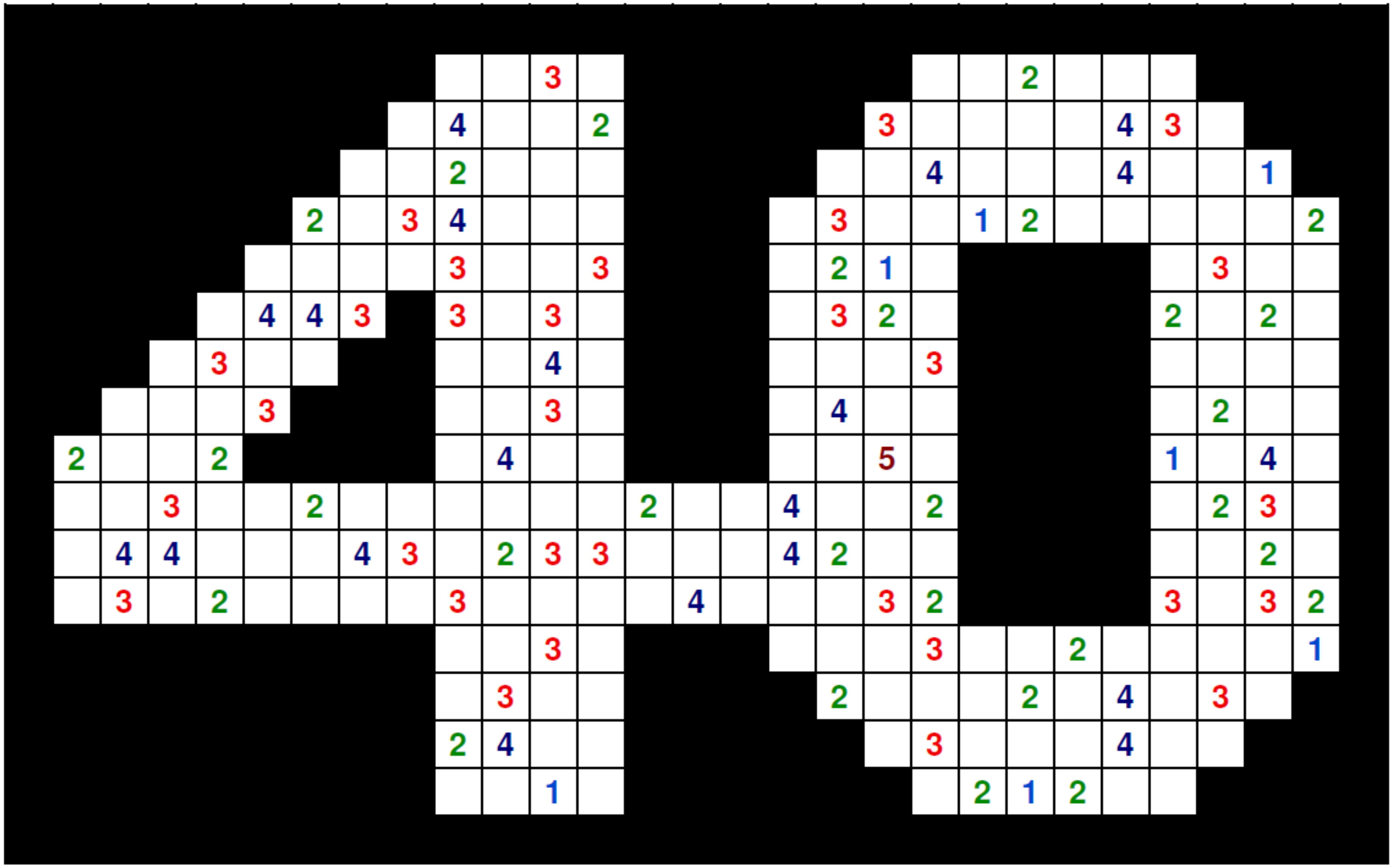
The password to the next puzzle is 16 characters long, with the k^{th} character being the number of mines in the k^{th} row of the puzzle from top to bottom. As in previous puzzles, use hexadecimal notation (A = 10, B = 11, etc.) if the number of mines in a row is greater than nine. In the example below, the password would be 12231.



For more Minesweeper puzzles, visit the webpage of the puzzle's author, Christian Steinruecken http://cstein.kings.cam.ac.uk/~chris/mines/index.html.

Puzzle 8: Ways to 40 – Minesweeper Edition

91 Mines



Puzzle 9: Ways to 4-D

Given five points in R^4:

$x_1 = (0,0,0,0)$
$x_2 = (1,10,7,3)$
$x_3 = (2,-6,2,10)$
$x_4 = (-3,9,9,5)$
$x_5 = (8,-1,6,8)$

From \ To	x_1	x_2	x_3	x_4	x_5
x_1		A	L	A	N
x_2	A		R	E	A
x_3	F	A		M	I
x_4	R	E	A		L
x_5	O	N	L	Y	

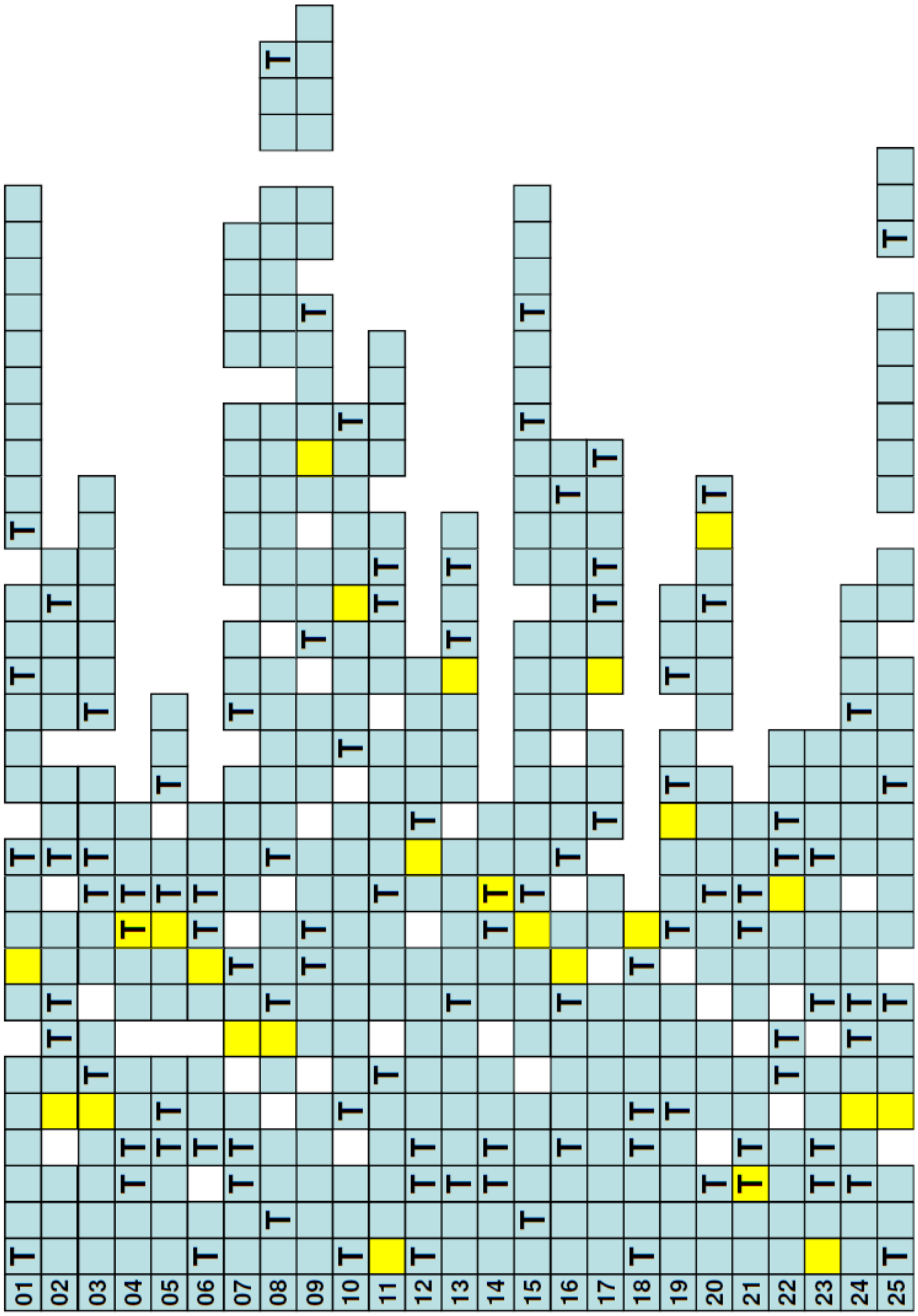
Find the shortest path starting at x_1 and visiting every other point exactly once. The path does *not* need to return to x_1. The table above associates a letter to each possible edge of the path. The password for the next puzzle is four characters, where each character is in the cell in the table associated to each edge of the shortest path. For example, if the shortest path was $x_1 \to x_2 \to x_3 \to x_4 \to x_5$, the password would be ARML.

Puzzle 10: Ways to 4 T's

The answers to the clues below contains exactly four T's, which are already placed in their correct positions in the grid on the next page. Fill in the remaining blank squares. The password to the final document is the answer to the question formed by the highlighted letter of each answer (the password has eleven capital letters, no spaces)

Q01: She covered his song "Cry Me A River" in concerts in 2011
Q02: The White Rabbit did not want to be late for this very important date
Q03: For a prime p, $a^p - a$ is a multiple of p
Q04: Delicious legume-flavored product from Nabisco
Q05: The abbreviated title of a Douglas Adams novel, and the answer to life, the universe, and everything
Q06: Exactly as written
Q07: In this 2014 song, they "went from San Berdoo to Kalamazoo"
Q08: It has been used to describe a yellow bikini in song
Q09: Spoiler: Cedric Diggory is killed by Wormtail in this book/movie
Q10: Follow-on to Q9: It happens at the end of this
Q11: Their accomplishment here is shown on the North Carolina state quarter
Q12: They play football at White Hart Lane
Q13: "e raised to the i times pi equals negative one", for one
Q14: The sound little feet make
Q15: His Hero Power in this game is to generate a 1/1 Silver Hand Recruit
Q16: Euclid used his first four of these to prove his first 28 propositions in this text
Q17: Natural selection
Q18: The 140-character elite
Q19: Andrew Jackson beat Henry Clay in this year
Q20: Joey Chestnut won one of these on July 4th, 2014
Q21: A grumpy wagerer
Q22: Top selling German manufactured all-wheel drive car
Q23: Kirk Cameron's 2014 movie *Saving Christmas* got a 0% rating on this website
Q24: Enterprise engines rely on this
Q25: This, literally?

Puzzle 10: Ways to 4 T's

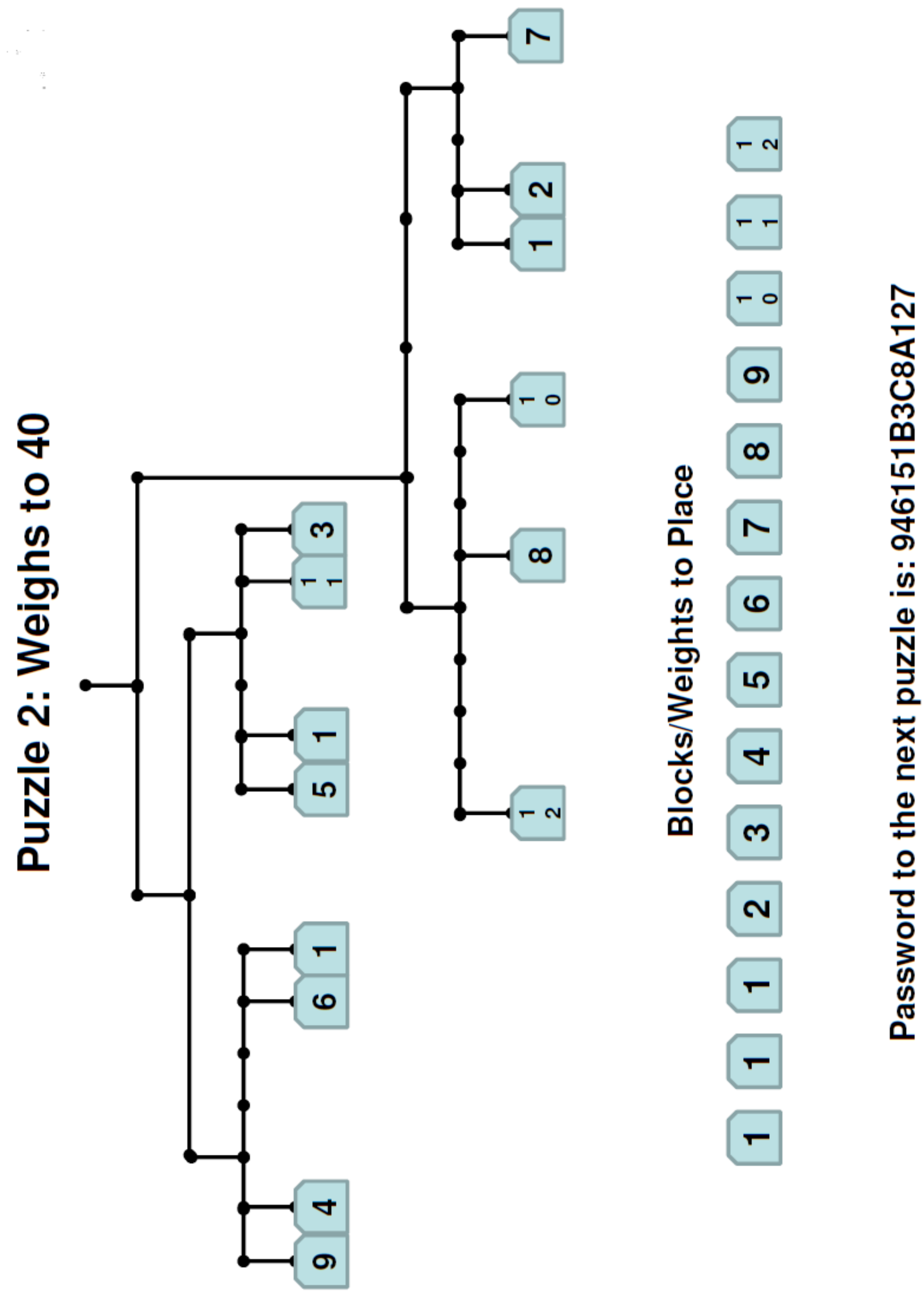


Answers to the Puzzles

Puzzle 1: Ways to 40

										Code
1	x	3	+	5	x	5	=	4	0	4
1	-	5	+	9	x	8	=	4	0	0
2	/	2	+	7	x	5	=	4	0	I
2	x	3	x	7	-	2	=	4	0	S
3	+	9	x	3	+	4	=	4	0	T
4	+	1	x	2	x	4	=	4	0	H
4	x	5	/	2	x	4	=	4	0	E
4	+	5	x	5	-	5	=	4	0	N
5	-	3	x	4	x	5	=	4	0	E
6	x	4	+	8	+	8	=	4	0	W
7	-	1	x	7	-	2	=	4	0	3
8	x	6	-	9	+	1	=	4	0	0

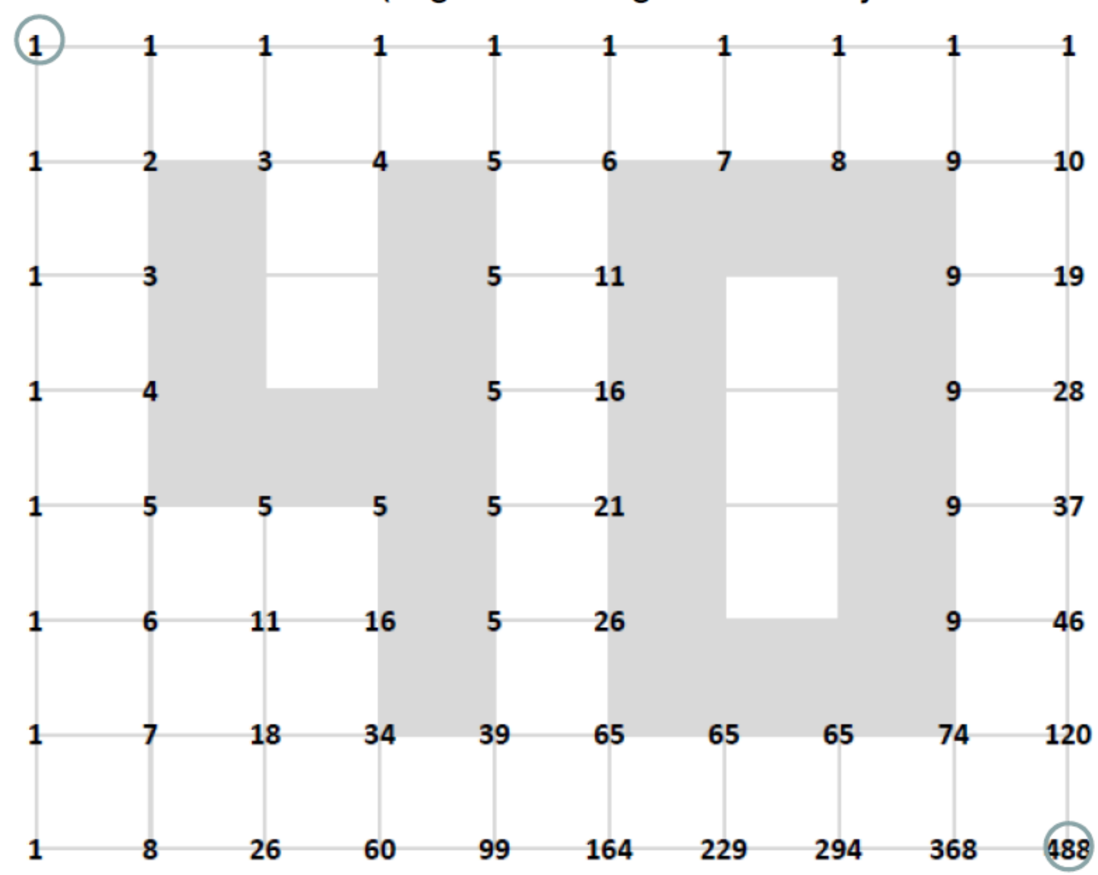
Password to the next puzzle is: 40ISTHENEW30


Puzzle 2: Weighs to 40
Blocks/Weights to Place
1 1 1 2 3 4 5 6 7 8 9 10 11 12
Password to the next puzzle is: 946151B3C8A127

Password to the next puzzle is: 488

Puzzle 3: Ways Around 40

The password to the next puzzle is the number of routes exactly 16 segments long from the upper left corner of the grid to the lower right corner of the grid that do *not* intersect the interior of the black boxes (segments along the boundary of the "40" are allowed).



Puzzle 4: Ways to 40, 4T, 4, and T

Reminders:
No cage may consist of a single cell

No two cages with the same arithmetic operation can contain the same number of cells

No two cages with the same target number can share an edge

A = T+ (1 cage)

B = T× (1 cage)

C = 4+ (1 cage)

D = 40x (3 cages)

E = 4T+ (1 cage)

4^D	5^D	2^D	1^D	3^E
3^B	2^B	1^D	5^E	4^E
2^D	1^D	4^E	3^E	5^E
5^D	4^D	3^A	2^A	1^A
1^C	3^C	5^D	4^D	2^D

4^D	5^D	2^D	1^D	3^E
3^A	2^A	1^D	5^E	4^E
2^D	1^A	4^E	3^E	5^E
5^D	4^D	3^B	2^B	1^D
1^C	3^C	5^D	4^D	2^D

T = 6, Password to the next puzzle is: A3B2C2DCE6

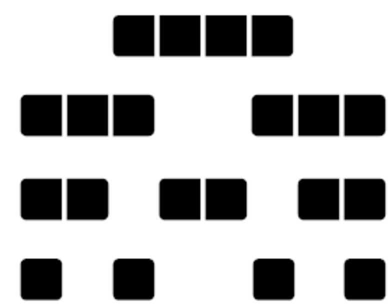
Note: There are two solutions to the puzzle, both of which result in the same password

Puzzle 5: Ways to 4 Seas – Sum Battleships

Locate the fleet of ships in the grid. Each ship piece occupies a single cell. Ships can be rotated 90 degrees, but do not touch each other, not even diagonally. Each number to the right or below the grid is equal to the sum of the numbers in the cells containing ship pieces in that row or column. To get the password for the next puzzle, for each row from top to bottom, write the letter corresponding to the column in the cell where the left-most ship piece in that row appears. In the example, the password would be AACACAC.

A	B	C	D	E	F	G	
5	5	2	2	2	2	5	13
1	2	3	1	2	2	6	
2	3	4	3	2	3	4	14
3	4	5	2	3	2	3	3
4	5	6	2	1	2	2	
5	6	7	2	3	1	3	
6	7	8	1	2	1	1	
14		27	5	5	5		

Ships to Place



R	O	T	D	F	S	E	I	M	
3	2	7	4	5	2	2	4	3	5
2	7	2	3	2	7	4	3	3	16
3	3	8	4	5	1	3	6	4	6
4	6	2	4	2	8	5	2	4	6
4	2	3	2	2	2	3	3	3	3
3	2	7	7	3	8	8	6	4	12
3	2	2	4	4	1	4	4	6	11
3	3	9	4	2	8	3	3	5	6
6	4	8	9	5	3	4	3	5	9
5	9	9	13	9	0	11	14	4	

Password to the next puzzle is: FRITTERED

Puzzle 6: Ways to XL

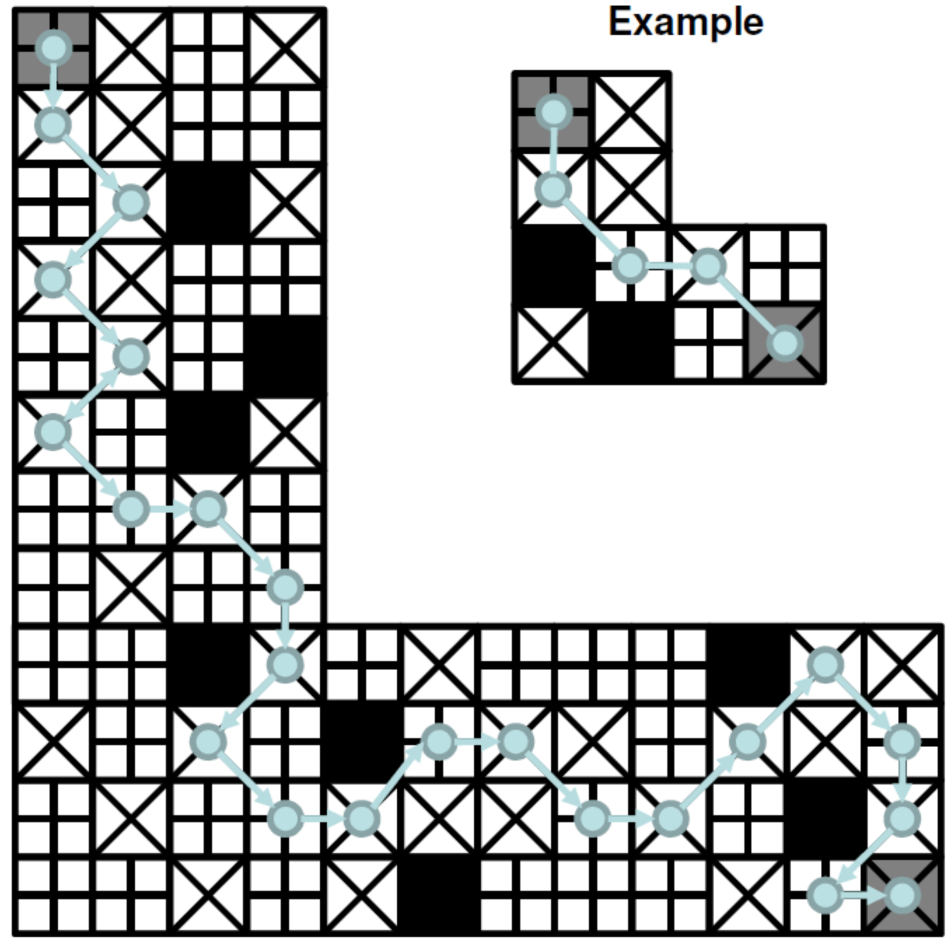
In the grid to the right, find the ***shortest*** path from the top left square to the bottom right square, subject to the following rules:

The path can only leave a square horizontally or vertically

The path can only leave a square diagonally

The path cannot enter a square at all.

The password to the next puzzle is a 12 digit number with the k^{th} digit equal to the number of squares in the shortest path in row k, going from top to bottom. In the example the password would be 1121.

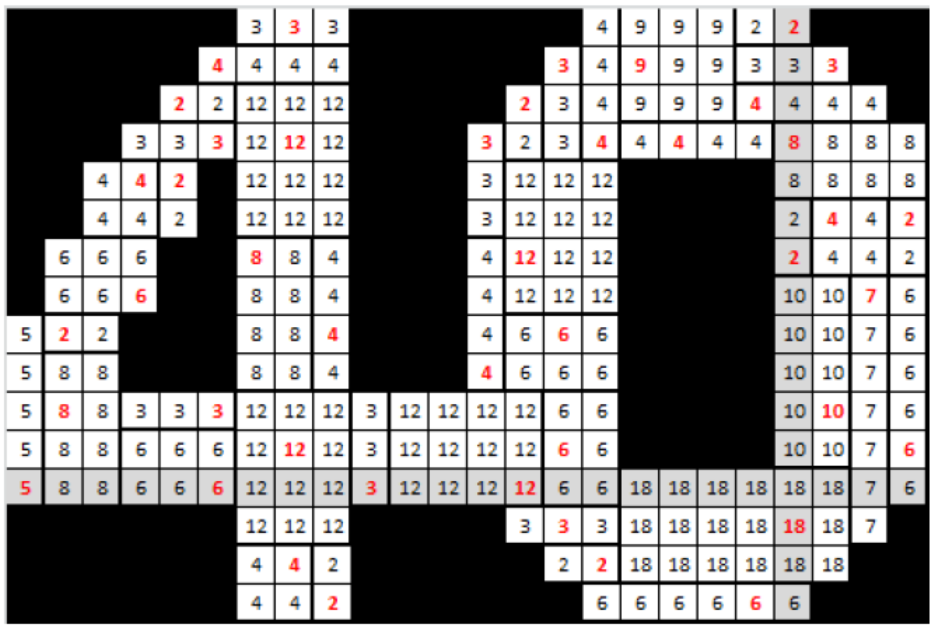


Example

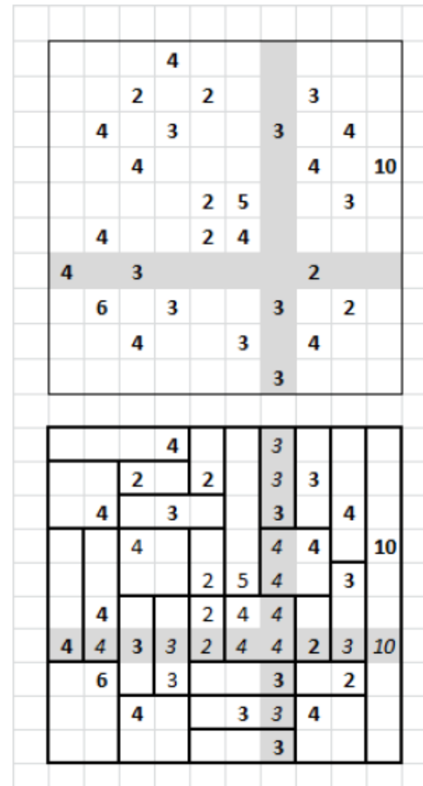
Password to the next puzzle is: 111111212552

Puzzle 7: Ways to 40 – Shikaku Edition

Partition the non-black cells in the grid into rectangles such that each rectangle contains exactly one cell containing a number and the number is equal to the number of cells in the rectangle. To get the password to the next puzzle, in each cell in the fourth row from the bottom and the fourth column from the right, write the number of the rectangle containing it. The password to the next puzzle is the sum of these numbers. In the example to the right, the password would be 69.



Example



Password to the next puzzle is: 380

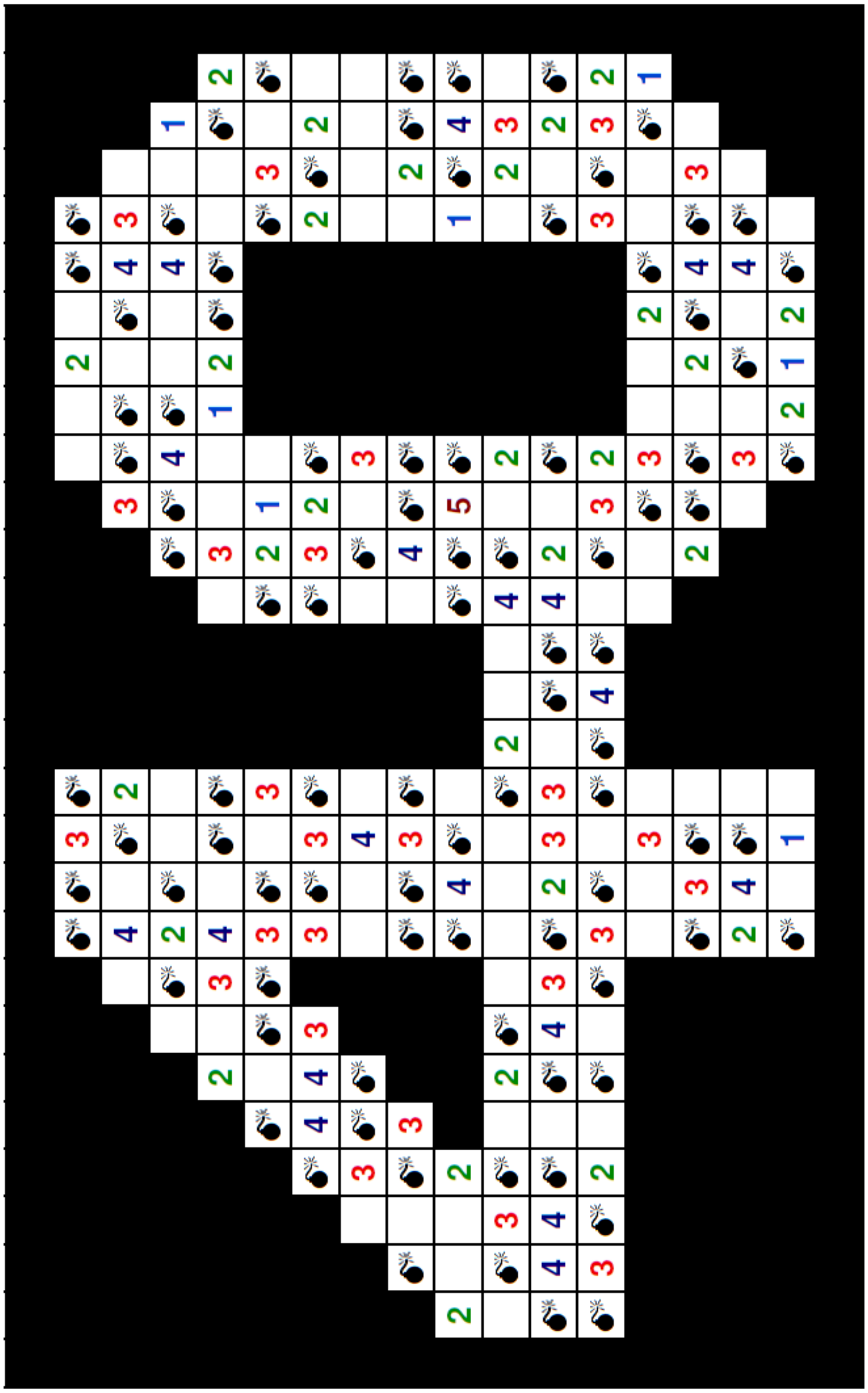
More information about Shikaku puzzles: http://www.nikoli.com/en/puzzles/shikaku/

Puzzle 8: Ways to 40 – Minesweeper Edition

Password to the next puzzle is:

54657639759A3633

91 Mines



Puzzle 9: Ways to 4-D

Given five points in R^4:

$x_1 = (0,0,0,0)$
$x_2 = (1,10,7,3)$
$x_3 = (2,-6,2,10)$
$x_4 = (-3,9,9,5)$
$x_5 = (8,-1,6,8)$

From \ To	x_1	x_2	x_3	x_4	x_5
x_1		A	L	A	N
x_2	A		R	E	A
x_3	F	A		M	I
x_4	R	E	A		L
x_5	O	N	L	Y	

Find the shortest path starting at x_1 and visiting every other point exactly once. The path does *not* need to return to x_1. The table above associates a letter to each possible edge of the path. The password for the next puzzle is four characters, where each character is in the cell in the table associated to each edge of the shortest path. For example, if the shortest path was $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5$, the password would be ARML.

Password to the next puzzle is: LINE ($x_1 \rightarrow x_3 \rightarrow x_5 \rightarrow x_2 \rightarrow x_4$) has length $12 + 9 + 14 + 5 = 40$. Also, if A = 1, B = 2, …, Z = 26 , then 12-9-14-5 is LINE.

Puzzle 10: Ways to 4 T's

Password to the last document is: NEWYORKCITY (WHATREGIONWONTHEFIRSTARML)

2016 Contest

Contents

2016 Team Problems

T-1. Compute all values of a for which the intersection of the graphs of $y \geq \left|\frac{x}{2}\right|$ and $y \leq a|x| + 17$ is a region having area 51.

T-2. Let $f(x) = \log_b x$ and let $g(x) = x^2 - 4x + 4$. Given that $f(g(x)) = g(f(x)) = 0$ has exactly one solution and that $b > 1$, compute b.

T-3. Compute all solutions to the following system of equations:

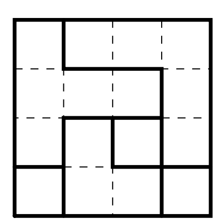
$$|y| - \frac{2x}{|x|} = -1$$
$$x|x| + y|y| = 24.$$

T-4. Two players are playing a game as follows. Integers N and M are each selected independently at random from the set $\{1, 2, \ldots, 50\}$. The players begin with an $N \times M$ grid of empty squares. A turn consists of filling in a single empty square, then filling in all the squares in the same row, and then filling in all the squares in the same column. The players alternate taking turns until all squares are filled in. Compute the probability that the player who goes first fills in the last square.

T-5. A sequence $T_1, T_2, \ldots$ is called (a, b)-*nacci* if $T_1 = a$, $T_2 = b$, and for $n \geq 3$, $T_n = T_{n-1} + T_{n-2}$. Compute the number of (a, b)-nacci sequences of positive integers such that $b > a$ and there exists an integer $k \geq 3$ with $T_k = 50$.

T-6. Complete the number puzzle below. Clues are given for the four rows. (Answers may not begin with a zero.) Cells inside a region must all contain the same digit, and each region contains a different digit.

1. Product of two primes
2. Multiple of 19
3. A perfect square
4. Multiple of 1-Across



T-7. Real numbers x, y, and z are chosen at random from the unit interval $[0, 1]$. Compute the probability that $\max\{x, y, z\} - \min\{x, y, z\} \leq \frac{2}{3}$.

T-8. Square $ARML$ has sides of length 11. Collinear points X, Y, and Z lie in the interior of $ARML$ so that $\triangle LXY$ and $\triangle RXZ$ are equilateral. Given that $YZ = 2$, compute $[RLX]$.

T-9. Compute the number of permutations $x_1, x_2, \ldots, x_{10}$ of the integers $-3, -2, -1, \ldots, 6$ that satisfy the chain of inequalities

$$x_1x_2 \leq x_2x_3 \leq \cdots \leq x_9x_{10}.$$

T-10. Compute the greatest integer n for which the decimal representation of $\frac{1000}{n}$ is of the form $\underline{1}\,.\,\underline{T}\,\underline{A}\,\underline{S}\,\underline{T}\,\underline{Y}\,\underline{7}\ldots$, where the digits T, A, S, and Y are not necessarily distinct.

2016 Team Answers

T-1. $-\dfrac{31}{6}$

T-2. $\sqrt{3}$

T-3. $(5, -1)$ and $(\sqrt{23}, 1)$

T-4. $\dfrac{51}{100}$

T-5. 37

T-6.

4	6	6	6
4	4	4	6
4	3	5	6
9	3	3	2

T-7. $\dfrac{20}{27}$

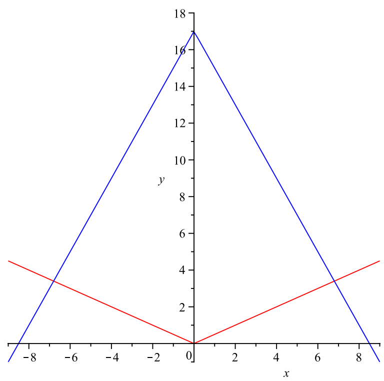
T-8. $\dfrac{119\sqrt{3}}{6}$

T-9. 240

T-10. 968

2016 Team Solutions

T-1. For $a < 0$, the graphs of the two functions are shown in the figure below.



In the first quadrant, the two bounding functions are $y = \frac{x}{2}$ and $y = ax + 17$, so they intersect where $\frac{x}{2} = ax + 17$. Moreover, because the triangle in the first quadrant is half of the total area it has area $\frac{51}{2}$. Taking the side along the y-axis as the base, the height is the x-value where the two lines intersect, hence $\frac{1}{2} \cdot 17 \cdot x = \frac{51}{2}$, thus $x = 3$. Now $\frac{3}{2} = 3a + 17$, so $a = -\mathbf{\frac{31}{6}}$.

T-2. Consider the equation $f(g(x)) = 0$. Because $f(x) = \log_b(x)$, this equation holds precisely when $g(x) = 1$, no matter what b is. Setting $g(x) = x^2 - 4x + 4$ equal to 1 yields the quadratic equation $x^2 - 4x + 3 = 0$, which has two solutions, $x = 3$ and $x = 1$. Now consider the equation $g(f(x)) = 0$: $g(x)$ has one root at 2, so $f(x) = 2$, which implies that $x = b^2$. Therefore in order for the equation $f(g(x)) = g(f(x)) = 0$ to have exactly one solution, b must equal $\boldsymbol{\sqrt{3}}$.

T-3. Note that x must be real because otherwise, the first equation would not be satisfied.

Case 1: x is positive. Then $\frac{2x}{|x|} = 2$, and the first equation yields $y = \pm 1$. The second equation then becomes $x^2 \pm 1 = 24$, so $x = 5$ (when $y = -1$) or $x = \sqrt{23}$ (when $y = 1$).

Case 2: x is negative. Then $\frac{2x}{|x|} = -2$, but in this case, the first equation becomes $|y| = -3$, which is impossible.

Thus the two solutions are $\mathbf{(5, -1)}$ and $\boldsymbol{(\sqrt{23}, 1)}$.

T-4. The outcome of the game is determined by the parity of $\min(N, M)$. This is because each time a player takes a turn, a row and a column that are previously not completely filled in (i.e., the row and column that contain the square selected on that turn) are now both completely filled in. Therefore, as soon as $\min(N, M)$ turns are taken, either all the rows will be completely filled, or all the columns, which means that the entire grid is filled in. Therefore the first player will fill in the last square if and only if $\min(N, M)$ is odd.

To compute the probability than $\min(N, M)$ is odd, count the number of pairs (N, M) satisfying this condition, and then divide by $50 \times 50 = 2500$, the total number of possible choices of N and M. First consider the case when $N < M$. There are 25 choices for N, from 1 to 49, and each results in $50 - N$ choices for M (i.e., the number of numbers larger than N). So there are $49 + 47 + \cdots + 3 + 1 = 25^2 = 625$ such pairs (N, M). The case when $M < N$ is symmetric, so there are another 625 cases there. Then there are another 25 cases where $N = M$, because N and M can equal any odd number between 1 and 50. This is a total of $625 + 625 + 25 = 1275$ pairs, so the desired probability is $\frac{1275}{2500} = \mathbf{\frac{51}{100}}$.

T-5. The sequence T_k must grow at least as fast as the $(1, 2)$-nacci sequence $1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$, thus for $k \geq 9$, $T_k > 50$.

If $k = 3$, then $a + b = 50$, which has 24 solutions with a being any integer from 1 to 24 inclusive.
If $k = 4$, then $a+2b = 50$. Now a must be even and $a \leq 16$, so there are 8 solutions for $k = 4$.
If $k = 5$, then $2a+3b = 50$. Now $a \equiv 1 \pmod 3$ and $a < 10$, so there are 3 solutions for $k = 5$.
If $k = 6$, then $3a + 5b = 50$. Now a must be a multiple of 5 and the only solution is $(5, 7)$.
If $k = 7$, then $5a + 8b = 50$. Now b must be a multiple of 5 and the only solution is $(2, 5)$.
If $k = 8$, then $8a + 13b = 50$, which has no solutions with $a < b$.

Thus the desired number of (a, b)-nacci sequences is $24 + 8 + 3 + 1 + 1 = \mathbf{37}$.

Alternate Solution: Note that $T_3 = a + b, T_4 = a + 2b, T_5 = 2a + 3b, T_6 = 3a + 5b, T_7 = 5a + 8b, T_8 = 8a + 13b$, and $T_9 = 13a + 21b$. Because $b \geq 2$ and $a \geq 1$, it follows that $T_9 > 50$. Suppose that $T_8 = 50 = 8a + 13b$. By parity, b must be even. The only possibility is $b = 2$, but then $a = 3 > b$, so this solution fails. Similar reasoning leads to the following:

$$\begin{aligned}
T_7 = 50 &\quad \text{when} \quad (a, b) = (2, 5);\\
T_6 = 50 &\quad \text{when} \quad (a, b) = (5, 7);\\
T_5 = 50 &\quad \text{when} \quad (a, b) = (1, 16), (4, 14), \text{ or } (7, 12);\\
T_4 = 50 &\quad \text{when} \quad (a, b) = (2, 24), (4, 23), \ldots, \text{ or } (16, 17);\\
T_3 = 50 &\quad \text{when} \quad (a, b) = (1, 49), (2, 48), \ldots, \text{ or } (24, 26).
\end{aligned}$$

Thus the desired number of ordered pairs is $1 + 1 + 3 + 8 + 24 = \mathbf{37}$.

T-6. Consider first 1-Across, which is of the form $\underline{A}\,\underline{B}\,\underline{B}\,\underline{B}$. 4-Across is a multiple of 1-Across, and of a different form, so they are not equal. Because both are four-digit numbers, A must be less than 5.

Next look at 2-Across. It is a four-digit multiple of 19 of the form $\underline{A}\,\underline{A}\,\underline{A}\,\underline{B}$, where A and B

are the same digits as above. Therefore 2-Across equals $1110A + B$, which is equivalent to $8A + B \bmod 19$. Because 2-Across is a multiple of 19 and $A < 5$, the possibilities for (A, B) are $(2, 3)$ and $(4, 6)$.

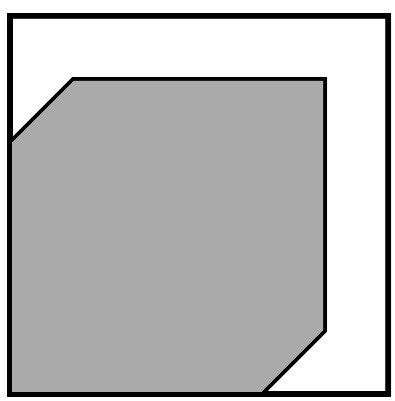
Therefore 1-Across is either 2333 or 4666. The four-digit multiples of 2333 are 4666, 6999, and 9332. Only 9332 fits the pattern of 4-Across, so 4-Across is 9332. Because the digits in each region are distinct, 1-Across must be 4666.

Then 2-Across is 4446. Filling in the digits that are the same in each region, 3-Across must be $43\underline{X}6$, where X is 0, 1, 5, 7, or 8, because these are the only unused digits. Noting that 3-Across is a perfect square and that $4356 = 66^2$, 3-Across is 4356.

4	6	6	6
4	4	4	6
4	3	5	6
9	3	3	2

T-7. Consider points lying in a unit cube in the first octant with one corner at the origin. To find the volume of the region of points where the inequality holds, take cross-sections for each value of z from 0 to 1. For any particular value of z, the values of x and y must satisfy $0 \le x \le 1$, $0 \le y \le 1$, $-\frac{2}{3} \le y - x \le \frac{2}{3}$, $z - \frac{2}{3} \le x \le z + \frac{2}{3}$, and $z - \frac{2}{3} \le y \le z + \frac{2}{3}$.

Depending on the value of z, the resulting region is a square of area $\frac{4}{9}$ (when $z = 0$ or 1), a hexagon of area $\frac{4}{9} + \frac{4z}{3}$ (when $0 < z < \frac{1}{3}$), a hexagon of area $\frac{8}{9}$ (when $\frac{1}{3} \le z \le \frac{2}{3}$), or a hexagon of area $\frac{4}{9} + \frac{4(1-z)}{3}$ (when $\frac{2}{3} < z < 1$). The region where $z = \frac{1}{6}$ is shown.



In the cases where z is not between $\frac{1}{3}$ and $\frac{2}{3}$, the area changes linearly with z and hence can be averaged to $\frac{1}{2}\left(\frac{4}{9} + \frac{8}{9}\right) = \frac{2}{3}$. This area corresponds to $\frac{2}{3}$ of the z-values, with the other $\frac{1}{3}$ having area $\frac{8}{9}$, for a total probability of $\frac{2}{3}(\frac{2}{3}) + \frac{1}{3}(\frac{8}{9}) = \mathbf{\frac{20}{27}}$.

T-8. First note that X cannot be between Y and Z, because then the side lengths of triangles LXY and RXZ would each be less than $YZ = 2$, which would mean that XL and XR would each be less than 2, which is impossible because $XL + XR \geq LR = 11\sqrt{2}$. Without loss of generality, let Y lie between X and Z. Let $XY = YL = LX = x$, so that $XZ = ZR = RX = x + 2$. In triangle RLX, angle LXR has measure $120°$ because angles LXY and ZXR are each $60°$. Applying the Law of Cosines then gives

$$\frac{x^2 + (x+2)^2 - (11\sqrt{2})^2}{2x(x+2)} = \cos 120° = -\frac{1}{2}.$$

Solving this quadratic for x gives $x = -1 \pm \sqrt{\frac{241}{3}}$, and the positive solution for x is used because it is a side length. So $x = -1 + \sqrt{\frac{241}{3}}$ and $x + 2 = 1 + \sqrt{\frac{241}{3}}$, and the area $[RLX]$ is

$$\frac{1}{2}x(x+2)\sin 120° = \frac{1}{2} \cdot \left(\frac{241}{3} - 1\right) \cdot \frac{\sqrt{3}}{2} = \mathbf{\frac{119\sqrt{3}}{6}}.$$

T-9. First note that no two negative numbers can occur consecutively because otherwise, either the resulting positive product would have to be followed by a non-positive product once the negatives are exhausted, or it would have to occur at the end but there are too many positive terms for that to be possible. Thus the sequence must start with alternating positive and negative values. Each inequality $x_i x_{i+1} \leq x_{i+1} x_{i+2}$ implies that $x_i < x_{i+2}$ if $x_{i+1} > 0$ and $x_i > x_{i+2}$ if $x_{i+1} < 0$ (the inequalities are strict because the x_is are distinct). The two main cases to consider are $x_1 = -3$ and $x_2 = -3$.

If $x_1 = -3$, then $x_3 = -2$, $x_5 = -1$, and either $x_6 = 0$ or $x_7 = 0$. If $x_6 = 0$, then the positive entries (x_2, x_4, x_7, x_8, x_9, and x_{10}) must satisfy $x_2 > x_4$, $x_7 < x_9$, and $x_8 < x_{10}$. There are $\frac{6!}{2!\cdot 2!\cdot 2!} = 90$ such sequences. If $x_7 = 0$, then the positive entries (x_2, x_4, x_6, x_7, x_8, and x_{10}) must satisfy $x_2 > x_4 > x_6$, and $x_8 < x_{10}$ for $\frac{6!}{3!\cdot 2!\cdot 1!} = 60$ more.

If $x_2 = -3$, then $x_4 = -2$, $x_6 = -1$, and either $x_7 = 0$ or $x_8 = 0$. If $x_7 = 0$, then the positive entries (x_1, x_3, x_4, x_8, x_9, and x_{10}) must satisfy $x_1 > x_3 > x_5$, and $x_8 < x_{10}$ for another 60. If $x_8 = 0$, then the positive entries (x_1, x_3, x_5, x_7, x_9, and x_{10}) must satisfy $x_1 > x_3 > x_5 > x_7$, for $\frac{6!}{4!\cdot 1!\cdot 1!} = 30$ more.

The desired number of permutations is therefore $90 + 60 + 60 + 30 = \mathbf{240}$.

T-10. In order for the integral part of $\frac{1000}{n}$ to be 1, n must be between 501 and 1000. Let $m = 1000 - n$. Then

$$\frac{1000}{n} = \frac{1000}{1000 - m} = \frac{1}{1 - \frac{m}{1000}} = 1 + \frac{m}{1000} + \frac{m^2}{1000^2} + \frac{m^3}{1000^3} + \cdots.$$

For example, if $n = 998$, then $\frac{1000}{n}$ begins $1.002004008016032064\ldots$ and each block of 3 digits following the decimal point is a power of 2 until they begin to overlap.

To maximize n, first consider attempting to find a solution with $T = 0$, which implies $m < 100$, by the previous analysis, because the first block of 3 digits equals either m or m plus the thousands digit of m^2. Because the ones digit of the second block of 3 digits is 7, and m^2 cannot end in 7 (as no squares end in 7), the m^3 block must overlap into the m^2 block. Therefore $m^3 > 1000$, so $m > 10$.

Because $T = 0$ by (optimistic) assumption, and therefore the part of m^2 in the second block must have a zero in the hundreds place, the goal is to find four-digit squares with a zero in the hundreds place. (Note that m cannot have a two-digit square because $m > 10$, and the m^3 block will not overlap onto the 0 from the hundreds place of the second block as long as $m^3 < 100{,}000$.)

The first square with a zero in the hundreds place is $32^2 = 1024$, which corresponds to $m = 32$, $m^2 = 1024$, $m^3 = 32768$. Then the expansion for $\frac{1000}{1000-m}$ begins

$$\begin{aligned} & 1 + \frac{32}{1000} + \frac{1024}{1000^2} + \frac{32768}{1000^3} + \cdots \\ = \; & 1 + 0.032 + 0.001024 + 0.000032768 + \cdots \\ = \; & 1.033056768 + \cdots . \end{aligned}$$

This fits the pattern, except the sixth decimal digit is a 6 instead of a 7. However, $m^4 = 1024^2$ is just over 1 million, so its first digit overlaps onto the 6, making it a 7. Therefore $m = 32$, which corresponds to $n = \mathbf{968}$ giving the maximal n satisfying the given criteria. This can be confirmed by computing $\frac{1000}{968} = 1.033057\ldots$.

Note: The same method as in the solution can be used for negative integers m that are close to 0; the geometric series would have alternating signs. Also, replacing the 1000s with 10^ps yields a similar result, but instead of blocks of 3 digits apiece, each of the blocks will have p digits.

2016 Individual Problems

I-1. Given that a, b, and c are positive integers such that $a^b \cdot b^c$ is a multiple of 2016, compute the least possible value of $a + b + c$.

I-2. Triangle ABC is isosceles. An ant begins at A, walks exactly halfway along the perimeter of $\triangle ABC$, and then returns directly to A, cutting through the interior of the triangle. The ant's path surrounds exactly 90% of the area of $\triangle ABC$. Compute the maximum possible value of $\tan A$.

I-3. Compute $\dfrac{\lfloor \sqrt[4]{1} \rfloor \cdot \lfloor \sqrt[4]{3} \rfloor \cdot \lfloor \sqrt[4]{5} \rfloor \cdots \lfloor \sqrt[4]{2015} \rfloor}{\lfloor \sqrt[4]{2} \rfloor \cdot \lfloor \sqrt[4]{4} \rfloor \cdot \lfloor \sqrt[4]{6} \rfloor \cdots \lfloor \sqrt[4]{2016} \rfloor}$.

I-4. Compute the number of permutations $x_1, \ldots, x_6$ of the integers $1, \ldots, 6$ such that $x_{i+1} \leq 2x_i$ for all i, $1 \leq i < 6$.

I-5. Compute the least possible non-zero value of $A^2 + B^2 + C^2$ such that A, B, and C are integers satisfying $A \log 16 + B \log 18 + C \log 24 = 0$.

I-6. In $\triangle LEO$, point J lies on $\overline{LO}$ so that $\overline{JE} \perp \overline{EO}$, and point S lies on $\overline{LE}$ so that $\overline{JS} \perp \overline{LE}$. Given that $JS = 9, EO = 20$, and $JO + SE = 37$, compute the perimeter of $\triangle LEO$.

I-7. Compute the least possible area of a non-degenerate right triangle with sides of length $\sin x$, $\cos x$, and $\tan x$, where x is a real number.

I-8. Let $P(x)$ be the polynomial $x^3 + Ax^2 + Bx + C$ for some constants A, B, and C. There exist constants D and E such that for all x, $P(x + 1) = x^3 + Dx^2 + 54x + 37$ and $P(x + 2) = x^3 + 26x^2 + Ex + 115$. Compute the ordered triple (A, B, C).

I-9. An n-sided die has the integers between 1 and n (inclusive) on its faces. All values on the faces of the die are equally likely to be rolled. An 8-sided die, a 12-sided die, and a 20-sided die are rolled. Compute the probability that one of the values rolled is equal to the sum of the other two values rolled.

I-10. Compute the greatest of the three prime divisors of $13^4 + 16^5 - 172^2$.

2016 Individual Answers

I-1. 14

I-2. $3\sqrt{11}$

I-3. $\dfrac{5}{16}$

I-4. 144

I-5. 105

I-6. 120

I-7. $\dfrac{\sqrt[4]{2}}{2} - \dfrac{\sqrt[4]{8}}{4}$

I-8. $(20, 11, 5)$

I-9. $\dfrac{23}{240}$

I-10. 1321

2016 Individual Solutions

I-1. Factoring yields $2016 = 2^5 \cdot 3^2 \cdot 7$, which means that $2 \cdot 3 \cdot 7 = 42 \mid ab$. To minimize $a+b$, first look for cases where $ab = 42$. If $a = 2$, then $b = 21$, so setting $c = 2$ is sufficient, yielding 25 for the sum. If $a = 3$, then $b = 14$, but c must be 5 (or greater) to satisfy the condition that $2^5 \mid a^b b^c$, yielding $a+b+c = 22$. If $a = 6$ and $b = 7$, then $c = 1$ is sufficient, yielding $a+b+c = 14$. This value is minimal because $a = 7$ forces $b = 6$ and $c \geq 5$, while the next larger value of a is 14. Thus the minimal value of $a+b+c$ is $\mathbf{14}$.

I-2. Point A cannot be the vertex of the isosceles triangle, because if it were, the ant would reach the midpoint of the base, and then the area surrounded by the ant's path would be half the triangle's area rather than 90%. So A is one of the two base angles. Let P be the point reached by the ant and, without loss of generality, let $\mathrm{m}\angle B$ be the triangle's vertex angle. Note that by the triangle inequality, P lies on $\overline{BC}$. There are two cases to consider: either the ant passes through vertex B first, making $[APB]$ greater than $[APC]$, or the ant passes through vertex C first, making $[APC]$ greater than $[APB]$. In the first case, because $\triangle APB$ and $\triangle APC$ have collinear bases (along $\overline{BC}$) and the same altitude (from point A), $\frac{[APB]}{[APC]} = \frac{PB}{PC}$. Because $\frac{[APB]}{[APC]} = \frac{90\%}{10\%} = 9$, $\frac{PB}{PC} = 9$. Without loss of generality, set $PB = 9$ and $PC = 1$. Then $AB = 10$, and from $AB + PB = \frac{1}{2}(AB + BC + AC)$, obtain $AC = 18$. By the Pythagorean Theorem, the altitude to $\overline{AC}$ is $\sqrt{19}$, and $\tan A = \frac{\sqrt{19}}{9} \approx 0.5$. In the second case, analogous reasoning yields $PB = 1$, $PC = 9$, and $AC = 2$, so that $\tan A = \frac{\sqrt{99}}{1} > 0.5$. Thus the answer is $\mathbf{3\sqrt{11}}$.

I-3. Notice that $\lfloor \sqrt[4]{n-1} \rfloor = \lfloor \sqrt[4]{n} \rfloor$ except when n is a perfect fourth power. So all of the factors in the product can be ignored except for $\frac{\lfloor \sqrt[4]{15} \rfloor}{\lfloor \sqrt[4]{16} \rfloor}, \frac{\lfloor \sqrt[4]{255} \rfloor}{\lfloor \sqrt[4]{256} \rfloor}, \frac{\lfloor \sqrt[4]{1295} \rfloor}{\lfloor \sqrt[4]{1296} \rfloor}$. These fractions simplify to $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$, respectively, and their product is $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}$, or $\mathbf{\frac{5}{16}}$.

I-4. First notice that some pairs are impossible: 1 can only be followed by 2, and 2 can only be followed by 1, 3, or 4. These observations suggest that the problem can be divided into several cases:

Case 1: $x_6 = 1$. If $x_5 = 2$, then there are 4! arrangements of the other numbers. If $x_5 \neq 2$, then holding either 23 or 24 as a consecutive pair yields 4! permutations. So there are $3 \cdot 24 = 72$ possible arrangements.

Case 2: $x_5 = 1$ and $x_6 = 2$. Then there are $4! = 24$ arrangements of the other numbers.

Case 3: Either 123 or 124 occur in that order. Then there are 4! permutations of each cluster and the other three numbers, yielding $2 \cdot 4! = 48$ possible arrangements.

Hence the desired number of permutations of the numbers is $\mathbf{144}$.

I-5. Use the laws of logarithms to obtain $(4\log 2)A + (\log 2 + 2\log 3)B + (3\log 2 + \log 3)C = 0$. Collecting terms with common logarithm coefficients yields $(4A + B + 3C)\log 2 + (2B + C)\log 3 = 0$. Because 3 is not a rational power of 2, in order to satisfy this equation with integral A, B, C, both $4A+B+3C = 0$ and $2B+C = 0$. The second equation is equivalent to $C = -2B$, yielding $4A + B - 6B = 0$ or $4A = 5B$. Thus A is divisible by 5 and B is divisible by 4. Set $A = 5$, $B = 4$, and $C = -8$ to obtain $5^2 + 4^2 + (-8)^2 = \mathbf{105}$.

I-6. Optimistically, one might hope that $SE = 12$ and $JO = 25$, and in fact this turns out to be the case! Let $JO = x$ and $SE = y$. Then $JE^2 = 9^2 + y^2 = 81 + y^2$ and $JE^2 + 20^2 = x^2$, hence $y^2 + 481 = x^2$. Thus $x^2 - y^2 = 481$. Because $x + y = 37$, $x - y = 481/37 = 13$. Hence $2x = 50$, $x = 25$, and $y = 12$. That information by itself does not solve the problem, but the fact that the sides of right triangles JES and JOE are known makes a combination of angle-chasing and trigonometry feasible.

Let $m\angle JOE = \theta$. Then $m\angle JES = \theta$ because triangles JSE and JEO are both $3-4-5$ right triangles. Then $m\angle L = 180° - (m\angle LEO + m\angle LOE) = 180° - (90° + \theta + \theta) = 90° - 2\theta$. Hence $\tan L = \frac{1}{\tan 2\theta}$ and $\sin L = \cos 2\theta$. Using double-angle identities for tangent and sine yields $\tan L = \frac{1-\tan^2\theta}{2\tan\theta}$ and $\sin L = \cos^2\theta - \sin^2\theta$. In this case, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, and $\tan\theta = \frac{3}{4}$. Substitute these values into the preceding equations to obtain $\tan L = \frac{7/16}{6/4} = \frac{7}{24}$, while $\sin L = \frac{7}{25}$. Then $LS = \frac{9}{\tan L} = \frac{216}{7}$ and $LJ = \frac{9}{\sin L} = \frac{225}{7}$. Thus $LS + LJ = \frac{441}{7} = 63$. With $SE = 12, EO = 20, JO = 25$, the perimeter is $\mathbf{120}$.

Alternate Solution: Let $t = SE$. Then $JE^2 = t^2 + 9^2 = (37 - t)^2 - 20^2$, which implies that $t = 12$. Note that $\triangle JEO \sim \triangle JSE$. Let $\alpha = m\angle EOJ$; then $m\angle JES = \alpha$ and $m\angle LJS = 2\alpha$. Thus $\sin 2\alpha = 2(\sin\alpha)(\cos\alpha) = \frac{24}{25}$, and $\triangle LJS$ is similar to a 7–24–25 triangle. Letting $x = \frac{JS}{9}$, it follows that $LJ + LS = 25x + 24x = 49x = 63$. Thus the perimeter of $\triangle LEO$ is $63 + 37 + 20 = \mathbf{120}$.

I-7. Let $\triangle ABC$ be a right triangle with hypotenuse $\overline{BC}$ and side lengths $\sin x, \cos x$, and $\tan x$. There are three cases to consider, depending on the length chosen for BC. If $BC = \tan x$, then $\tan^2 x = \sin^2 x + \cos^2 x = 1 \Rightarrow \tan x = 1 \Rightarrow x = 45°$. In this case, ABC would be an isosceles right triangle with legs of length $\sin 45° = \frac{\sqrt{2}}{2}$, so $[ABC]$ would be $\frac{AB \cdot AC}{2} = \frac{1}{4}$.

Because $\sin x$, $\cos x$, and $\tan x$ must all be positive,

$$\tan x = \frac{\sin x}{\cos x} \geq \frac{\sin x}{1} = \sin x.$$

Thus it cannot be the case that $BC = \sin x$, because the hypotenuse is the longest side of the triangle. Finally, if $BC = \cos x$, then $[ABC] = \frac{AB \cdot AC}{2} = \frac{\sin^2 x}{2\cos x}$ and $\cos^2 x = \sin^2 x + \tan^2 x$.

The latter equation can be rewritten as

$$\begin{aligned}\cos^4 x &= (\sin^2 x)(\cos^2)x + \sin^2 x\\ \cos^4 x &= (1-\cos^2 x)\cos^2 x + (1-\cos^2 x)\\ 2\cos^4 x &= 1,\end{aligned}$$

and so $\cos x = \frac{1}{\sqrt[4]{2}} = \frac{\sqrt[4]{8}}{2}$, $\sin^2 x = 1 - \frac{\sqrt{2}}{2}$. Thus

$$[ABC] = \frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{8}}{4} = \frac{1}{4}\left(2\sqrt[4]{2} - \sqrt[4]{8}\right).$$

The last step is to compare the value above with $\frac{1}{4}$. To see that the value above is less than $\frac{1}{4}$, observe that $4\sqrt[4]{2} < 5$ because $(4\sqrt[4]{2})^4 = 512 < 625 = 5^4$. Next, observe that $2\sqrt[4]{8} > 3$ because $(2\sqrt[4]{8})^4 = 2^7 = 128 > 81 = 3^4$. Put these facts together to see that

$$\frac{1}{4}\left(2\sqrt[4]{2} - \sqrt[4]{8}\right) = \frac{1}{8}\left(4\sqrt[4]{2} - 2\sqrt[4]{8}\right) < \frac{1}{8}(5-3) = \frac{1}{4}.$$

Thus the answer is $\boldsymbol{\frac{\sqrt[4]{2}}{2} - \frac{\sqrt[4]{8}}{4}}$.

I-8. Plug $x = 0$ into the given equation involving $P(x+2)$ to obtain $P(2) = 115$. Next, plug in $x = 1$ into the given equation involving $P(x+1)$ to obtain $P(2) = 1 + D + 54 + 37$. Thus $D = 115-(1+54+37) = 23$. Then $P(x) = P((x-1)+1) = (x-1)^3+23(x-1)^2+54(x-1)+37 = x^3 + 20x^2 + 11x + 5$, hence $(A, B, C) = \mathbf{(20, 11, 5)}$. It can be verified that $E = 103$.

I-9. Let a, b, and c be the values rolled on the 8-, 12-, and 20-sided dice, respectively. Call a triple (a, b, c) of rolls *great* if one roll equals the sum of the other two.

Imagine that the first two dice are rolled (so that a and b are determined), and then the 20-sided die is rolled. Then the triple (a, b, c) is great if either $c = a+b$, $c = a-b$, or $c = b-a$. The first case is possible for any pair of rolls (a, b), and the probability is $\frac{1}{20}$ for each such pair. The second and third cases are only possible if $a > b$ or if $b > a$, respectively, but again, in each of these cases, the probability of obtaining the proper value of c is $\frac{1}{20}$. So the total probability of rolling a great triple satisfying either $c = a - b$ or $c = b - a$ is simply $\frac{1}{20} \cdot P(a > b \text{ or } b > a)$. But $P(a > b \text{ or } b > a) = 1 - P(a = b) = 1 - \frac{8}{96} = 1 - \frac{1}{12} = \frac{11}{12}$. (This value is simply the probability that whatever number was rolled on the 8-sided die also comes up on the 12-sided die.) So the total probability of rolling a great triple is $\frac{1}{20} + \frac{11}{240} = \boldsymbol{\frac{23}{240}}$.

I-10. Let $N = 13^4 + 16^5 - 172^2$. Notice that $16^5 = (2^4)^5 = 2^{20}$ and $13^4 - 172^2 = 169^2 - 172^2 = (169-172)(169+172) = -3 \cdot 341 = -1023 = 1 - 2^{10}$. Thus $N = 2^{20} - 2^{10} + 1$. Multiply by $1025 = 2^{10} + 1$ to obtain

$$\begin{aligned}1025 \cdot N &= (2^{20} - 2^{10} + 1)(2^{10} + 1)\\ &= 2^{30} + 1.\end{aligned}$$

The right-hand side can be refactored via the identity $4u^4+1=(2u^2+2u+1)(2u^2-2u+1)$, using $u=2^7$:

$$\begin{aligned}
1025\cdot N &= (2\cdot 2^{14}+2\cdot 2^7+1)(2\cdot 2^{14}-2\cdot 2^7+1)\\
5^2\cdot 41\cdot N &= (32768+256+1)(32768-256+1)\\
&= 33025\cdot 32513\\
&= (25\cdot 1321)(41\cdot 793)\\
N &= 1321\cdot 793.
\end{aligned}$$

Searching for small prime divisors leads to $793=13\cdot 61$, so **1321** is the greatest prime divisor of N.

Power Question 2016: Taxicab Geometry

In standard coordinate (Euclidean) geometry, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the distance between A and B is given by the Euclidean distance function: $d_{\mathrm{E}}(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In this Power Question, we investigate what happens when distance is measured using the *taxicab* distance function,

$$d_{\mathrm{T}}(A, B) = |x_2 - x_1| + |y_2 - y_1|.$$

This distance is called the *taxi-distance* between A and B.

1. This problem will help you become familiar with taxi-distances.

   a. Compute the taxi-distance between the points $(2, 6)$ and $(7, -2)$. [1 pt]

   b. Compute the coordinates of a point whose Euclidean distance from $(3, 2)$ is 5 and whose taxi-distance from $(3, 2)$ is also 5. [1 pt]

   c. Compute the coordinates of a point whose Euclidean distance from $(3, 2)$ is 5 and whose taxi-distance from $(3, 2)$ is 7. [1 pt]

In Euclidean geometry, one possible definition of the line segment $\overline{AB}$ is the set of points P such that $AP + PB = AB$, or $d_{\mathrm{E}}(A, P) + d_{\mathrm{E}}(P, B) = d_{\mathrm{E}}(A, B)$. This definition does not work as well in taxicab geometry!

2. Let $A = (0, 4)$ and $B = (4, 0)$.

   a. Show that $P = (1, 3)$ satisfies $d_{\mathrm{T}}(A, P) + d_{\mathrm{T}}(P, B) = d_{\mathrm{T}}(A, B)$. [1 pt]

   b. Find the coordinates of a point P *not* on the line $x + y = 4$ satisfying $d_{\mathrm{T}}(A, P) + d_{\mathrm{T}}(P, B) = d_{\mathrm{T}}(A, B)$. [1 pt]

   c. Find the set of all points P such that $d_{\mathrm{T}}(A, P) + d_{\mathrm{T}}(P, B) = d_{\mathrm{T}}(A, B)$, and justify your answer. [3 pts]

As in Euclidean geometry, two triangles are congruent if and only if the lengths of corresponding sides and the measures of corresponding angles are equal. In Euclidean geometry, this *measure-congruence* is equivalent to *transformation-congruence*: two triangles are transformation-congruent if and only if one is the image of the other under a composition of reflections, rotations, and translations. (Actually, reflections alone suffice, but that result is beyond the scope of this Power Question.) For the remainder of this Power Question, the symbol $\cong$ will be used exclusively to denote measure-congruence; we write $\cong_{\mathrm{T}}$ to emphasize that we are using taxicab measure.

3. Let's investigate whether the equivalence between measure- and transformation-congruence holds in taxicab geometry. Suppose that $A = (x_A, y_A)$ and similarly for other named points.

   a. If $\triangle A'B'C'$ is the image of $\triangle ABC$ under the translation $(x, y) \mapsto (x + h, y + k)$, are the taxi-lengths of the sides of $\triangle A'B'C'$ equal to those of $\triangle ABC$? Justify your answer. [1 pt]

b. If $\triangle A'B'C'$ is the image of $\triangle ABC$ under a reflection in the line $y = k$, are the taxi-lengths of the sides of $\triangle A'B'C'$ equal to those of $\triangle ABC$? Justify your answer. [1 pt]

c. Let $O = (0,0)$, $R = (8,0)$, $D = (0,4)$, and let $\triangle O'R'D'$ be the image of $\triangle ORD$ under a (Euclidean) rotation of 45° counterclockwise about the origin. Is $\triangle ORD \cong_{\mathrm{T}} \triangle O'R'D'$? Justify your answer. [2 pts]

4. Let O have coordinates $(0,0)$, let P have coordinates $(1,0)$, and let A be the point $(4,2)$. The point B has coordinates (m,n), where m and n are both positive integers, m is relatively prime to n, and $\angle BOA \cong \angle AOP$ (considered as Euclidean angles).

   a. Compute the ordered pair (m,n). [2 pts]

   b. Compute the taxi-lengths of the segments intercepted by $\angle BOA$ and $\angle AOP$ on the line $x + y = 1$. [2 pts]

The computation in the previous item is problematic, for the following reason. A *taxi-circle* is the set of all points that are a fixed taxi-distance from a given point, and the segments whose lengths you computed in 4b are actually arcs on a unit taxi-circle centered at the origin. So angles congruent in the Euclidean sense do not necessarily intercept taxi-congruent arcs.

One way to fix this problem is to define the measure of an angle analogously to the way the radian measure of an angle is defined in Euclidean geometry, namely, as a quotient. The numerator is the length of the arc that the angle intercepts on a circle centered at the angle's vertex. The denominator is the radius of that circle. If P and R are equidistant from Q, then the *measure of $\angle PQR$ in taxi-radians* is given by the formula

$$\mathrm{m_T}\angle PQR = \frac{\text{taxi-length of the arc from } P \text{ to } R \text{ on the taxi-circle centered at } Q \text{ passing through } P}{d_{\mathrm{T}}(Q,P)}.$$

5. Let's flesh out the above definition with an example.

   a. Find an equation to describe the taxi-circle centered at the origin of radius 17. [1 pt]

   b. Compute the value of *taxi-pi*, π_{T}, by dividing the semi-taxi-circumference of the taxi-circle from part 5a by its radius. [1 pt]

   c. Show that the value you computed in part 5b is independent of the taxi-circle's radius. [2 pts]

   d. Given that $C = (3,1)$, $O = (0,0)$, and $B = (-1,3)$, compute $\mathrm{m_T}\angle COB$. [1 pt]

Using taxi-circles and taxi-angles, define a *rotation by θ taxi-radians about the origin* as follows. The point P' is the image of P under a counterclockwise rotation of θ taxi-radians around $O = (0,0)$ if and only if $d_{\mathrm{T}}(O,P) = d_{\mathrm{T}}(O,P')$ and the measure of the *taxi-arc* from P to P' (going counterclockwise around the taxi-circle centered at O) is $d_{\mathrm{T}}(O,P) \cdot \theta$.

6. Let $O = (0, 0)$, $A = (8, 0)$, and $R = (3, 1)$.

   a. Let $\triangle O'A'R'$ be the image of $\triangle OAR$ under a rotation of 1 taxi-radian about the origin. Compute the coordinates of O', A', and R'. [3 pts]

   b. Is $\triangle O'A'R' \cong_{\mathrm{T}} \triangle OAR$? Justify your answer. [3 pts]

7. We now consider conditions for two triangles to be measure-congruent in Euclidean geometry and apply them to taxicab geometry to discover whether or not measure-congruence still holds. Let $\triangle MOD$ have vertices $M = (0, 2)$, $O = (0, 0)$, and $D = (2, 0)$.

   a. Show that there exists a triangle $\triangle CAB$ such that $d_{\mathrm{T}}(C, A) = d_{\mathrm{T}}(M, O)$, $d_{\mathrm{T}}(A, B) = d_{\mathrm{T}}(O, D)$, and $\mathrm{m_T}\angle CAB = \mathrm{m_T}\angle MOD$, but $\triangle MOD \not\cong_{\mathrm{T}} \triangle CAB$. (Thus SAS congruence does not hold in taxicab geometry.) [2 pts]

   b. Show that there exists a triangle $\triangle PIG$ such that $d_{\mathrm{T}}(P, I) = d_{\mathrm{T}}(M, O)$, $d_{\mathrm{T}}(I, G) = d_{\mathrm{T}}(O, D)$, and $d_{\mathrm{T}}(P, G) = d_{\mathrm{T}}(M, D)$, but $\triangle MOD \not\cong_{\mathrm{T}} \triangle PIG$. (Thus SSS congruence does not hold in taxicab geometry.) [2 pts]

8. Define the *taxi-cosine* and *taxi-sine* of an angle as follows:

$$\begin{aligned} \cos_{\mathrm{T}} \angle ACB &= \frac{d_{\mathrm{E}}(C, A) \cdot d_{\mathrm{E}}(C, B)}{d_{\mathrm{T}}(C, A) \cdot d_{\mathrm{T}}(C, B)} \cos \angle ACB \\ \sin_{\mathrm{T}} \angle ACB &= \frac{d_{\mathrm{E}}(C, A) \cdot d_{\mathrm{E}}(C, B)}{d_{\mathrm{T}}(C, A) \cdot d_{\mathrm{T}}(C, B)} \sin \angle ACB, \end{aligned}$$

   where $\cos \angle ACB$ and $\sin \angle ACB$ are the ordinary (Euclidean) cosine and sine functions, respectively.

   a. Given $A = (3, 0)$, $O = (0, 0)$, and $B = (3, 4)$, compute $\cos_{\mathrm{T}} \angle AOB$. [1 pt]

   b. Show that if $\overrightarrow{OA}$ is parallel to the x-axis, then $\cos_{\mathrm{T}} \angle AOB = \dfrac{\cos \angle AOB}{|\cos \angle AOB| + |\sin \angle AOB|}$. [3 pts]

   c. Let $\overrightarrow{QP}$ be parallel to the x-axis, and let A and B be points not on $\overrightarrow{QP}$. Show that

$$\cos_{\mathrm{T}} \angle AQB = \frac{\cos \angle AQB}{(|\cos \angle AQP| + |\sin \angle AQP|)(|\cos \angle BQP| + |\sin \angle BQP|)}.$$

   [3 pts]

9. Suppose that triangle ABC is given with $\overline{AB}$ parallel to the x-axis, and that the perpendicular dropped from C to $\overleftrightarrow{AB}$ intersects $\overline{AB}$ at C' and has taxi-length h_{T}. As with conventional trigonometry, let $a_{\mathrm{T}} = d_{\mathrm{T}}(B, C)$, $b_{\mathrm{T}} = d_{\mathrm{T}}(A, C)$, and $c_{\mathrm{T}} = d_{\mathrm{T}}(A, B)$. Prove the following:

   a. $a_{\mathrm{T}} = b_{\mathrm{T}} + c_{\mathrm{T}} - 2b_{\mathrm{T}} \cos_{\mathrm{T}} A$. [3 pts]

   b. $b_{\mathrm{T}} = a_{\mathrm{T}} + c_{\mathrm{T}} - 2a_{\mathrm{T}} \cos_{\mathrm{T}} B$. [2 pts]

c. $c_\mathrm{T} = \dfrac{a_\mathrm{T}^2 + b_\mathrm{T}^2 - 2a_\mathrm{T}b_\mathrm{T}\cos_\mathrm{T} C}{a_\mathrm{T} + b_\mathrm{T}}$. [2 pts]

10. Given $\triangle ABC$, prove that $\dfrac{a_\mathrm{T}}{\sin_\mathrm{T} A} = \dfrac{b_\mathrm{T}}{\sin_\mathrm{T} B} = \dfrac{c_\mathrm{T}}{\sin_\mathrm{T} C}$, where a_T, b_T, and c_T are defined as in the previous problem. [4 pts]

Solutions to 2016 Power Question

1. Compute from the given definition:

   a. $|7-2|+|-2-6|=13$.

   b. There are four points that can found by moving from $(3,2)$ along vectors parallel to the coordinate axes. These points are $(8,2),(-2,2),(3,7)$, and $(3,-3)$. *One of these is sufficient to answer the question.*

   c. The classic $3-4-5$ right triangle gives many more possibilities: $(3\pm 3, 2\pm 4)$ and $(3\pm 4, 2\pm 3)$. (One of these is sufficient to answer the question.)

2. a. $d_{\mathrm{T}}(A,P)+d_{\mathrm{T}}(P,B)=|1-0|+|3-4|+|4-1|+|0-3|=8=|4-0|+|0-4|=d_{\mathrm{T}}(A,B)$.

   b. Many solutions exist (see part c below), but $(2,4)$ is one such point.

   c. Because $d_{\mathrm{T}}(A,B)=8$, to move from A to B, find all points $P(x,y)$ such that $|x-0|+|y-4|+|4-x|+|0-y|=8$. If $0\le x\le 4$ and $0\le y\le 4$, the left hand side of the equation simplifies to $x+(4-y)+(4-x)+y$, which equals 8 for all such x and y. On the other hand, if $x<0$ and $0\le y\le 4$, then the left hand side of the equation simplifies to $4-2x+(4-y)+y=8-2x$, and for $x<0$, $8-2x>8$. Analogous arguments show that no points with $x>4$, or $y<0$, or $y>4$ satisfy the condition. Hence the solution set is the square $0\le x\le 4, 0\le y\le 4$.

   Alternate Solution: Let $P=(x,y)$, so that

   $$\begin{aligned} d_{\mathrm{T}}(A,P)+d_{\mathrm{T}}(P,B) &= |x-0|+|y-4|+|4-x|+|0-y| \\ &= |4-x|+|x-0|+|4-y|+|y-0|. \end{aligned}$$

   For any real number r, $|r|\ge r$, and equality holds if and only if $r\ge 0$. Applying this observation to each term in the above sum, $d_{\mathrm{T}}(A,P)+d_{\mathrm{T}}(P,B)\ge(4-x)+(x-0)+(4-y)+(y-0)=8=d_{\mathrm{T}}(A,B)$, with equality if and only if each of $4-x$, $x-0$, $4-y$, $y-0$ is nonnegative. Hence the solution set is the square $0\le x\le 4$ and $0\le y\le 4$.

3. a. Let $A=(x_1,y_1)$ and let $B=(x_2,y_2)$. Then $d_{\mathrm{T}}(A,B)=|x_2-x_1|+|y_2-y_1|$. On the other hand, $A'=(x_1+h,y_1+k)$ and $B'=(x_2+h,y_2+k)$, yielding $d_{\mathrm{T}}(A',B')=|(x_2+h)-(x_1+h)|+|(y_2+k)-(y_1+k)|=|x_2-x_1|+|y_2-y_1|=d_{\mathrm{T}}(A,B)$. The same argument would apply to the taxi-lengths of $\overline{BC}$ and $\overline{AC}$. Therefore the taxi-lengths of the sides are preserved under translation.

   b. Let $A=(x_1,y_1)$ and let $B=(x_2,y_2)$. The reflection takes $(x,y)=(x,k+(y-k))$ to $(x,k-(y-k))=(x,2k-y)$. Hence $A'=(x_1,2k-y_1)$ and $B'=(x_2,2k-y_2)$. Thus $d_{\mathrm{T}}(A',B')=|x_2-x_1|+|(2k-y_2)-(2k-y_1)|=|x_2-x_1|+|y_1-y_2|=d_{\mathrm{T}}(A,B)$. Analogous arguments apply to the taxi-lengths of $\overline{BC}$ and $\overline{AC}$, so the taxi-lengths of the sides are preserved under reflection in a horizontal line. (Note: It would be straightforward to show the same result for reflection in a vertical line. What might be difficult about reflecting in an oblique line?)

c. $O' = (0,0)$, $R' = (4\sqrt{2}, 4\sqrt{2})$, $D' = (-2\sqrt{2}, 2\sqrt{2})$. Notice that $d_{\mathrm{T}}(O,D) = 4$, $d_{\mathrm{T}}(D,R) = 12$, and $d_{\mathrm{T}}(O,R) = 8$, but $d_{\mathrm{T}}(O',R') = 8\sqrt{2}$. Therefore $\triangle ORD \not\cong_{\mathrm{T}} \triangle O'R'D'$.

4. a. Let θ be the angle between the positive x-axis and the ray $\overrightarrow{OA}$. Then $\tan\theta = \frac{1}{2}$. The ray $\overrightarrow{OB}$ makes an angle of 2θ with the positive x-axis, so using the double-angle identity from trigonometry yields $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{1}{1-\left(\frac{1}{2}\right)^2} = \dfrac{4}{3}$. Hence B lies on the line $y = \frac{4}{3}x$. Because m and n can have no prime factors in common, $m = 3$ and $n = 4$. The desired point is $(3,4)$.

b. The rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ intersect the line $x + y = 1$ at $\left(\frac{2}{3}, \frac{1}{3}\right)$ and $\left(\frac{3}{7}, \frac{4}{7}\right)$ respectively. Note that $d_{\mathrm{T}}\left(P, \left(\frac{2}{3}, \frac{1}{3}\right)\right) = |1-\frac{2}{3}| + |0-\frac{1}{3}| = \frac{2}{3}$. However, $d_{\mathrm{T}}\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{3}{7}, \frac{4}{7}\right)\right) = |\frac{3}{7}-\frac{2}{3}| + |\frac{4}{7}-\frac{1}{3}| = \frac{10}{21}$. Therefore central angles with the same Euclidean angle measure can intercept arcs of different taxi-lengths on the unit taxi-circle.

5. a. The taxi-circle is the set of all points (x,y) such that $|x| + |y| = 17$. This equation describes a square centered at the origin with vertices $(\pm 17, 0)$ and $(0, \pm 17)$; the sides of the square lie on the lines $x + y = 17$, $x + y = -17$, $x - y = 17$, and $x - y = -17$.

b. The semi-taxi-circumference of this circle is 68, because

$$d_{\mathrm{T}}((17,0),(0,17)) = d_{\mathrm{T}}((0,17),(-17,0)) = 34.$$

(Note that the distance along the arc is, as usual, longer than the straight-line taxi-distance $d_{\mathrm{T}}((17,0),(-17,0))$, which is simply 34.) The radius of the circle is 17. So $\pi_{\mathrm{T}} = 68/17 = 4$.

c. In general, the taxi-circle of radius r centered at the origin is given by the equation $|x| + |y| = r$, and one semicircle has endpoints $(r,0)$ and $(-r,0)$. Hence the arc length is simply $d_{\mathrm{T}}((r,0),(0,r)) + d_{\mathrm{T}}((0,r),(-r,0)) = 2r + 2r = 4r$. Therefore $\pi_{\mathrm{T}} = 4r/r = 4$. (As an interesting exercise, you might try to show that this result holds for *any* taxi-semicircle with endpoints (x,y) and $(-x,-y)$.)

d. These points are both found on a taxi-circle of radius 4.

$$\mathrm{m_T}\angle COB = \frac{\text{taxi-length of the arc from } C \text{ to } B \text{ on taxi-circle } O}{d_{\mathrm{T}}(O,C)} = \frac{8}{4} = 2 \text{ taxi-radians.}$$

6. a. First note that $O' = O$ because O is the center of rotation. To find A', note that $d_{\mathrm{T}}(O,A) = 8$, so the arc from A to A' on the circle centered at O passing through A must also have length 8, and if A' is in the first quadrant, then it lies on the line $x + y = 8$. These conditions are satisfied by the point $(4,4)$. Similarly, R' must lie on the line $x + y = 4$ and $d_{\mathrm{T}}(R,R') = 4$. Assuming that $0 < x < 3$ and $1 < y < 4$ yields $d_{\mathrm{T}}(R,R') = (3-x) + (y-1) = y - x + 2$. Hence the desired point satisfies both $x + y = 4$ and $y - x + 2 = 4$, yielding $x = 1, y = 3$ as the unique solution. Thus $R' = (1,3)$.

Alternate Solution: First note that $O' = O$ because O is the center of rotation. Before considering the other points, let $P = (x, y)$ be any point in the first quadrant and let $0 \le r \le x$. Then $Q = (x - r, y + r)$ is also in the first quadrant, Q and P are on the same taxi-circle centered at O, and $d_\mathrm{T}(P, Q) = 2r$.

Apply this observation to $A = (8, 0)$ with $r = \frac{1}{2}d_\mathrm{T}(O, A) = 4$ to find $A' = (8-4, 0+4) = (4, 4)$. Similarly, take $P = B = (3, 1)$ and $r = \frac{1}{2}d_\mathrm{T}(O, B) = 2$ to find $B' = (3-2, 1+2) = (1, 3)$.

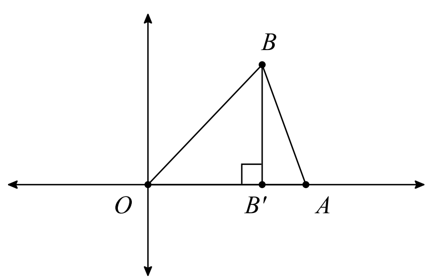
b. By construction, $d_\mathrm{T}(O, A) = d_\mathrm{T}(O, A')$ and $d_\mathrm{T}(O, B) = d_\mathrm{T}(O, B')$. However, $d_\mathrm{T}(A, B) = |3 - 8| + |1 - 0| = 6$ while $d_\mathrm{T}(A', B') = |1 - 4| + |3 - 4| = 4$. Hence the two triangles are *not* measure-congruent.

7. a. There are many examples. One such example is $\triangle CAB$ with vertices $C(-1, 1)$, $A(0, 0)$, and $B(1, 1)$. Then $d_\mathrm{T}(C, A) = d_\mathrm{T}(M, O) = 2$, $d_\mathrm{T}(A, B) = d_\mathrm{T}(O, D) = 2$, and $\mathrm{m_T}\angle A \cong \mathrm{m_T}\angle O = \pi_\mathrm{T}/2$, but $\triangle MOD \ncong_\mathrm{T} \triangle CAB$ because $d_\mathrm{T}(C, B) = 2 \neq d_\mathrm{T}(M, D)$.

b. There are many examples. One such example is $\triangle PIG$ with vertices $P(-1, 1)$, $I(0, 0)$, and $G(2, 0)$. Then $d_\mathrm{T}(P, I) = d_\mathrm{T}(M, O) = 2$, $d_\mathrm{T}(I, G) = d_\mathrm{T}(O, D) = 2$, and $d_\mathrm{T}(P, G) = d_\mathrm{T}(M, D) = 4$, but $\triangle MOD \ncong_\mathrm{T} \triangle PIG$ because $\mathrm{m_T}\angle PIG > \mathrm{m_T}\angle MOD$.

8. a. To use the formula, note that $d_\mathrm{E}(O, A) = 3$, $d_\mathrm{E}(O, B) = 5$, $d_\mathrm{T}(O, A) = 3$, $d_\mathrm{T}(O, B) = 7$, $\cos\angle AOB = \frac{3}{5}$, and $\sin\angle AOB = \frac{4}{5}$. Then $\cos_\mathrm{T}\angle AOB = \frac{3\cdot 5}{3\cdot 7}\cdot\frac{3}{5} = \frac{3}{7}$. (Similarly, $\sin_\mathrm{T}\angle AOB = \frac{3\cdot 5}{3\cdot 7}\cdot\frac{4}{5} = \frac{4}{7}$.)

b. Consider the diagram below.



Because $\overrightarrow{OA}$ is parallel to the x-axis, $d_\mathrm{E}(O, A) = d_\mathrm{T}(O, A)$. Then

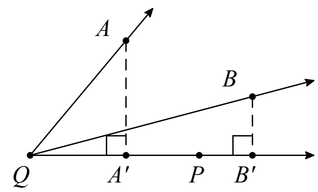
$$\cos_\mathrm{T}\angle AOB = \frac{d_\mathrm{E}(O, A)\cdot d_\mathrm{E}(O, B)}{d_\mathrm{T}(O, A)\cdot d_\mathrm{T}(O, B)}\cos\angle AOB = \frac{d_\mathrm{E}(O, B)}{d_\mathrm{T}(O, B)}\cos\angle AOB.$$

Let B' be the foot of the perpendicular from B to $\overleftrightarrow{OA}$, so that $d_\mathrm{T}(O, B) = d_\mathrm{E}(O, B') + d_\mathrm{E}(B', B)$. Then

$$\cos_\mathrm{T}\angle AOB = \frac{d_\mathrm{E}(O, B)}{d_\mathrm{E}(O, B') + d_\mathrm{E}(B', B)}\cos\angle AOB = \frac{1}{\frac{d_\mathrm{E}(O,B')}{d_\mathrm{E}(O,B)} + \frac{d_\mathrm{E}(B,B')}{d_\mathrm{E}(O,B)}}\cos\angle AOB.$$

Substituting $|\cos\angle AOB|$ for $\frac{d_\mathrm{E}(O,B')}{d_\mathrm{E}(O,B)}$ and $|\sin\angle AOB|$ for $\frac{d_\mathrm{E}(B,B')}{d_\mathrm{E}(O,B)}$ yields the desired result. (The reader may wish to show that, under these conditions, $\cos_\mathrm{T}\angle AOB + \sin_\mathrm{T}\angle AOB = 1$.)

c. Let the feet of the altitudes from A and B to $\overleftrightarrow{PQ}$ be A' and B', respectively, as shown in the diagram below.



Then $d_{\mathrm{T}}(Q,A) = d_{\mathrm{E}}(Q,A') + d_{\mathrm{E}}(A',A)$ and $d_{\mathrm{T}}(Q,B) = d_{\mathrm{E}}(Q,B') + d_{\mathrm{E}}(B',B)$. Using the definition yields

$$\begin{aligned}
\cos_{\mathrm{T}} \angle AQB &= \frac{d_{\mathrm{E}}(Q,A)\cdot d_{\mathrm{E}}(Q,B)}{d_{\mathrm{T}}(Q,A)\cdot d_{\mathrm{T}}(Q,B)} \cos\angle AQB \\
&= \frac{1}{\left(\dfrac{d_{\mathrm{E}}(Q,A') + d_{\mathrm{E}}(A,A')}{d_{\mathrm{E}}(Q,A)}\right)\cdot\left(\dfrac{d_{\mathrm{E}}(Q,B') + d_{\mathrm{E}}(B,B')}{d_{\mathrm{E}}(Q,B)}\right)} \cos\angle AQB \\
&= \frac{1}{\left(\dfrac{d_{\mathrm{E}}(Q,A')}{d_{\mathrm{E}}(Q,A)} + \dfrac{d_{\mathrm{E}}(A,A')}{d_{\mathrm{E}}(Q,A)}\right)\cdot\left(\dfrac{d_{\mathrm{E}}(Q,B')}{d_{\mathrm{E}}(Q,B)} + \dfrac{d_{\mathrm{E}}(B,B')}{d_{\mathrm{E}}(Q,B)}\right)} \cos\angle AQB.
\end{aligned}$$

On the other hand, $\frac{d_{\mathrm{E}}(Q,A')}{d_{\mathrm{E}}(Q,A)} = |\cos\angle AQP|$, $\frac{d_{\mathrm{E}}(A',A)}{d_{\mathrm{E}}(Q,A)} = |\sin\angle AQP|$, and similarly $\frac{d_{\mathrm{E}}(Q,B')}{d_{\mathrm{E}}(Q,B)} = |\cos\angle BQP|$, $\frac{d_{\mathrm{E}}(B',B)}{d_{\mathrm{E}}(Q,B)} = |\sin\angle BQP|$. Hence the right side can be rewritten as

$$\frac{\cos\angle AQB}{(|\cos\angle AQP| + |\sin\angle AQP|)(|\cos\angle BQP| + |\sin\angle BQP|)},$$

which is the desired result.

Alternate Solution: Let the feet of the altitudes from A and B to $\overleftrightarrow{PQ}$ be A' and B', respectively. Then

$$\begin{aligned}
d_{\mathrm{T}}(Q,A) &= d_{\mathrm{E}}(Q,A') + d_{\mathrm{E}}(A',A) \\
&= d_{\mathrm{E}}(Q,A)\cdot|\cos\angle AQP| + d_{\mathrm{E}}(Q,A)\cdot|\sin\angle AQP| \\
&= d_{\mathrm{E}}(Q,A)\cdot\big(|\cos\angle AQP| + |\sin\angle AQP|\big),
\end{aligned}$$

so $d_{\mathrm{E}}(Q,A)/d_{\mathrm{T}}(Q,A) = \big(|\cos\angle AQP| + |\sin\angle AQP|\big)^{-1}$. The analogous formula for $d_{\mathrm{E}}(Q,B)/d_{\mathrm{T}}(Q,B)$ is derived in the same way. Substituting these expressions into the definition of the taxi-cosine yields

$$\begin{aligned}
\cos_{\mathrm{T}} \angle AQB &= \frac{d_{\mathrm{E}}(Q,A)\cdot d_{\mathrm{E}}(Q,B)}{d_{\mathrm{T}}(Q,A)\cdot d_{\mathrm{T}}(Q,B)} \cos\angle AQB \\
&= \frac{\cos\angle AQB}{(|\cos\angle AQP| + |\sin\angle AQP|)(|\cos\angle BQP| + |\sin\angle BQP|)},
\end{aligned}$$

which is the desired result.

9. For these three problems, the following lemma will be useful:

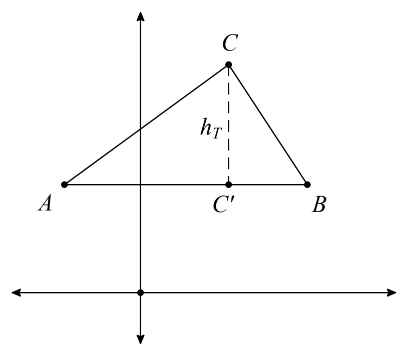
Lemma: Suppose that $\triangle PQR$ is a right triangle with right angle at Q and $\overline{PQ}$ parallel to the x-axis. Then $\cos_{\mathrm{T}} \angle P = \frac{d_{\mathrm{T}}(P,Q)}{d_{\mathrm{T}}(P,R)}$.

Proof of Lemma: Because $\overline{PQ}$ is parallel to the x-axis, the result from 8b applies, and $\cos_{\mathrm{T}} \angle P = \frac{\cos \angle P}{|\cos \angle P|+|\sin \angle P|}$. In this situation, $\cos \angle P = \frac{d_{\mathrm{E}}(P,Q)}{d_{\mathrm{E}}(P,R)}$ and $\sin \angle P = \frac{d_{\mathrm{E}}(Q,R)}{d_{\mathrm{E}}(P,R)}$, so substitute to obtain the following:

$$\begin{aligned} \cos_{\mathrm{T}} \angle P &= \frac{\frac{d_{\mathrm{E}}(P,Q)}{d_{\mathrm{E}}(P,R)}}{\left|\frac{d_{\mathrm{E}}(P,Q)}{d_{\mathrm{E}}(P,R)}\right| + \left|\frac{d_{\mathrm{E}}(Q,R)}{d_{\mathrm{E}}(P,R)}\right|} \\ &= \frac{d_{\mathrm{E}}(P,Q)}{|d_{\mathrm{E}}(P,Q)| + |d_{\mathrm{E}}(Q,R)|}. \end{aligned}$$

However, P and Q lie on the same horizontal line and Q and R lie on the same vertical line, so $d_{\mathrm{E}}(P,Q) = d_{\mathrm{T}}(P,Q)$ and $d_{\mathrm{T}}(P,R) = d_{\mathrm{E}}(P,Q) + d_{\mathrm{E}}(Q,R)$, yielding the desired result. □

Consider the diagram below.



a. Because $d_{\mathrm{T}}(A,C) = d_{\mathrm{T}}(A,C') + d_{\mathrm{T}}(C',C)$, it follows that $d_{\mathrm{T}}(C,C') = d_{\mathrm{T}}(A,C) - d_{\mathrm{T}}(A,C')$ and analogously that $d_{\mathrm{T}}(C,C') = d_{\mathrm{T}}(B,C) - d_{\mathrm{T}}(B,C')$. Then $d_{\mathrm{T}}(B,C) = d_{\mathrm{T}}(A,C) + d_{\mathrm{T}}(A,B) - 2 \cdot d_{\mathrm{T}}(A,C') = b_{\mathrm{T}} + c_{\mathrm{T}} - 2d_{\mathrm{T}}(A,C')$. The lemma applies to angle A in $\triangle AC'C$, and $\cos_{\mathrm{T}} \angle CAC' = \dfrac{d_{\mathrm{T}}(A,C')}{d_{\mathrm{T}}(A,C)}$ implies $d_{\mathrm{T}}(A,C') = d_{\mathrm{T}}(A,C) \cdot \cos_{\mathrm{T}} \angle A$. Substituting this expression for $d_{\mathrm{T}}(A,C')$ into the equation for $d_{\mathrm{T}}(B,C)$ yields the first equation.

b. The second equation is derived similarly. As before, $d_{\mathrm{T}}(A,C) = d_{\mathrm{T}}(B,C) + d_{\mathrm{T}}(A,B) - 2 \cdot d_{\mathrm{T}}(A,C') = a_{\mathrm{T}} + c_{\mathrm{T}} - 2d_{\mathrm{T}}(A,C')$. Again, apply the lemma to angle B in $\triangle BC'C$ to obtain $\cos_{\mathrm{T}} \angle CBC' = \dfrac{d_{\mathrm{T}}(B,C')}{d_{\mathrm{T}}(B,C)}$, yielding $d_{\mathrm{T}}(B,C') = d_{\mathrm{T}}(B,C) \cdot \cos_{\mathrm{T}} \angle B$. Substituting this result into the previous equation yields $d_{\mathrm{T}}(A,C) = a_{\mathrm{T}} + c_{\mathrm{T}} - 2a_{\mathrm{T}} \cos_{\mathrm{T}} \angle B$.

c. Apply the definition of taxi-cosine to obtain $\cos_{\mathrm{T}} \angle ACB = \dfrac{d_{\mathrm{E}}(A,C) \cdot d_{\mathrm{E}}(B,C)}{d_{\mathrm{T}}(A,C) \cdot d_{\mathrm{T}}(B,C)} \cos \angle ACB$.
By the Law of Cosines applied to $\triangle ABC$,

$$\cos \angle C = \frac{(d_{\mathrm{E}}(B,C))^2 + (d_{\mathrm{E}}(A,C))^2 - (d_{\mathrm{E}}(A,B))^2}{2 \cdot d_{\mathrm{E}}(A,C) \cdot d_{\mathrm{E}}(B,C)}.$$

Hence

$$\begin{aligned} \cos_{\mathrm{T}} \angle ACB &= \frac{d_{\mathrm{E}}(A,C) \cdot d_{\mathrm{E}}(B,C)}{d_{\mathrm{T}}(A,C) \cdot d_{\mathrm{T}}(B,C)} \cdot \frac{(d_{\mathrm{E}}(B,C))^2 + (d_{\mathrm{E}}(A,C))^2 - (d_{\mathrm{E}}(A,B))^2}{2 \cdot d_{\mathrm{E}}(A,C) \cdot d_{\mathrm{E}}(B,C)} \\ &= \frac{(d_{\mathrm{E}}(B,C))^2 + (d_{\mathrm{E}}(A,C))^2 - (d_{\mathrm{E}}(A,B))^2}{2 \cdot d_{\mathrm{T}}(A,C) \cdot d_{\mathrm{T}}(B,C)}. \end{aligned}$$

Use the Pythagorean Theorem in $\triangle AC'C$ and $\triangle BC'C$ to obtain

$$\cos_{\mathrm{T}} \angle ACB = \frac{(d_{\mathrm{E}}(B,C'))^2 + (d_{\mathrm{E}}(C,C'))^2 + (d_{\mathrm{E}}(A,C'))^2 + (d_{\mathrm{E}}(C,C'))^2 - (d_{\mathrm{E}}(A,C') + d_{\mathrm{E}}(B,C'))^2}{2 \cdot d_{\mathrm{T}}(A,C) \cdot d_{\mathrm{T}}(B,C)}.$$

By algebra, this equation is equivalent to $\cos_{\mathrm{T}} \angle ACB = \dfrac{h_{\mathrm{T}}^2 - p_{\mathrm{T}}(c_{\mathrm{T}} - p_{\mathrm{T}})}{a_{\mathrm{T}} \cdot b_{\mathrm{T}}}$, where $p_{\mathrm{T}} = d_{\mathrm{T}}(A,C')$.

Now, obtain expressions for h_{T}, p_{T} and $c_{\mathrm{T}} - p_{\mathrm{T}}$ by algebraic manipulation of previously-found equations relating lengths of sides.

First, find an expression for p_{T}. Note that $d_{\mathrm{T}}(A,C) = b_{\mathrm{T}} = a_{\mathrm{T}} + c_{\mathrm{T}} - 2a_{\mathrm{T}} \cos_{\mathrm{T}} \angle B$, which implies $b_{\mathrm{T}} = a_{\mathrm{T}} + c_{\mathrm{T}} - 2a_{\mathrm{T}} \left(\dfrac{d_{\mathrm{E}}(B,C) \cdot (c_{\mathrm{T}} - p_{\mathrm{T}})}{a_{\mathrm{T}} \cdot (c_{\mathrm{T}} - p_{\mathrm{T}})} \cdot \dfrac{c_{\mathrm{T}} - p_{\mathrm{T}}}{d_{\mathrm{E}}(B,C)} \right)$. After cancellation, $b_{\mathrm{T}} = a_{\mathrm{T}} + c_{\mathrm{T}} - 2(c_{\mathrm{T}} - p_{\mathrm{T}}) \rightarrow b_{\mathrm{T}} = a_{\mathrm{T}} - c_{\mathrm{T}} + 2p_{\mathrm{T}} \rightarrow p_{\mathrm{T}} = \dfrac{b_{\mathrm{T}} + c_{\mathrm{T}} - a_{\mathrm{T}}}{2}$.

A similar manipulation to the equation $b_{\mathrm{T}} = a_{\mathrm{T}} + c_{\mathrm{T}} - 2(c_{\mathrm{T}} - p_{\mathrm{T}})$ yields $c_{\mathrm{T}} - p_{\mathrm{T}} = \dfrac{a_{\mathrm{T}} + c_{\mathrm{T}} - b_{\mathrm{T}}}{2}$.

Now, notice that $h_{\mathrm{T}} = a_{\mathrm{T}} - (c_{\mathrm{T}} - p_{\mathrm{T}}) = a_{\mathrm{T}} - \dfrac{a_{\mathrm{T}} + c_{\mathrm{T}} - b_{\mathrm{T}}}{2}$, which simplifies to $\dfrac{2a_{\mathrm{T}}}{2} - \dfrac{a_{\mathrm{T}} + c_{\mathrm{T}} - b_{\mathrm{T}}}{2} = \dfrac{a_{\mathrm{T}} - c_{\mathrm{T}} + b_{\mathrm{T}}}{2}$, as needed. Substitution yields

$$\cos_{\mathrm{T}} \angle ACB = \frac{\left(\dfrac{b_{\mathrm{T}} + c_{\mathrm{T}} - a_{\mathrm{T}}}{2}\right)^2 - \left(\dfrac{b_{\mathrm{T}} + c_{\mathrm{T}} - a_{\mathrm{T}}}{2}\right)\left(\dfrac{a_{\mathrm{T}} + c_{\mathrm{T}} - b_{\mathrm{T}}}{2}\right)}{a_{\mathrm{T}} \cdot b_{\mathrm{T}}},$$

which simplifies to $\cos_{\mathrm{T}} \angle ACB = \dfrac{a_{\mathrm{T}}^2 + b_{\mathrm{T}}^2 - c_{\mathrm{T}}(a_{\mathrm{T}} + b_{\mathrm{T}})}{2 \cdot a_{\mathrm{T}} \cdot b_{\mathrm{T}}}$.
This equation can be algebraically manipulated to obtain the desired result.

Alternate Solution: To simplify the computations, introduce notation for various Euclidean lengths. Let

$$a = d_{\mathrm{E}}(B,C), \qquad b = d_{\mathrm{E}}(A,C), \qquad c = d_{\mathrm{E}}(A,B) = c_{\mathrm{T}};$$
$$p = d_{\mathrm{E}}(A,C') = d_{\mathrm{T}}(A,C') = b\cos A;$$
$$q = d_{\mathrm{E}}(B,C') = d_{\mathrm{T}}(B,C') = a\cos B.$$

Then

$$a_{\mathrm{T}} = q + h_{\mathrm{T}}, \qquad b_{\mathrm{T}} = p + h_{\mathrm{T}}, \qquad c_{\mathrm{T}} = p + q.$$

a. From the definition of the taxi-cosine, $b_{\mathrm{T}}\cos_{\mathrm{T}} A = b_{\mathrm{T}}\cdot\frac{b\cdot c}{b_{\mathrm{T}}\cdot c_{\mathrm{T}}}\cos A = b\cos A = p$. Therefore $b_{\mathrm{T}} + c_{\mathrm{T}} - 2b_{\mathrm{T}}\cos_{\mathrm{T}} A = (p+h_{\mathrm{T}}) + (p+q) - 2p = q + h_{\mathrm{T}} = a_{\mathrm{T}}$.

b. Similarly, $a_{\mathrm{T}}\cos_{\mathrm{T}} B = q$, so $a_{\mathrm{T}} + c_{\mathrm{T}} - 2a_{\mathrm{T}}\cos_{\mathrm{T}} B = (q+h_{\mathrm{T}}) + (p+q) - 2q = b_{\mathrm{T}}$.

c. The Euclidean radian measures of the angles add up to π, so $\cos C = \cos(\pi - A - B) = -\cos(A+B) = \sin A\,\sin B - \cos A\,\cos B$. Substitute into the definition of the taxi-cosine and multiply by $a_{\mathrm{T}}b_{\mathrm{T}}$:

$$\begin{aligned} a_{\mathrm{T}}b_{\mathrm{T}}\cos_{\mathrm{T}} C &= a_{\mathrm{T}}b_{\mathrm{T}}\frac{ab}{a_{\mathrm{T}}b_{\mathrm{T}}}(\sin A\,\sin B - \cos A\,\cos B) \\ &= (b\sin A)(a\sin B) - (b\cos A)(a\cos B) \\ &= h_{\mathrm{T}}^2 - pq \\ &= h_{\mathrm{T}}^2 - (b_{\mathrm{T}} - h_{\mathrm{T}})(a_{\mathrm{T}} - h_{\mathrm{T}}) \\ &= (a_{\mathrm{T}} + b_{\mathrm{T}})h_{\mathrm{T}} - a_{\mathrm{T}}b_{\mathrm{T}}. \end{aligned}$$

Therefore $a_{\mathrm{T}}^2 + b_{\mathrm{T}}^2 - 2a_{\mathrm{T}}b_{\mathrm{T}}\cos_{\mathrm{T}} C = a_{\mathrm{T}}^2 + b_{\mathrm{T}}^2 - 2(a_{\mathrm{T}}+b_{\mathrm{T}})h_{\mathrm{T}} + 2a_{\mathrm{T}}b_{\mathrm{T}} = (a_{\mathrm{T}}+b_{\mathrm{T}})(a_{\mathrm{T}} + b_{\mathrm{T}} - 2h_{\mathrm{T}}) = (a_{\mathrm{T}}+b_{\mathrm{T}})c_{\mathrm{T}}$. Solving for c_{T} gives the desired result.

10. In general, the area of a triangle ABC is equal to half of the area of the parallelogram determined by any two sides of triangle ABC. The definition of taxi-sine, namely that

$$\sin_{\mathrm{T}}\angle AOB = \left(\frac{d_{\mathrm{E}}(O,A)\cdot d_{\mathrm{E}}(O,B)}{d_{\mathrm{T}}(O,A)\cdot d_{\mathrm{T}}(O,B)}\right)(\sin\angle AOB),$$

implies that

$$d_{\mathrm{T}}(O,A)\cdot d_{\mathrm{T}}(O,B)\cdot\sin_{\mathrm{T}}\angle AOB = d_{\mathrm{E}}(O,A)\cdot d_{\mathrm{E}}(O,B)\cdot\sin\angle AOB.$$

The latter expression is equal to $|\overrightarrow{OA}\times\overrightarrow{OB}|_{\mathrm{T}}$. Thus the cross-product of two vectors can be interpreted in a taxicab space as it is in a Euclidean space!

Now, express the area of the triangle ABC in three different ways:

$$[ABC] = \frac{|\overrightarrow{AB}\times\overrightarrow{AC}|_{\mathrm{T}}}{2} = \frac{|\overrightarrow{AB}\times\overrightarrow{BC}|_{\mathrm{T}}}{2} = \frac{|\overrightarrow{AC}\times\overrightarrow{BC}|_{\mathrm{T}}}{2}.$$

Also notice that $|\overrightarrow{AB} \times \overrightarrow{AC}|_{\mathrm{T}} = c_{\mathrm{T}} \cdot b_{\mathrm{T}} \cdot \sin_{\mathrm{T}} A$ and $|\overrightarrow{AB} \times \overrightarrow{BC}|_{\mathrm{T}} = c_{\mathrm{T}} \cdot a_{\mathrm{T}} \cdot \sin_{\mathrm{T}} B$ and $|\overrightarrow{AC} \times \overrightarrow{BC}|_{\mathrm{T}} = b_{\mathrm{T}} \cdot a_{\mathrm{T}} \cdot \sin_{\mathrm{T}} C$. By substitution and multiplication by 2, $c_{\mathrm{T}} \cdot b_{\mathrm{T}} \cdot \sin_{\mathrm{T}} A = c_{\mathrm{T}} \cdot a_{\mathrm{T}} \cdot \sin_{\mathrm{T}} B = b_{\mathrm{T}} \cdot a_{\mathrm{T}} \cdot \sin_{\mathrm{T}} C$. Dividing by the nonzero quantity $a_{\mathrm{T}} b_{\mathrm{T}} c_{\mathrm{T}}$ and taking reciprocals leads to the desired result.

Alternate Solution: As in the solution to problem 9, let a, b, and c denote the Euclidean lengths of the sides. With this notation, the definition of the taxi-sine gives

$$\sin_{\mathrm{T}} C = \frac{ab}{a_{\mathrm{T}} b_{\mathrm{T}}} \sin C = \frac{ab \sin C}{a_{\mathrm{T}} b_{\mathrm{T}}}, \quad \text{thus} \quad \frac{c_{\mathrm{T}}}{\sin_{\mathrm{T}} C} = \frac{a_{\mathrm{T}} b_{\mathrm{T}} c_{\mathrm{T}}}{abc} \frac{c}{\sin C}.$$

Hence the Taxicab Law of Sines follows from the Euclidean version, multiplying by $\dfrac{a_{\mathrm{T}} b_{\mathrm{T}} c_{\mathrm{T}}}{abc}$.

Note: Taxicab geometry is an active area of mathematical research, so many important questions remain unanswered. Students who are interested in taxicab geometry might find the following sources helpful; both were inspirational in the creation of this Power Question.

- Eugene Krause, *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*, Dover, 1975.
- Ayşe Bayar, Süheyla Ekmekçi and Münevver Özcan, *On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane*, International Electronic Journal of Geometry, **2** (2009), 17–24.

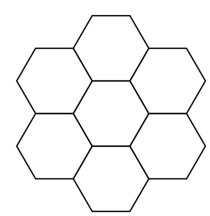
2016 Relay Problems

R1-1. Compute the number of lattice points on the graph of $y = (x - 153)^{(153 - x^2)}$.

R1-2. Let T = TNYWR. Let r and s be the zeros of $x^2 - 4Tx + T^2$. Compute $\sqrt{r} + \sqrt{s}$.

R1-3. Let T = TNYWR. Among all triples of integers (a, b, c) satisfying $2^a + 4^b = 8^c$, compute the least value of $a + b + c$ greater than T^2.

R2-1. Compute the number of ways to assign the integers $\{1, 2, \ldots, 7\}$ to each of the hexagons in the figure to the right so that every pair of numbers in edge-adjacent hexagons is relatively prime.



R2-2. Let T = TNYWR. In right triangle ABC with hypotenuse $\overline{BC}$, $AB = T$ and $BC = \frac{2T}{\sqrt{3}}$. A circle passes through A and is tangent to $\overline{BC}$ at its midpoint. Compute the radius of the circle.

R2-3. Let T = TNYWR. Compute the number of ways to make \$$T$ from an unlimited supply of \$10 bills, \$5 bills, and \$1 bills.

2016 Relay Answers

R1-1. 27

R1-2. $9\sqrt{2}$

R1-3. 169

R2-1. 72

R2-2. 24

R2-3. 9

2016 Relay Solutions

R1-1. For an integer x, $(x-153)^{(153-x^2)}$ will be an integer if and only if one of the following three conditions is satisfied:

(1) $153 - x^2 \geq 0$,

(2) $x - 153 = -1$ or 1,

(3) $x - 153 = 0$ and $153 - x^2 > 0$.

For the first case, there are 25 solutions for $-12 \leq x \leq 12$. For the second case, there are two solutions ($x = 152$ or 154). There are no solutions for the third case, as $153 - 153^2 < 0$. Hence there are **27** lattice points on the graph.

R1-2. Let $u = \sqrt{r} + \sqrt{s}$. Then $u^2 = r + s + 2\sqrt{rs}$. The sum of the zeros of $x^2 - 4Tx + T^2$ is $4T$ and their product is T^2. Thus $u^2 = 4T + 2|T|$. With $T = 27$, $u^2 = 6 \times 27$, so $u = \mathbf{9\sqrt{2}}$.

R1-3. Rewrite the equation with common bases to obtain $2^a + 2^{2b} = 2^{3c}$. The only case in which the sum of two integer powers of 2 equals a third integer power of 2 occurs when the summands have equal exponents, so $a = 2b$, $3c = a+1$, and $a+b+c = a + \frac{a}{2} + \frac{a+1}{3} = \frac{11a+2}{6}$. For all three values to be integers, a must be an even number of the form $3k-1$. As $T = 9\sqrt{2}$, $T^2 = 162$ and $\frac{11a+2}{6} \geq 162 \Rightarrow 11a + 2 \geq 972 \Rightarrow a \geq \frac{970}{11} > 88$. The least even integer value a of the form $3k-1$ greater than 88 is 92, and the sum is $92 + 46 + 31 = \mathbf{169}$.

R2-1. No two even numbers can be adjacent, so the three even numbers must be in alternating hexagons on the boundary. There are two ways to select alternating hexagons on the boundary, and six ways to assign the even numbers to the three hexagons. The number 3 cannot be adjacent to 6, hence the 3 must be opposite the 6. Then the remaining odd numbers can be assigned in any way in the remaining three hexagons. Hence the total number of arrangements is $2 \times 6 \times 6 = \mathbf{72}$.

R2-2. Note that $\triangle ABC$ is a $30-60-90$ right triangle, so $AC = \frac{T}{\sqrt{3}}$. Let point C' be obtained by reflecting point C about $\overline{AB}$ so that $\triangle CBC'$ is equilateral. Note that the circle which passes through A and which is tangent to $\overline{BC}$ at its midpoint happens to be the incircle of $\triangle CBC'$! The radius of the incircle is one-third the altitude of $\triangle CBC'$, so the radius is $\frac{T}{3}$. As $T = 72$, the radius is **24**.

R2-3. Let $T = 10t + 5f + n$, where t, f, and n are non-negative integers representing the number of tens, fives, and ones, respectively. Then $2t + f \leq \lfloor \frac{T}{5} \rfloor$, with the remaining total consisting of \$1 bills. The number of ways to make \$$T$ is equal to the number of lattice points (t, f) contained in the triangle bounded by the lines $t = 0$, $f = 0$, and $2t + f = \lfloor \frac{T}{5} \rfloor$. When $t = 0$, there are $\lfloor \frac{T}{5} \rfloor + 1$ possible values of f. Incrementing t by one reduces by two the number of values of f for which there is a solution. As $T = 24$, $\lfloor \frac{T}{5} \rfloor + 1 = 5$, and there are $5 + 3 + 1 = \mathbf{9}$ solutions: $(t, f, n) = (0,0,24), (0,1,19), (0,2,14), (0,3,9), (0,4,4), (1,0,14), (1,1,9), (1,2,4)$, and $(2,0,4)$.

2016 Tiebreaker Problems

TB-1. Compute the least value of N such that there are exactly 43 ordered quadruples of positive integers (a, b, c, d) satisfying $N = a^2 + b^2 + c^2 + d^2$.

TB-2. Regular octagon $HEPTAGON$ has legs of length 8. Segments $\overline{ET}$ and $\overline{PA}$ intersect at X. Compute the area of heptagon $HEXAGON$.

TB-3. Compute the base-10 value of $111_2 + 222_3 + 333_4 + \cdots + 888_9$.

2016 Tiebreaker Answers

TB-1. 100

TB-2. $112 + 112\sqrt{2}$

TB-3. 2016

2016 Tiebreaker Solutions

TB-1. To understand how to count the number of ordered quadruples that satisfy the given equation, it is useful to consider a particular value of N and a particular solution to the resulting equation. So, for example, with $N = 18$, one solution is $(1, 2, 2, 3)$. Because of the symmetry of the expression $a^2 + b^2 + c^2 + d^2$, any permutation of $(1, 2, 2, 3)$ is also a solution, so there are $\frac{4!}{2!} = 12$ different quadruples corresponding to the same values; call the set of all such quadruples a *solution class*. This observation suggests starting by identifying different possible sizes of solution classes. Assuming that the initial values satisfy $a \le b \le c \le d$, the different solution classes and their sizes are given in the following table.

Solution class	Size
$a < b < c < d$	$4! = 24$
$a = b < c < d$ $a < b = c < d$ $a < b < c = d$	$\frac{4!}{2!} = 12$
$a = b < c = d$	$\frac{4!}{2!2!} = 6$
$a = b = c < d$ $a < b = c = d$	$\frac{4!}{3!} = 4$
$a = b = c = d$	1

The goal is then to identify the least value of N such that the sizes of the different solution classes sum to 43. Note that 43 is odd, while all but one of the class sizes are even. So for some positive integer m, $N = m^2 + m^2 + m^2 + m^2 = 4m^2 = (2m)^2$. Thus N is not only even, it is the square of an even number. Furthermore, note that all other solution classes except the case $a = b < c = d$ have sizes divisible by four. Because 42 is not divisible by four, there must also be a solution of the form $N = a^2 + a^2 + c^2 + c^2 = 2a^2 + 2c^2$, which implies that half of N is a sum of squares of distinct integers.

These two insights rule out several small cases, because for $m = 1, 2, 3, 4$, $N/2 = 2, 8, 18, 32$ respectively, and none of these is the sum of two *distinct* squares. On the other hand, $m = 5$ yields $N = 100$, and $N/2 = 50 = 1^2 + 7^2$, which is promising! In fact,

$$\begin{aligned} 100 &= 9^2 + 3^2 + 3^2 + 1^2 \\ &= 8^2 + 4^2 + 4^2 + 2^2 \\ &= 7^2 + 5^2 + 5^2 + 1^2, \end{aligned}$$

yielding a total of $3 \cdot 12 + 6 + 1 = 43$ solutions. For $N = 100$, note that if the resulting equation had a solution of the form $a = b = c < d$, then because 100 is not a multiple of 3, d cannot be a multiple of 3. Similarly, if a solution of the form $a < b = c = d$ existed, then a cannot be a multiple of 3. But because none of $(100 - 1^2)/3$, $(100 - 2^2)/3$, $(100 - 4^2)/3$, $(100 - 7^2)/3$, or $(100 - 8^2)/3$ is a perfect square, and because $(100 - 5^2)/3 = 25 = 5^2$, corresponds to the case $100 = 5^2 + 5^2 + 5^2 + 5^2$, which has already been accounted for, it follows that there are no solutions of the form $a = b = c < d$ or $a < b = c = d$.

Lastly, for $N = 100$, the equation has no solutions of the form $a < b < c < d$. To see this, note that a perfect square is either congruent to 1 or 0 modulo 4, depending on its parity. Given that $a^2 + b^2 + c^2 + d^2 = 100 \equiv 0 \pmod 4$, it follows that a, b, c, and d must all have the same parity. If all are even, then the least possible value of $a^2 + b^2 + c^2 + d^2$ with $a < b < c < d$ would be $2^2 + 4^2 + 6^2 + 8^2 = 120$ which is larger than 100. If all are odd, then the least value is $1^2 + 3^2 + 5^2 + 7^2 = 84 \neq 100$. The second smallest value is $1^2 + 3^2 + 5^2 + 9^2 = 116$. Thus there are no solutions in the solution class $a < b < c < d$, and so the least value of N for which the equation $a^2 + b^2 + c^2 + d^2 = N$ has 43 solutions is **100**.

TB-2. Dissect heptagon $HEXAGON$ into hexagon $HEAGON$ and $\triangle AXE$. First compute the area of hexagon $HEAGON$. Extend $\overline{HE}$ and $\overline{ON}$ to meet at Q, and extend $\overline{ON}$ and $\overline{AG}$ to meet at R. Note that $\triangle HNQ$ and $\triangle GOR$ are isosceles right triangles each with hypotenuse of length 8. Thus each triangle has legs of length $4\sqrt{2}$ and therefore $[HNQ] = [GOR] = \frac{1}{2}(4\sqrt{2})^2 = 16$. Therefore

$$\begin{aligned}[HEAGON] &= [AEQR] - [HNQ] - [GOR] \\ &= QR \cdot QE - 16 - 16 \\ &= (QN + NO + OR)(QH + HE) - 32 \\ &= (8 + 8\sqrt{2})(8 + 4\sqrt{2}) - 32 \\ &= 96 + 96\sqrt{2}.\end{aligned}$$

Next, compute $[AXE]$. Draw in altitude $\overline{XY}$ from X to $\overline{EA}$, so that

$$[AXE] = \tfrac{1}{2}EA \cdot XY = (4 + 4\sqrt{2}) \cdot XY.$$

To complete this computation, determine XY as follows. Draw $\overline{HA}$. Because $\overline{HA} \parallel \overline{ET}$ and $\overline{HE} \parallel \overline{XY}$, triangles HEA and XYE are similar. Specifically, $EY = \frac{1}{2}EA$, so $XY = \frac{1}{2}HE = 4$. Thus

$$[AXE] = (4 + 4\sqrt{2}) \cdot 4 = (16 + 16\sqrt{2}).$$

Finally, combine these computations to obtain

$$\begin{aligned}[HEXAGON] &= [HEAGON] + [AXE] \\ &= (96 + 96\sqrt{2}) + (16 + 16\sqrt{2}) \\ &= \mathbf{112 + 112\sqrt{2}}.\end{aligned}$$

TB-3. Note that $\underline{m}\,\underline{m}\,\underline{m}_{m+1} = (m+1)^3 - 1$. So the desired sum S is

$$\begin{aligned}S &= (2^3 - 1) + \cdots + (9^3 - 1) \\ &= (2^3 + \cdots + 9^3) - 8 \\ &= (1^3 + 2^3 + \cdots + 9^3) - 8 - 1^3.\end{aligned}$$

Use the identity $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$ to obtain

$$S = \left(\frac{9 \cdot 10}{2}\right)^2 - 9 = 45^2 - 9 = \mathbf{2016}.$$

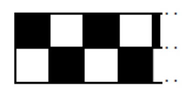
2016 Super Relay Problems

1. Given that a, b, and c are positive integers with $ab = 20$ and $bc = 16$, compute the positive difference between the maximum and minimum possible values of $a + b + c$.

2. Let T = TNYWR. Mary circles a number in one of the first six rows of Pascal's Triangle, shown at right. Jim then circles another number—possibly having the same value as Mary's number—but Jim's circle cannot coincide with Mary's circle. For example, if Mary circles the top "1", then Jim can circle any number other than the top "1". The product of the two circled numbers is then computed. Compute the number of possible distinct products that are **greater** than T.

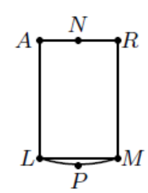
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

3. Let T = TNYWR. The diagram at right shows part of a $2 \times T$ checkerboard pattern. A 1×3 tile is randomly placed on the board so that it completely covers three unit squares. Compute the probability that two of the covered squares are black.



4. Let T = TNYWR. Leo rides his bike at $16T$ miles per hour and Carla and Rita both ride their bikes at 12 miles per hour. Leo and Carla start at the points $(0, 0)$ and $(60, 0)$, respectively, and they begin biking towards each other (along the x-axis), where the units of both coordinate axes are in miles. At the same time, Rita starts biking from the point (a, b), where $a > 0$, and she is riding along the line $x = a$, towards the x-axis. Compute the value of a such that Leo, Carla, and Rita will meet at the same moment.

5. Let T = TNYWR. In rectangle $ARML$, diagrammed at right (not drawn to scale), N is the midpoint of $\overline{AR}$, $NL = 40$, and $AN = T$. Arc $\widehat{LPM}$ is an arc of a circle centered at N, P is the midpoint of arc $\widehat{LPM}$, and arc $\widehat{LP'M}$ is the reflection of $\widehat{LPM}$ across $\overline{LM}$. Given that P' is the midpoint of arc $\widehat{LP'M}$, compute NP'.



6. Let T = TNYWR. Let p and q be distinct primes and for each positive integer n, let $d(n)$ be the number of positive divisors of n. Compute the least possible positive integer k such that the quotient $\dfrac{d(p^{T+2}q^k)}{d(p^{T+1}q^2)}$ is an integer.

7. Let T = TNYWR. An ARML team of 15 students contains k boys and $15-k$ girls. The value of k satisfies the equation $\binom{T+1}{3} = 325k$. Compute the probability that a randomly chosen student from the team is a boy.

8. Let t be the number you will receive from position 7 and let s be the number you will receive from position 9. In $\triangle ABC$, point H lies on $\overline{BC}$ so that $\overline{AH} \perp \overline{BC}$. Given that the sides of $\triangle ABC$ are integers, $\tan\angle B = t$, and $\sin\angle CAH = s$, compute the least possible perimeter of $\triangle ABC$.

9. Let T = TNYWR, and let $K = \left\lfloor \frac{T}{\pi} \right\rfloor$. Consider the sequence defined by $a_1 = 20$, $a_2 = 16$, and for $n \geq 3$, a_n is the units digit of $a_{n-1} + a_{n-2}$. Compute $\frac{a_K}{10}$.

10. Let T = TNYWR. In circle O, perpendicular chords $\overline{AR}$ and $\overline{ML}$ intersect at N. Given that $AN = T$, $RN = 5$, and $MN = 25$, compute the area of circle O.

11. Let T = TNYWR. John has T distinct baseball cards and Benson has $2T$ distinct baseball cards, which include the same T cards that John has. John randomly chooses two cards from Benson's deck. The probability that exactly one of the chosen cards is **not** in John's deck can be expressed in the form $\frac{T}{K}$. Compute K.

12. Let T = TNYWR. Compute the value of k such that the following system of equations has a solution.

$$\begin{aligned} x + y &= T \\ 20x + 16y &= 8 \\ kx + 20y &= 16. \end{aligned}$$

13. Let T = TNYWR. Compute $2^{\log_T 8} - 8^{\log_T 2}$.

14. Let T = TNYWR. Let $A = 2^K, R = A^{20}, M = R^{16}$, and $L = M^T$. Given that K and L are positive integers, compute the least possible value of A.

15. A dartboard is made up of two concentric circles that have radii 20 and 16. A dart is thrown at random and hits the board. Compute the probability that the dart lands in the circle of radius 16.

2016 Super Relay Answers

1. 24

2. 6

3. $\frac{1}{2}$

4. 24

5. 24

6. 25

7. $\frac{8}{15}$

8. 48

9. $\frac{3}{5}$

10. 949π

11. 55

12. 28

13. 0

14. 32

15. $\frac{16}{25}$

2016 Super Relay Solutions

1. Note that 5 must divide a, hence a must be either $5, 10$, or 20. Thus the possible solutions (a, b, c) are $(5, 4, 4)$, $(10, 2, 8)$, and $(20, 1, 16)$. The first of these gives the minimum possible sum $a+b+c$ of 13 while the last of these gives the maximum possible sum $a+b+c$ of 37. Thus the answer is $37 - 13 = \mathbf{24}$.

2. Before receiving a value of T, it makes sense to list out all the possible products. There are **6** products that are greater than T $(= 24)$: 25 (5×5), 30 $(5 \times 6 = 3 \times 10)$, 40 (4×10), 50 (5×10), 60 (6×10), and 100 (10×10).

3. Note that for any possible position of the 1×3 tile's left boundary, there are exactly two placements of the tile (in the first row or in the second row). Exactly one of these placements will cover two white squares and one black square, and the other will cover two black squares and one white square. Because these options are equally likely, the probability that the 1×3 tile covers two black squares is therefore $\mathbf{\frac{1}{2}}$ (independent of the value of T).

4. First note that because Carla and Rita ride at the same speed, Carla should ride a distance of b (i.e., Rita's initial y-coordinate) and Leo should ride a distance of $a = 60 - b$ in order for them to meet at the same time. If k is the ratio of Leo's speed to Carla's and Rita's speed, then $\frac{60-b}{b} = \frac{a}{60-a} = r$, hence $a = \frac{60r}{1+r} = \frac{240T}{3+4T}$. With $T = \frac{1}{2}$, it follows that $a = \mathbf{24}$.

5. Let $\overline{NP}$ intersect $\overline{LM}$ at Q. Because $NP = NL = 40$, it follows that $PQ = 40 - AL$. Because $P'Q = PQ$, it follows that $NP' = AL - (40 - AL) = 2 \cdot AL - 40$. With $T = 24$, it follows that $\triangle NAL$ is similar to a $3{-}4{-}5$ triangle, hence $AL = 32$, and $NP' = \mathbf{24}$.

6. Note that $d(p^{T+2}q^k) = (T+3)(k+1)$ because a positive divisor of $p^{T+2}q^k$ is of the form $p^x q^y$, where x and y are integers satisfying $0 \le x \le T+2$ and $0 \le y \le k$. Similarly, $d(p^{T+1}q^2) = 3(T+2)$. Thus the quotient is $\dfrac{(T+3)(k+1)}{3(T+2)}$. Note that $T+3$ and $T+2$ are relatively prime, so either $k = T+1$ if 3 divides $T+3$ or $k = 3T+5$ if 3 does not divide $T+3$. With $T = 24$, the former case applies, thus $k = \mathbf{25}$.

7. With $T = 25$, $\binom{26}{3} = \frac{26 \cdot 25 \cdot 24}{3!} = 2600$. Thus $k = 2600/325 = 8$, and the desired probability is $\mathbf{\frac{8}{15}}$.

8. Let $BH = x$ and $AC = y$. Then $AH = tx$ and $CH = sy$. By the Pythagorean Theorem, $AB = x\sqrt{1+t^2}$ and $t^2x^2 + s^2y^2 = y^2$, hence $AC = y = \frac{tx}{\sqrt{1-s^2}}$. With $t = \frac{8}{15}$ and $s = \frac{3}{5}$, note that triangles AHB and CHA are similar to $8{-}15{-}17$ and $3{-}4{-}5$ triangles, respectively.

Setting $x = 15$ gives $AH = 8$, $AB = 17$, $CH = 6$, and $AC = 10$. Thus the minimum perimeter of $\triangle ABC$ is $17 + 10 + (15 + 6) = \mathbf{48}$.

9. Ignoring the tens digits of the first two terms, the sequence is

$$0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0, 2, 2, 4, 6, 0, 6, 6, 2, \ldots .$$

Note that this sequence is periodic with period 20. Thus when m is an integer, $a_{20m} = 6$. With $T = 949\pi$, $K = 949$, and $a_{940} = 6$. Counting 9 terms into the next block of 20 terms, conclude that $a_{949} = 6$, hence the desired ratio is $\mathbf{\frac{3}{5}}$.

10. Note that by Power of a Point, $LN = \frac{AN \cdot RN}{MN} = \frac{T}{5}$. With $T = 55$, $LN = 11$. Let P be the foot of the perpendicular from O to $\overline{AR}$ and let Q be the foot of the perpendicular from O to $\overline{ML}$. Note that $AP = \frac{1}{2}(55 + 5) = 30$ and $MQ = \frac{1}{2}(25 + 11) = 18$. Because chords $\overline{AR}$ and $\overline{ML}$ are perpendicular, it follows that $OP = QN = MN - MQ = 7$ and that $OQ = PN = PR - NR = 30 - 5 = 25$. Letting r be the radius of circle O, it follows that $r^2 = 30^2 + 7^2 = 18^2 + 25^2 = 949$, hence the answer is $\mathbf{949\pi}$.

11. There are $\binom{2T}{2} = T(2T-1)$ possible selections of two cards that John can choose from Benson's deck. Of those, there will be T^2 pairs of cards, exactly one of which is not in John's deck. Hence the desired probability is $\frac{T^2}{T(2T-1)} = \frac{T}{2T-1}$, hence $K = 2T - 1$. With $T = 28$, $K = \mathbf{55}$.

12. Subtract sixteen times the first equation from the second equation to obtain $4x = 8 - 16T$. Thus $x = 2 - 4T$. Plug this into the first equation to get $y = T - x = 5T - 2$. Plug in these values for x and y into the third equation to obtain $k(2 - 4T) + 20(5T - 2) = 16 \rightarrow k = \dfrac{28 - 50T}{1 - 2T}$. With $T = 0$, it follows that $k = \frac{28}{1} = \mathbf{28}$.

13. Note that $2^{\log_T 8} = 2^{\log_T 2^3} = 2^{3 \log_T 2} = 8^{\log_T 2}$. Thus, so long as T is positive and $T \neq 1$ (to ensure that the function $\log_T$ is defined), the expression $2^{\log_T 8} - 8^{\log_T 2}$ equals 0. Because $T = 32$ satisfies these constraints, the answer is $\mathbf{0}$.

14. Note that $R = (A^K)^{20} = A^{20K}$, $M = R^{16} = A^{320K}$, and $L = M^T = A^{320KT}$. If T is a positive integer, then $K = 1$ suffices. On the other hand, if T is a rational number (say $\frac{c}{d}$ in lowest terms), but not a positive integer, then $K = \frac{d}{\gcd(320,d)}$. With $T = \frac{16}{25}$, $K = \frac{25}{\gcd(320,25)} = 5$. Thus $A = 2^5 = \mathbf{32}$.

15. The inner circle has radius 16 and therefore has area $\pi \cdot 16^2$. The outer circle has radius 20 and therefore has area $\pi \cdot 20^2$. The probability that a dart lands in the inner circle is the ratio of these areas, so the desired probability is $\frac{\pi \cdot 16^2}{\pi \cdot 20^2} = \left(\frac{16}{20}\right)^2 = \left(\frac{4}{5}\right)^2 = \mathbf{\frac{16}{25}}$.

2017 Contest

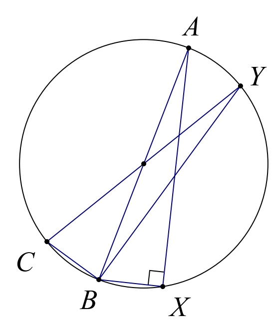
Contents

2017 Team Problems

T-1. Compute the number of ordered triples of positive integers (a, b, c) such that $\sqrt{a^b + c!} = 28$.

T-2. Chris the frog begins on a number line at 0. Chris takes jumps of lengths $1, 2, 3, \ldots, 2017$, in that order. If Chris's current location is an even integer, he jumps in the positive direction; otherwise, he jumps in the negative direction. Let $P(n)$ denote Chris's location after the n^{th} jump. Compute $\sum_{j=1}^{2017} P(j)$.

T-3. The diagram below shows arc $\widehat{ABC}$, which has a measure of $210°$. Points X and Y lie on the arc so that $\mathrm{m}\angle AXB = 90°$ and $\triangle ABX \cong \triangle YCB$. Given that $AX = 8$, the value of $[ABC]$ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are integers and c is not divisible by any perfect square greater than 1. Compute the ordered triple (a, b, c).



T-4. Ari repeatedly rolls a standard, fair, six-sided die. Let $R(n)$ be the n^{th} number rolled, and let $Q(n) = R(1) \cdot R(2) \cdot \ldots \cdot R(n)$. Compute the probability that there exists an n such that $Q(n) = 100$ and for all $m < n$, $Q(m)$ is not a perfect square.

T-5. Given that C, A, T, F, I, S, and H are digits, not necessarily distinct, and that

$$3 \cdot \underline{C}\ \underline{A}\ \underline{T} \cdot \underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = \underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H},$$

compute the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$.

T-6. Let $\{a_n\}$ be a sequence with $a_0 = 1$, and for all $n > 0$, $a_n = \dfrac{1}{2}\sum_{i=0}^{n-1} a_i$. Compute the greatest value of n for which $a_n < 2017$.

T-7. On July 17, 2017, the nation of Armlandia will turn n^2 years old and the nation of Nysmlistan will turn n years old. The next four anniversaries for which Nysmlistan's age divides Armlandia's age will occur, in order, on July 17 in the years 2027, 2032, 2038, and M. Compute M.

T-8. Ellipse $\mathcal{E}$ has center O, major axis of length 10, and minor axis of length 4. Ellipse $\mathcal{E}'$ is obtained by rotating $\mathcal{E}$ counterclockwise about O by $60°$. Proceeding clockwise around the perimeter of $\mathcal{E}$, the intersection points of $\mathcal{E}$ and $\mathcal{E}'$ are labeled A, R, M, L. Compute $[AOL]$.

T-9. Let $E(n)$ denote the least integer strictly greater than n whose base-10 representation contains only even digits, and let $O(n)$ denote the least integer strictly greater than n whose base-10 representation contains only odd digits. Compute the least positive integer N for which $E(N) + O(N)$ is a multiple of 2017.

T-10. Compute the number of tilings of a 4×7 rectangle using only 1×1, 2×2, 3×3, and 4×4 tiles.

2017 Team Answers

T-1. 9

T-2. 1

T-3. $(64, -32, 3)$

T-4. $\dfrac{3}{250}$

T-5. 6673335

T-6. 21

T-7. 2062

T-8. $\dfrac{200\sqrt{2923}}{2923}$

T-9. 64026

T-10. 1029

2017 Team Solutions

T-1. Squaring both sides of the given equation gives $a^b + c! = 784$. Accordingly, only values of c up to 6 need be considered, as $7! > 784$.

c	$784 - c!$	prime factorization of $784 - c!$
1	783	$3^3 \cdot 29$
2	782	$2 \cdot 17 \cdot 23$
3	778	$2 \cdot 389$
4	760	$2^3 \cdot 5 \cdot 19$
5	664	$2^3 \cdot 83$
6	64	2^6

For $1 \leq c \leq 5$, the only ordered triple satisfying the equation is $(784 - c!, 1, c)$. For $c = 6$, there are four triples: $(2, 6, 6)$, $(4, 3, 6)$, $(8, 2, 6)$, and $(64, 1, 6)$, for a total of **9** ordered triples.

T-2. List where Chris has landed after his first few jumps.

Jump	Before	Direction	After
1	0	$+$	1
2	1	$-$	-1
3	-1	$-$	-4
4	-4	$+$	0

In fact, this pattern continues: Chris lands back at the origin after every fourth jump. Suppose that $P(4n) = 0$. Then the next four jumps are as follows.

Jump	Before	Direction	After
$4n+1$	0	$+$	$4n+1$
$4n+2$	$4n+1$	$-$	-1
$4n+3$	-1	$-$	$-4n-4$
$4n+4$	$-4n-4$	$+$	0

This inductive argument shows that Chris lands back at the origin after every four jumps. Moreover, the sum of the four values of $P(j)$ is constant:

$$\begin{aligned} P(4n+1) + P(4n+2) + P(4n+3) + P(4n+4) \\ &= (4n+1) + (-1) + (-4n-4) + 0 \\ &= -4. \end{aligned}$$

Compute $P(1)+\cdots+P(2017)$ by grouping terms in groups of four:

$$\begin{aligned}\sum_{j=1}^{2017} P(j) &= \left(\sum_{k=0}^{503}\Big(P(4k+1)+P(4k+2)+P(4k+3)+P(4k+4)\Big)\right)+P(2017)\\ &= \left(\sum_{k=0}^{503} -4\right)+P(2017)\\ &= -2016+P(2017).\end{aligned}$$

Because 2016 is divisible by 4, after jump 2016, Chris will be at the origin. After jump 2017, Chris will be at 2017, so $P(2017)=2017$, and the sum is

$$\sum_{j=1}^{2017} P(j) = -2016+2017 = \mathbf{1}.$$

T-3. Because $\angle AXB$ is a right angle, it follows that $\overline{AB}$ is a diameter of the circle containing points A, B, and C. Thus $\angle ACB$ is also right angle. Also $\angle BAC \cong \angle BYC$ because both angles subtend $\widehat{BC}$. Because $\angle BYC \cong \angle XAB$, conclude that $\triangle ABC \cong \triangle ABX$. Next, note that $\mathrm{m}\angle XAB = \mathrm{m}\angle BYC = \frac{1}{2}\cdot \mathrm{m}\widehat{BC} = \frac{1}{2}(210^\circ - 180^\circ) = 15^\circ$. Thus $[ABC] = [ABX] = \frac{1}{2}\cdot AX\cdot BX = \frac{1}{2}\cdot 8\cdot(8\tan 15^\circ) = 32\tan 15^\circ$. The value of $\tan 15^\circ$ can be computed by using the tangent half-angle identity: $\tan(\frac{x}{2}) = \frac{\sin x}{1+\cos x}$ with $x = 30^\circ$, yielding $\tan 15^\circ = \frac{\sin 30^\circ}{1+\cos 30^\circ} = \frac{1/2}{1+\sqrt{3}/2} = 2-\sqrt{3}$. Thus $[ABC] = 32(2-\sqrt{3}) = 64-32\sqrt{3}$, and the desired ordered triple (a,b,c) is $\mathbf{(64, -32, 3)}$.

Note: The value of $\tan 15^\circ$ can also be computed using the identity $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$, with $A = 45^\circ$ and $B = 30^\circ$.

Alternate Solution: Proceed as in the previous solution to conclude that $\triangle ABC \cong \triangle ABX$. Then $AB = \frac{AX}{\cos 15^\circ}$ and $[ABC] = \frac{1}{2}\cdot AX\cdot \frac{AX}{\cos 15^\circ}\cdot \sin 15^\circ$. Use the subtraction identities to compute $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ\cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ\cos 30^\circ - \cos 45^\circ\sin 30^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$. Dividing, $\frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2})(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \frac{6-2\sqrt{12}+2}{4} = 2-\sqrt{3}$. Substituting into the expression for $[ABC]$ yields $\frac{1}{2}\cdot 8^2\cdot\left(2-\sqrt{3}\right) = 64-32\sqrt{3}$, as in the previous solution.

T-4. In order for the conditions in the problem to be met, the first roll must be either 2 or 5. After the first roll, rolls of 1 can be ignored, because they are essentially equivalent to simply re-rolling; there are five other possibilities. Because of the requirement that no partial product before 100 be a perfect square, there are only five sequences of non-1 rolls that result in success: 2-5-2-5, 2-5-5-2, 5-4-5, 5-2-5-2, and 5-2-2-5. The probability of obtaining each sequence of length four is $\frac{1}{6}\cdot\left(\frac{1}{5}\right)^3 = \frac{1}{750}$, while the probability of obtaining the sequence of length three is $\frac{1}{6} \cdot \left(\frac{1}{5}\right)^2 = \frac{5}{750}$. Because there are four sequences of length four and one sequence of length three, the desired probability is $4 \cdot \frac{1}{750} + 1 \cdot \frac{5}{750} = \frac{9}{750} = \mathbf{\frac{3}{250}}$.

T-5. First rewrite the given equation as

$$3 \cdot \underline{C}\ \underline{A}\ \underline{T} \cdot \underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = 10000 \cdot \underline{C}\ \underline{A}\ \underline{T} + \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$$

which gives

$$3 = \frac{10000}{\underline{F}\ \underline{I}\ \underline{S}\ \underline{H}} + \frac{1}{\underline{C}\ \underline{A}\ \underline{T}}.$$

Therefore, in order to compute the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}$, and hence of $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$, $\frac{10000}{\underline{F}\ \underline{I}\ \underline{S}\ \underline{H}}$ must be as close as possible to 3 without going over, which means $\underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$ must be as close to 3333 as possible.

Trying $\underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = 3334$ gives $\underline{C}\ \underline{A}\ \underline{T} = 1667$, which is impossible. Trying $\underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = 3335$ gives $\underline{C}\ \underline{A}\ \underline{T} = 667$, which is the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}$.

Thus the greatest possible value for $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$ is **6673335**.

T-6. Examine the first few terms in the sequence until a pattern emerges.

$$\begin{aligned} a_1 &= \frac{1}{2} = \frac{3^0}{2^1} \\ a_2 &= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} = \frac{3^1}{2^2} \\ a_3 &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{3}{4}\right) = \frac{9}{8} = \frac{3^2}{2^3} \end{aligned}$$

The following argument shows that for $n \geq 1$, $a_n = \frac{3^{n-1}}{2^n}$. The base case is shown above. For the induction hypothesis, assume that $a_k = \frac{3^{k-1}}{2^k}$ for some $k \geq 1$; then use the geometric series sum formula with $r = \frac{3}{2}$ to compute a_{k+1}:

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_0 + a_1 + \cdots + a_k) \\ &= \frac{1}{2}\left(1 + \frac{3^0}{2^1} + \frac{3^1}{2^2} + \frac{3^2}{2^3} + \cdots + \frac{3^{k-1}}{2^k}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2}\left(\frac{1 - \frac{3^k}{2^k}}{1 - \frac{3}{2}}\right)\right) = \frac{1}{2}\left(1 - \frac{1}{2}\left(\frac{1 - \frac{3^k}{2^k}}{\frac{1}{2}}\right)\right) = \frac{1}{2}\left(\frac{3^k}{2^k}\right) = \frac{3^k}{2^{k+1}}. \end{aligned}$$

Thus for $n > 0$, $a_n = \frac{3^{n-1}}{2^n} = \frac{1}{2} \cdot \left(\frac{3}{2}\right)^{n-1}$.

Consequently, the number which must be computed is the greatest value of n for which

$1.5^{n-1} < 2 \cdot 2017 = 4034$, i.e., $1.5^n < 6051$. Note that $1.5^4 = 2.25^2 = 5.0625 > 5$. Therefore $1.5^{20} = (1.5^4)^5 > 5^5 = 3125$, and so $1.5^{22} > 3125 \cdot 2.25$ which is clearly larger than 6051. Thus the desired value of n is no more than 21. Moreover, because 1.5^4 is only slightly larger than 5 and $1.5^{20} = \left(1.5^4\right)^5$, one should expect that 1.5^{20} will be only slightly larger than $5^5 = 3125$. This argument suggests that $3125 \cdot 1.5 = 4627.5$ is a reasonable approximation for 1.5^{21}, making 21 a reasonable guess for the greatest n with $1.5^n < 6051$.

To verify this guess, it suffices to verify that $1.5^{20} < 4034$. Expanding 1.5^{20} using the binomial theorem gives

$$\begin{aligned}
1.5^{20} &= (1.5^4)^5 \\
&= (5 + 0.0625)^5 \\
&= \binom{5}{0} \cdot 5^5 + \binom{5}{1} \cdot 5^4 \cdot 0.0625 + \binom{5}{2} \cdot 5^3 \cdot 0.0625^2 + \\
&\quad \binom{5}{3} \cdot 5^2 \cdot 0.0625^3 + \binom{5}{4} \cdot 5 \cdot 0.0625^4 + \binom{5}{5} \cdot 0.0625^5.
\end{aligned}$$

The largest binomial coefficient is $\binom{5}{2} = \binom{5}{3} = 10$, but $0.0625 < 0.1$, so when positive powers of 0.0625 are multiplied by binomial coefficients, the result is never more than 1. Replacing the binomial coefficients and powers of 0.0625 with 1 yields an upper bound of

$$1.5^{20} < 5^5 + 5^4 + 5^3 + 5^2 + 5^1 + 5^0 = \frac{5^6 - 1}{5 - 1} = 3906 < 4034.$$

Thus $1.5^{21} < 4034 \cdot 1.5 = 6051$, and so **21** is indeed the correct answer.

T-7. First note that the age of Nysmlistan in each of 2027, 2032, and 2038 is $n + 10$, $n + 15$, and $n + 21$, respectively. Similarly, the age of Armlandia in those years is $n^2 + 10$, $n^2 + 15$, and $n^2 + 21$. Next suppose that $n + k \mid n^2 + k$, for some integers n and k. Then using polynomial long division,

$$n^2 + k = (n - k)(n + k) + \frac{k(k + 1)}{n + k}$$

which implies that $n + k$ must divide $k(k + 1)$.

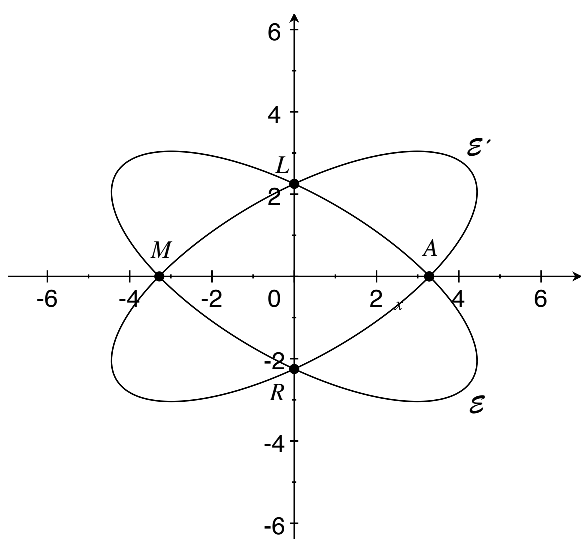
Therefore $n + 10$ must divide 110, $n + 15$ must divide 240, and $n + 21$ must divide 462. Factoring each value shows that only two values of n satisfy these three conditions: $n = 1$ and $n = 45$. The condition that 2027 is the first year after 2017 in which Nysmlistan's age divides Armlandia's age precludes $n = 1$, so $n = 45$.

Let $m = M - 2017$, so that Nysmlistan's age in year M is $m + n$. Because $M > 2038$, $m + n > 21$. Then, following the above, $n + m \mid (n^2 + m)$, so $n + m$ divides $m(m + 1)$. Substituting $n = 45$ gives $m + 45 \mid (m^2 + m)$, and using polynomial long division once again yields

$$m^2 + m = m - 44 + \frac{1980}{m + 45}.$$

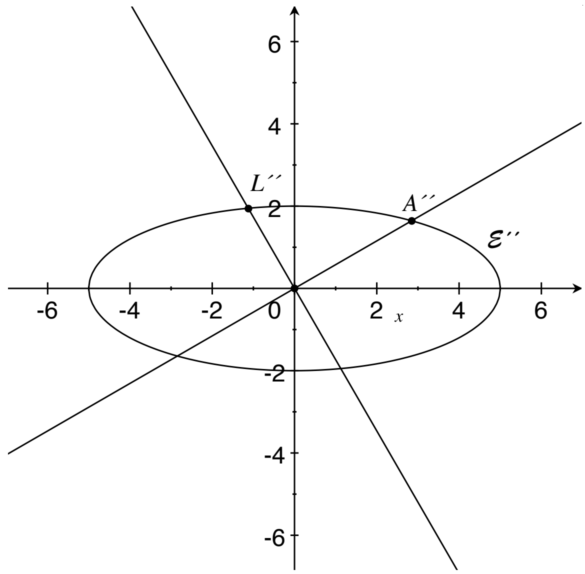
The first factors of 1980 greater than 45 are 55, 60, 66, and 90, corresponding to $m = 10$, $m = 15$, $m = 21$, and $m = 45$, respectively. Hence $m = 45$ and $M = \mathbf{2062}$.

T-8. Place the ellipse and its images on the coordinate plane. While ordinarily it is more convenient to place ellipses so that their major and minor axes are parallel to (or coincide with) the x- and y-axes, in this situation, it's more strategic to place the ellipses so that their *intersection points* lie on the axes. Accordingly, place $\mathcal{E}$ on the xy-plane so that its center is at the origin and its major axis is rotated 30° clockwise relative to the x-axis. Then $\mathcal{E}'$ will also have its center at the origin, but will have its major axis rotated 30° counterclockwise relative to the x-axis. Because this diagram is symmetric about both $x-$ and $y-$ axes, the points of intersection A, R, M, and L all lie on the coordinate axes, as shown in the diagram below.



Without loss of generality, let $A = (a, 0)$ be the point of intersection on the positive x-axis. Then $L = (0, \ell)$ is the point of intersection on the positive y-axis, and $[AOL] = \frac{a\ell}{2}$.

To compute the lengths a and ℓ, rotate $\mathcal{E}$ by 30° counterclockwise to a new ellipse $\mathcal{E}''$ so that its major axis coincides with the x-axis. The points A and L will also be rotated by 30° to new points A'' and L'', respectively. Because $\triangle AOL \cong \triangle A''OL''$, it follows that $[AOL] = [A''OL''] = \frac{1}{2}A''O \cdot L''O$. Now A'' is the intersection of $\mathcal{E}''$ and a line through the origin making a 30° angle with the x-axis, as shown in the diagram below.



The equation for $\mathcal{E}''$ is $\dfrac{x^2}{25}+\dfrac{y^2}{4}=1$, or $4x^2+25y^2=100$. Because $\tan 30^\circ=\frac{1}{\sqrt{3}}$, the equation for the line is $y=\frac{1}{\sqrt{3}}x$. Substitute to obtain

$$\begin{aligned} 4x^2+25\left(\frac{1}{\sqrt{3}}x\right)^2 &= 100 \\ 12x^2+25x^2 &= 300 \\ x^2 &= \frac{300}{37} \\ x &= 10\sqrt{\frac{3}{37}}. \end{aligned}$$

Because $a=A''O$, it suffices to compute $A''O$ using trigonometry: $\frac{x}{A''O}=\cos 30^\circ=\frac{\sqrt{3}}{2}$, so $A''O=\dfrac{10\sqrt{\frac{3}{37}}}{\frac{\sqrt{3}}{2}}=\frac{20}{\sqrt{37}}$.

Similarly, L'' is the intersection of $\mathcal{E}''$ with the line $y=-\sqrt{3}x$. Solve for the intersection point to obtain

$$\begin{aligned} 4x^2+25\left(-\sqrt{3}x\right)^2 &= 100 \\ 79x^2 &= 100 \\ x &= -\frac{10}{\sqrt{79}}. \end{aligned}$$

Because $\ell=L''O$ and $\frac{x}{L''O}=\cos 120^\circ=-\frac{1}{2}$, conclude that $\ell=\frac{20}{\sqrt{79}}$.

Then $[AOL]=[A''OL'']=\dfrac{a\ell}{2}=\dfrac{\left(\frac{20}{\sqrt{37}}\right)\left(\frac{20}{\sqrt{79}}\right)}{2}=\dfrac{\mathbf{200\sqrt{2923}}}{\mathbf{2923}}$.

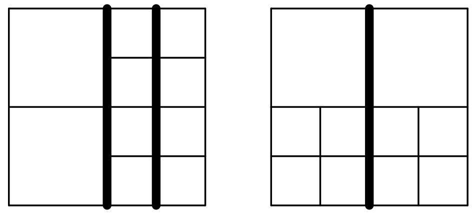
T-9. Let $S(n) = E(n) + O(n)$. In general, either $S(n)$ consists of only odd digits, or $S(n)$ has initial digit 2, and then only odd digits. To prove this observation, notice that one of $E(n)$ and $O(n)$ will consist of an initial digit, and then either all 0's or all 1's, respectively: if d is the first (leftmost) digit of n and d is odd, then the next larger number with all even digits must have first digit $d + 1$ (or 2 if $d = 9$) followed by all 0's, while if d is even, then the next larger number with all odd digits must have first digit $d + 1$ followed by all 1's. To compute the sum of $E(n)$ and $O(n)$, consider the sum of each pair of corresponding digits separately, beginning with the digits immediately to the right of their leftmost digits. If d is odd, this process is equivalent to adding 0's to $O(n)$'s digits. If d is even, this process is equivalent to adding 1's to $E(n)$'s digits, and none of $E(n)$'s digits is a 9. In neither case is there any need to carry. Thus when the $E(n)$ and $O(n)$ are added, there will be no carrying, except possibly in the initial digit. If $E(n)$ and $O(n)$ have the same number of digits, then the initial digit of $S(n)$ will be odd. If $E(n)$ and $O(n)$ do not have the same number of digits, then $O(n)$ must have initial digit 9 and $E(n)$ will have one more digit than $O(n)$, with an initial 2 and 0's everywhere else. Either way, $S(n)$ will fit the above form. This argument proves the lemma.

Therefore, to determine possible values of $S(N)$, look for multiples of 2017 that either consist of only odd digits or have an initial 2 and then all odd digits. Consider the thousands place of the first few multiples of 2017, which are $2017, 4034, 6051, \ldots$. The thousands place will continue to be even, until enough multiples of 17 are accumulated to roll over into the thousands place. As long as the thousands place is even, the multiple of 2017 cannot be a possible value of $S(N)$, because all the digits will not be even, and the thousands place is not the initial digit of the multiple (except for 2017, which does not work because 0 is not odd). Therefore, noting that $58 < 1000/17 < 59$, skip directly to $59 \cdot 2017 = 119003$. In order to force the hundreds place to be odd, note that $5 < 97/17 < 6$, so skip ahead by $6 \cdot 2017$ to $65 \cdot 2017 = 131105$. This value still doesn't work, so noting that $S(N)$ must be odd, consider $S(N) = 67 \cdot 2017 = 135139$.

Note that $S(N)$ does not begin with a 2, so $E(N)$ and $O(N)$ have the same number of digits, and the first digits of $E(N)$ and $O(N)$ must differ by 1, so $E(N)$ must begin with a 6 and $O(N)$ must equal 71111. Thus $E(N) = 135139 - 71111 = 64028$, and hence $N = \min(64028, 71111) - 2 = \mathbf{64026}$.

T-10. Let $T(n)$ be the number of ways to tile a $4 \times n$ rectangle with the given types of tiles, with $T(0) = 1$. Because there is only one way to tile a 4×1 rectangle, i.e., with 1×1 squares, $T(1) = 1$.

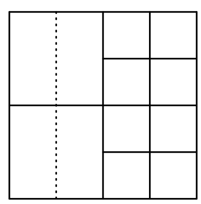
Use the term *fault line* to describe a vertical line from top to bottom that does not intersect any tiles and that is not either the left or the right border of the figure. For example, if a 4×4 region is tiled with two 2×2 tiles on the left, and then eight 1×1 tiles, as shown in the left figure below, then the tiling has two fault lines, while if the two 2×2 tiles are next to each other at the top of the region, then the tiling has only one fault line, as shown in the right figure below.



Two fault lines One fault line

Under this definition, it is possible for a tiling to have no fault lines; for example, if the region is completely filled by a single square, or if 2×2 squares are staggered so that the right half of one is directly above or below the left half of the other. Thus, in general, a $4 \times n$ rectangle can have between 0 and $n-1$ fault lines, inclusive. The number of tilings can be counted by conditioning on the location of the leftmost fault line.

For ease of notation, label the vertical lines (whether or not they are fault lines) one unit to the right of the leftmost edge, two units to the right of the leftmost edge, etc. as ℓ_1, ℓ_2, etc. respectively, as shown below. In that example, the leftmost fault line is at ℓ_2.

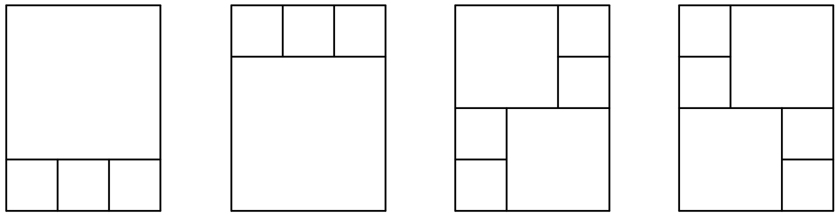


ℓ_1 ℓ_2 ℓ_3 ℓ_4

Let $F(n)$ represent the number of ways to tile a $4 \times n$ rectangle with *no* fault lines. Note that $F(0) = T(0) = 1$ and $F(1) = T(1) = 1$.

A 4×2 rectangle can be tiled with no fault line in four ways (two 2×2 squares or one 2×2 square and four 1×1 squares), so $F(2) = 4$. If there is one fault line, it is at ℓ_1, and divides the rectangle into two 4×1 strips, each of which can be tiled in $F(1) = T(1)$ ways, for $F(1) \cdot T(1) = 1$ tiling. So $T(2) = F(2) + F(1) \cdot T(1) = 5$.

A 4×3 rectangle can be tiled with no fault line in four ways: either there is a 3×3 square (and the rest of the area is filled in with 1×1 squares), which can happen in two ways, or there are two 2×2 squares, one above and to either the left or right of the other, with the other spaces filled in with 1×1 squares.



Hence $F(3) = 4$. If the leftmost fault line is at ℓ_1, then there are $F(1)$ ways to tile the left side and $T(2)$ ways to tile the right side. If the leftmost fault line is at ℓ_2, then there are $F(2)$ ways to tile the left side and $T(1)$ ways to tile the right side. Hence $T(3) = F(3) + F(1) \cdot T(2) + F(2) \cdot T(1) = 4 + 1 \cdot 5 + 4 \cdot 1 = 13$.

A 4×4 rectangle can be tiled with no fault line in three ways: either using a single 4×4 square, or by alternating 2×2 squares in the top and bottom row (and filling in with four 1×1 squares on the ends) as in the 4×3 case, which can happen in two ways. So $F(4) = 3$. Otherwise, with the first fault line occurring at ℓ_1, ℓ_2, or ℓ_3, there are $F(1) \cdot T(3)$, $F(2) \cdot T(2)$, and $F(3) \cdot T(1)$ possible tilings, respectively, so that

$$\begin{aligned} T(4) &= F(4) + F(1) \cdot T(3) + F(2) \cdot T(2) + F(3) \cdot T(1) \\ &= 3 + 1 \cdot 13 + 4 \cdot 5 + 4 \cdot 1 \\ \implies T(4) &= 40. \end{aligned}$$

The foregoing analysis shows that, in general,

$$T(n) = F(n) + F(1) \cdot T(n-1) + F(2) \cdot T(n-2) + \cdots + F(n-1) \cdot T(1).$$

Indeed, the proof is trivial, because if there is no fault line, there are $F(n)$ tilings, and if the leftmost fault line is at ℓ_i, then there are $F(i)$ ways of tiling the region to the left of the fault line without any fault lines, and there are $T(n-i)$ ways of tiling the region to the right of the fault line (with or without fault lines). Also, for $n > 4$, $F(n) = 2$, because the only way to tile such a region without fault lines is to alternate 2×2 squares in the top and bottom rows. So a simpler version of the formula above for $n > 4$ is simply

$$T(n) = 1 \cdot T(n-1) + 4 \cdot T(n-2) + 4 \cdot T(n-3) + 3 \cdot T(n-4) + 2 \cdot T(n-5) + \cdots + 2 \cdot T(0).$$

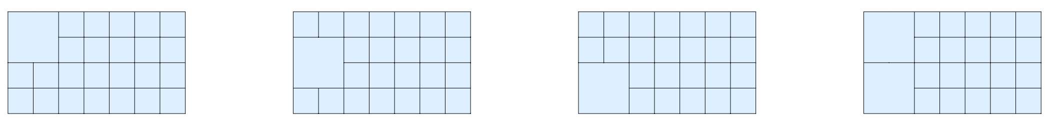
Using this formula, construct a table of values, as shown below.

n	0	1	2	3	4	5	6	7
$T(n)$	1	1	5	13	40	117	348	**1029**

Alternate Solution: Let $T(n)$ be the number of ways to tile an $4 \times n$ rectangle with square integer-length tiles, and let $S(n, k)$ be the number of ways to tile the same rectangle conditional on the largest tile touching the left-hand border being of size $k \times k$. Then $T(n) = S(n, 1) + S(n, 2) + S(n, 3) + S(n, 4)$.

Because there is only one way to arrange a strip of 1×1 squares along the left side, $S(n,1) = T(n-1)$. Because there is only one way to arrange a single 4×4 square on the left side, $S(n,4) = T(n-4)$. As the diagram below shows, there are two ways to arrange a 3×3 square and three 1×1 squares so that the 3×3 square touches the left side, so $S(n,3) = 2 \cdot T(n-3)$.

The remaining term to address is the $S(n,2)$ term. There are three ways to fit one 2×2 tile and four 1×1 tiles in the leftmost 4×2 subregion. There is also one way to fit two 2×2 tiles in that subregion, as illustrated below.



However, it is also possible for a series of two or more 2×2 squares to alternate across an $4 \times m$ range, where m can be any integer larger than 2. Note that for any such m, this alternating set of 2×2 squares can occur in two ways, depending on whether the leftmost 2×2 square is at the top or the bottom. The diagram below shows both examples with $m = 4$.

Thus $S(n,2) = 4 \cdot T(n-2) + 2 \cdot \sum_{m \geq 3} T(n-m)$.

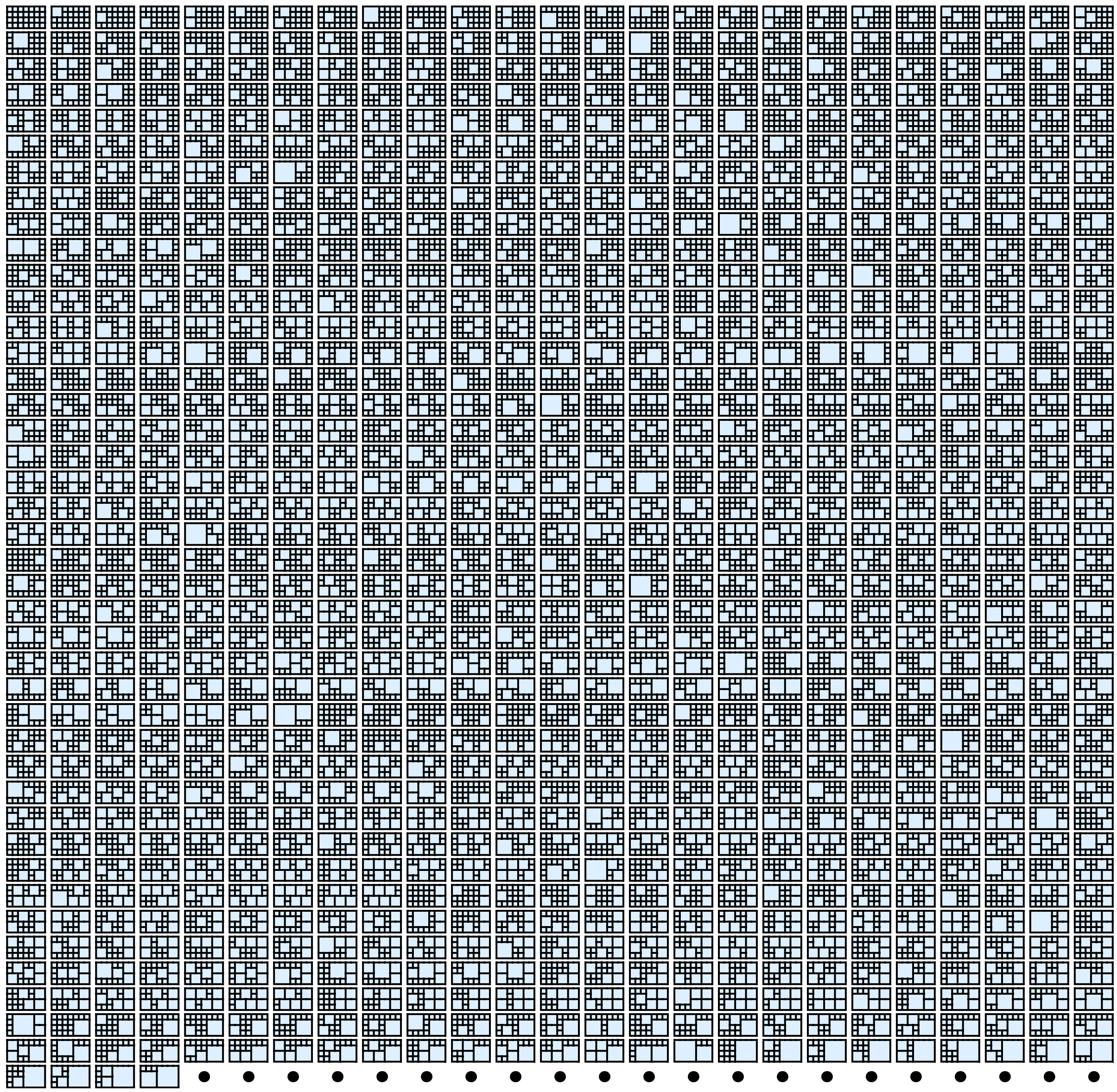
Combine the above results to obtain the following recursion:

$$T(n) = T(n-1) + 4 \cdot T(n-2) + 2 \cdot T(n-3) + T(n-4) + 2 \cdot \sum_{m \geq 3} T(n-m).$$

Use the recursion to build the table below.

n	0	1	2	3	4	5	6	7
$T(n)$	1	1	5	13	40	117	348	1029

Hence there are **1029** tilings. They are illustrated on the following page.



2017 Individual Problems

I-1. Compute the number of perfect squares in the set $\{1^1, 2^2, 3^3, \ldots, 2017^{2017}\}$.

I-2. Trapezoid $ARML$ has $\overline{AR} \parallel \overline{ML}$. Given that $AR = 4$, $RM = \sqrt{26}$, $ML = 12$, and $LA = \sqrt{42}$, compute AM.

I-3. Compute the number of ordered pairs of integers (a, b) such that the polynomials $x^2 - ax + 24$ and $x^2 - bx + 36$ have one root in common.

I-4. In $\triangle ABC$, $\mathrm{m}\angle A = 90°$, $AC = 1$, and $AB = 5$. Point D lies on ray $\overrightarrow{AC}$ so that $\mathrm{m}\angle DBC = 2\mathrm{m}\angle CBA$. Compute AD.

I-5. Given that $2^{-\frac{3}{2}+2\cos\theta} + 1 = 2^{\frac{1}{4}+\cos\theta}$, compute $\cos 2\theta$.

I-6. A diagonal of a regular 2017-gon is chosen at random. Compute the probability that the chosen diagonal is longer than the median length of all of the diagonals.

I-7. Given that $i = \sqrt{-1}$, compute

$$(i+1)^3(i-2)^3 + 3(i+1)^2(i+3)(i-2)^2 + 3(i+1)(i+3)^2(i-2) + (i+1)^3.$$

I-8. In triangle ABC, $\mathrm{m}\angle C = 90°$ and $BC = 17$. Point E lies on side $\overline{BC}$ so that $\mathrm{m}\angle CAE = \mathrm{m}\angle EAB$. The circumcircle of triangle ABE passes through a point F on side $\overline{AC}$. Given that $CF = 3$, compute AB.

I-9. Let S be the set of divisors of $67 \cdot 9! + 27 \cdot 8!$. Compute the median of S.

I-10. Rhombus $ARML$ has its vertices on the graph of $y = \lfloor x \rfloor - \{x\}$. Given that $[ARML] = 8$, compute the least upper bound for $\tan A$.

2017 Individual Answers

I-1. 1030

I-2. $\sqrt{66}$

I-3. 12

I-4. $\dfrac{37}{11}$

I-5. $\dfrac{1}{8}$

I-6. $\dfrac{503}{1007}$

I-7. $-20-24i$

I-8. $\dfrac{149}{3}$

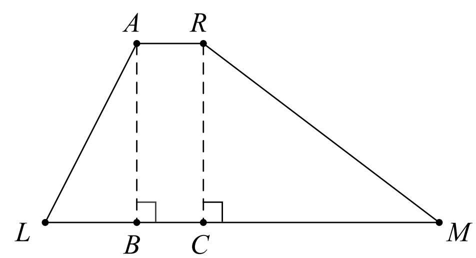
I-9. 5040

I-10. $\dfrac{100}{621}$

2017 Individual Solutions

I-1. For a positive integer k, consider whether k^k is even or odd. If k^k is even, then k is also even, and $k^k = (k^{k/2})^2$ is a perfect square. There are $\lfloor 2017/2 \rfloor = 1008$ even k^k in the set. If k^k is odd, then k is also odd, and k^k will be a perfect square if and only if k is a perfect square. Because $44^2 < 2017 < 45^2$, and 22 of the first 44 natural numbers are odd, there are 22 odd squares in the set. The total number of perfect squares in the given set is therefore $1008 + 22 = \mathbf{1030}$.

I-2. Draw lines perpendicular to $\overline{LM}$ through A and R, meeting $\overline{LM}$ at B and C, respectively, as shown in the diagram below. (As discussed in the note below, it can be shown that B and C must lie on the segment $\overline{LM}$.)



By the Pythagorean Theorem, $AB^2 + LB^2 = 42$ and $RC^2 + CM^2 = 26$. Because $AB = RC$, $LB + BC + CM = 12$, and $RC^2 + CM^2 = 26$, it follows that

$$\begin{aligned} AB^2 + (8 - LB)^2 &= 26 \\ \Longrightarrow \quad AB^2 + 64 - 16LB + LB^2 &= 26. \end{aligned}$$

Substitute 42 for $AB^2 + LB^2$ to obtain $64 - 16LB + 42 = 26$, from which it follows that $LB = 5$, and $AB = \sqrt{42 - 5^2} = \sqrt{17}$. Apply the Pythagorean Theorem once more to obtain

$$\begin{aligned} AM &= \sqrt{AB^2 + BM^2} \\ &= \sqrt{AB^2 + (LM - LB)^2} \\ &= \sqrt{AB^2 + (12 - 5)^2} \\ &= \sqrt{\mathbf{66}}. \end{aligned}$$

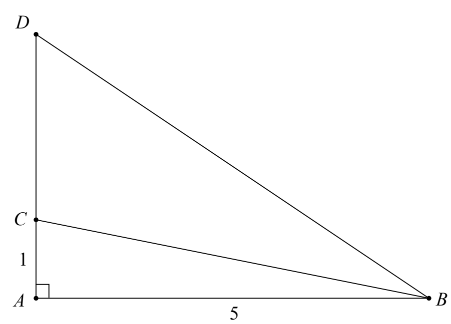
Note: In general, trapezoids with a given set of side lengths are unique, i.e., there is a SSSS congruence theorem for trapezoids. In this case, assuming that C does not lie on $\overline{LM}$ leads to a contradiction as follows. Let $CM = x$ and $RC = h$, implying $BL = (x + 12) - 4 = x + 8$. Then $x^2 + h^2 = 26$ and $(x+8)^2 + h^2 = 42$, yielding $x = -3$, but side lengths must be positive, so this is a contradiction.

I-3. In general, if r is a root of $f(x)$ and of $g(x)$, then r is a root of $f(x) - g(x)$. The common root of $x^2 - ax + 24$ and $x^2 - bx + 36$ is a root of $(b - a)x - 12$. It is evident that there are no solutions where $b = a$, because the constant polynomial $0x - 12$ has no roots. It follows that the common root of the two polynomials is $\dfrac{12}{b-a}$. Substitute this root into $x^2 - ax + 24$, and notice that

$$\begin{aligned} 0 &= \frac{144}{(b-a)^2} - a \cdot \frac{12}{b-a} + 24 \\ 0 &= 144 - 12a(b-a) + 24(b-a)^2 \\ -12 &= 3a^2 + 2b^2 - 5ab \\ -12 &= (a-b)(3a-2b). \end{aligned}$$

Because $a - b$ and $3a - 2b$ are integers, $a - b$ must be an integer that divides 12. If k divides 12, then $a - b = k$ implies that $a = k + b$, so $3(k + b) - 2b = -\frac{12}{k}$. Solve for b to obtain $b = -\frac{12}{k} - 3k$, which is also an integer. Thus choosing any integer that divides 12 for $a - b$ will result in a distinct ordered pair of integers (a, b). There are 12 integers that divide 12 (namely ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12), so there are **12** such ordered pairs of integers.

I-4. Consider the diagram below.



Because $\text{m}\angle DBC = 2\text{m}\angle CBA$, it follows that $\text{m}\angle DBA = 3\text{m}\angle CBA$. Thus $\dfrac{AD}{AB} = \dfrac{AD}{5} = \tan(3\text{m}\angle CBA)$. Using the triple-angle formula, $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$, substitute to obtain $\dfrac{AD}{5} = \dfrac{3 \cdot \frac{1}{5} - \frac{1}{125}}{1 - 3 \cdot \frac{1}{25}}$, so $AD = \dfrac{3 - \frac{1}{25}}{1 - \frac{3}{25}} = \dfrac{75 - 1}{25 - 3} = \boldsymbol{\dfrac{37}{11}}$.

I-5. Let $x = 2^{\cos\theta}$ and let $k = 2^{-3/2}$. Then $\frac{1}{k} = \sqrt{8} \Rightarrow \frac{1}{2k} = \sqrt{2} \Rightarrow \sqrt{\frac{1}{2k}} = \sqrt[4]{2}$. Then the given equation becomes $kx^2 + 1 = x\sqrt{\frac{1}{2k}}$, which is equivalent to $x^2 - \frac{x}{k}\sqrt{\frac{1}{2k}} + \frac{1}{k} = x^2 - x \cdot \sqrt{8} \cdot \sqrt{\frac{1}{2k}} + \frac{1}{k} = x^2 - 2x\sqrt{\frac{1}{k}} + \frac{1}{k} = \left(x - \sqrt{\frac{1}{k}}\right)^2 = 0$. Thus $2^{\cos\theta} = x = \sqrt{\frac{1}{k}} = 2^{3/4}$ and therefore $\cos\theta = \frac{3}{4}$. Hence $\cos 2\theta = 2\cos^2\theta - 1 = 2 \cdot \left(\frac{3}{4}\right)^2 - 1 = \boldsymbol{\frac{1}{8}}$.

Alternate Solution: The expressions in the exponents $\cos\theta$ and $2\cos\theta$ are suggestive of a quadratic equation. Let $x = 2^{\frac{1}{4}+\cos\theta}$. Then $x^2 = 2^{\frac{1}{2}+2\cos\theta}$; note that this exponent is 2 more than the exponent on the left-hand side. Because $2^2 = 4$, rewrite the left side as $\frac{1}{4}x^2 + 1$ to obtain the equation $\frac{1}{4}x^2 + 1 = x$. Hence $x^2 - 4x + 4 = 0$, yielding $x = 2$. Then $\frac{1}{4} + \cos\theta = 1 \Rightarrow \cos\theta = \frac{3}{4}$. Proceed as in the first solution to obtain $\cos 2\theta = \mathbf{\frac{1}{8}}$.

I-6. Consider the 2014 diagonals emanating from a particular vertex. By symmetry, the 1007 distinct lengths occur in pairs. The complete list of diagonal lengths consists of 2017 copies of the 1007 distinct lengths. As 1007 is odd, $\frac{1007-1}{2} = 503$ of the 1007 lengths exceed the median length and the probability is $\mathbf{\frac{503}{1007}}$. Considering that making 2017 copies of the list simply multiplies the numerator and denominator by 2017, it does not change the answer.

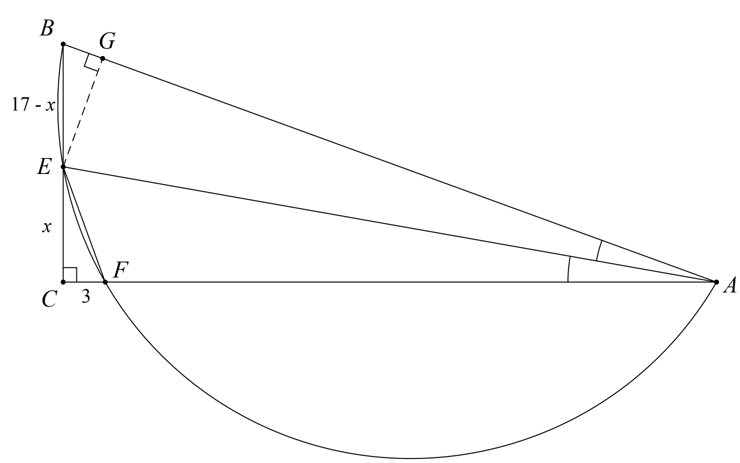
I-7. Let S denote the sum. Rather than expand each term, note that $[(i+1)(i-2)+(i+3)]^3 =$

$$(i+1)^3(i-2)^3 + 3(i+1)^2(i-2)^2(i+3) + 3(i+1)(i-2)(i+3)^2 + (i+3)^3$$

and $(i+1)(i-2)+(i+3) = i^2+1 = 0$. By subtraction,

$$\begin{aligned} S - 0 &= (i+1)^3 - (i+3)^3 \\ &= (i^3+3i^2+3i+1) - (i^3+9i^2+27i+27) \\ &= -6i^2 - 24i - 26 \\ &= \mathbf{-24}\boldsymbol{i}\mathbf{-20}. \end{aligned}$$

I-8. First note that because $m\angle CAE = m\angle EAB$, minor arcs $\widehat{FE}$ and $\widehat{EB}$ have the same length, and so $EF = EB$. Then if $CE = x$, $EB = 17 - x = EF$, so by the Pythagorean Theorem, $x^2 + 3^2 = (17-x)^2$, or $34x = 280$, yielding $x = \frac{140}{17}$ and $EB = 17 - x = \frac{149}{17}$. Proceed by constructing G on $\overline{AB}$ so that $\overline{EG} \perp \overline{AB}$, yielding the diagram below.

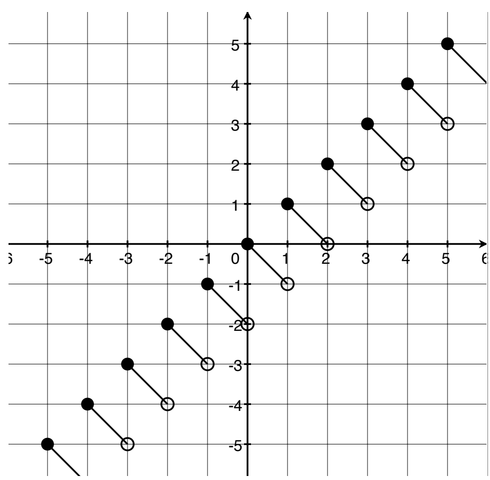


Then $\triangle ACE \cong \triangle AGE$ because both are right triangles sharing hypotenuse $\overline{AE}$ and with congruent angles at A. So $EG = CE$. Hence $\triangle CEF \cong \triangle GEB$ and $GB = CF = 3$. However, $AG = AC$ implies that $GB = AB - AC$. Then $AB^2 - AC^2 = BC^2 = 289$ and $AB^2 - AC^2 = (AB+AC)(AB-AC)$ yields $AB+AC = \frac{289}{3}$. Then $AB = \frac{(AB-AC)+(AB+AC)}{2} = \frac{3+289/3}{2} = \mathbf{\frac{149}{3}}$.

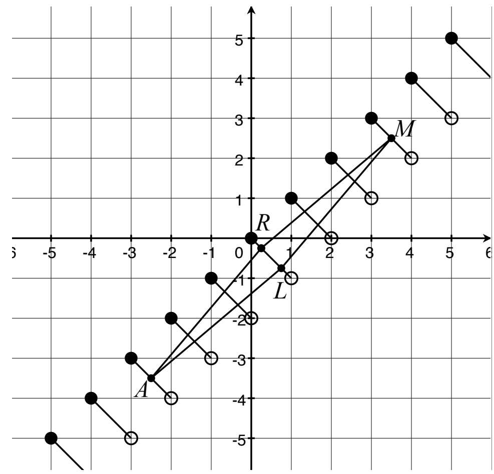
Alternate Solution: As in the first solution, compute $CE = \frac{140}{17}$ and $EB = \frac{149}{17}$. Then by the Power of the Point theorem, $CE \cdot CB = CF \cdot CA$, so $\frac{140}{17} \cdot 17 = 3 \cdot CA$, yielding $CA = \frac{140}{3}$. From this point there are at least two ways to compute AB. Using the Pythagorean Theorem is messy because it results in $AB^2 = \frac{22201}{9}$, although a reasonable guess at $\sqrt{22201}$ is 149 (because 22201 is just slightly less than $22500 = 150^2$ and the last digit of 22201 is 1), which turns out to be correct. Using the Angle Bisector Theorem yields $\frac{AB}{AC} = \frac{BE}{EC} = \frac{149}{140}$, and given that $AC = \frac{140}{3}$, $AB = \mathbf{\frac{149}{3}}$.

I-9. Rewrite $67 \cdot 9! + 27 \cdot 8!$ as $(67 \cdot 9 + 27)8! = 630 \cdot 8! = 5040 \cdot 7! = (7!)^2$. Divisors occur in pairs except for the central divisor, $7! = \mathbf{5040}$, which is the median of S.

I-10. As $x = \lfloor x \rfloor + \{x\}$, $y = 2\lfloor x \rfloor - x$ or $y = 2n - x$ on the interval $n \leq x < n+1$ for integers n. These line segments have length $\sqrt{2}$ and are separated by that same length as shown in the diagram below.



The following argument shows that for *any* rhombus to lie on this graph, one pair of opposite vertices must lie on the same diagonal segment. In fact, if U and V are points on different segments of the graph with coordinates (x_U, y_U) and (x_V, y_V) respectively, then the slope of segment $\overline{UV}$ must be positive. Suppose, without loss of generality, that $x_U < x_V$. Then $y_U < y_V$ because each segment is above the segment to its immediate left. Thus the slope of $\overline{UV}$ is $\frac{y_V - y_U}{x_V - x_U} > 0$. Because the diagonals of a rhombus are perpendicular and perpendicular segments have opposite reciprocal slopes, it is impossible that both diagonals of the rhombus have positive slopes. Hence one pair of opposite vertices of the rhombus must lie on the same diagonal segment. This argument yields a further conclusion: because the slope of the diagonal with negative slope is -1, the other two vertices must lie on a line parallel to the line $y = x$. A rhombus satisfying these conditions is shown in the diagram below.



In order for $\tan A$ to be positive, $\angle A$ must be acute. Hence A must be the lower-left (or upper-right) vertex of the rhombus, with R and L on the same diagonal segment; the slope of $\overline{AM}$ is 1. Because the distance between adjacent segments is $\sqrt{2}$, $AM = k\sqrt{2}$ for some integer k, and because R and L lie on the same diagonal segment, $RL < \sqrt{2}$. (The inequality is strict because the right endpoint of each segment is *not* included in the graph.) Because $[ARML] = \frac{1}{2}AM \cdot RL = 8$, $RL = \frac{16}{k\sqrt{2}} = \frac{8\sqrt{2}}{k}$. Hence $k > 8$, and to maximize $\angle A$, it suffices to maximize RL. If $k = 9$, however, the midpoint of $\overline{AM}$ does not lie on any segment. Thus $k = 10$ and $RL = \frac{8\sqrt{2}}{10} = \frac{4\sqrt{2}}{5}$ while $AM = 10\sqrt{2}$. Hence $\tan \frac{A}{2} = \frac{RL/2}{AM/2} = \frac{RL}{AM} = \frac{2}{25}$. Using the double-angle identity,

$$\tan A = \frac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}} = \frac{\frac{4}{25}}{1 - \frac{4}{625}} = \frac{100}{625 - 4} = \mathbf{\frac{100}{621}}.$$

Note that this least upper bound for $\tan A$ is in fact attainable, although the problem was stated to avoid having to assume that conclusion.

Power Question 2017: The Paral-Elle Universe

Elle is a student at Springfield High School. One day, she's escorted from her math class by a friendly space alien, who takes her to a parallel universe to learn math the way they do math, which is different than the way we do math. In the parallel universe, students are taught two operations: $\oplus$ and $\odot$. Elle notices that

$$a \oplus b = \min(a, b) \text{ and } a \odot b = a + b$$

for all real numbers a and b. Elle decides to investigate the properties of these two new operations.

Throughout this Power Question, $a \oplus b$ and $a \odot b$ will always denote parallel-universe addition and multiplication, respectively, while $a + b$ and $a \cdot b$ will denote addition and multiplication in the usual sense. The standard order of operations ($\odot$ first, then $\oplus$) will apply, although parentheses are sometimes included for clarification. All variables refer to real numbers unless otherwise specified.

1. a. Determine whether there is a real number $\mathcal{O}$ such that $\mathcal{O} \oplus x = x$ for all x. [2 pts]
   b. Show that there is a real number $\mathcal{I}$ such that $\mathcal{I} \odot x = x$ for all x. [2 pts]
2. Show that $\odot$ distributes over $\oplus$; that is, $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ for all a, b, c. [3 pts]
3. Find all solutions to each of the following equations:
   a. $(x \odot x) \oplus (2 \odot x) \oplus 1 = x \oplus 2$; [3 pts]
   b. $(x \odot x) \oplus (-1) = (x \oplus 1) \odot (x \oplus (-1))$. [3 pts]

Next, Elle investigates the properties of exponents and polynomials. She uses the notation

$$a^{\odot b} = \underbrace{a \odot a \odot a \odot \cdots \odot a}_{b \text{ copies of } a},$$

where a is a real number and b is a positive integer. Furthermore, by definition, $a^{\odot 0} = 0$. Elle wonders if exponentiation works in the parallel universe the way it works in ours.

4. Show that for any real numbers x and y, and any positive integer n:
   a. $(x \odot y)^{\odot n} = x^{\odot n} \odot y^{\odot n}$; [2 pts]
   b. $(x \oplus y)^{\odot n} = x^{\odot n} \oplus y^{\odot n}$; [2 pts]
   c. $(x \oplus y)^{\odot 2} = x^{\odot 2} \oplus y^{\odot 2} = (x^{\odot 2}) \oplus (x \odot y) \oplus (y^{\odot 2})$. [3 pts]
5. a. Show that for any a, there is exactly one solution to the equation $x^{\odot 2} = a$. This is called the **parallel square root of a** and is denoted by $\sqrt[\circ]{a}$. [2 pts]
   b. Show that for any a, there is exactly one solution to the equation $a \odot x = \mathcal{I}$, where $\mathcal{I}$ is defined in Problem 1b. This is called the **parallel reciprocal of a** and is denoted by $\widehat{a}$. [2 pts]

c. Show that for any a, $\sqrt[\circ]{\widehat{a}} = \widehat{\sqrt[\circ]{a}}$. [3 pts]

In the parallel universe, a **parallel-universe polynomial (PUP)** of degree n is a finite "sum" (i.e., parallel-universe addition) of $n+1$ parallel products whose exponents decrease from n to 0, inclusive:

$$f(x) = (a_n \odot x^{\odot n}) \oplus (a_{n-1} \odot x^{\odot(n-1)}) \oplus \cdots \oplus (a_1 \odot x^{\odot 1}) \oplus (a_0 \odot x^{\odot 0}),$$

where each a_i is a real number. As a convenient shorthand, $x^{\odot 1}$ will simply be written as x and $a_0 \odot x^{\odot 0}$ will simply be written as a_0. For example:

- $f(x) = (5 \odot x) \oplus 3$ is a degree-1 PUP, but not a degree-2 PUP.
- $f(x) = (1 \odot x^{\odot 2}) \oplus 1$ is not a PUP at all, because the x term is missing.
- $f(x) = (0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 0$ is a degree-2 PUP.
- $f(x) = (x^{\odot 2}) \oplus x \oplus 1$ is not a PUP; it should instead be written as $(0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 1$.

Elle now examines the properties of degree-2 (quadratic) PUPs. In the parallel universe, a **root** of a PUP f is a real number x such that $f(x) = 0$.

6. In our universe, a quadratic equation has either zero, one, or two distinct real roots.

   a. Find a quadratic PUP that has exactly one root. [2 pts]

   b. Could a quadratic PUP have zero roots? Justify your answer. [2 pts]

   c. Could a quadratic PUP have infinitely many distinct roots? Justify your answer. [2 pts]

   d. Suppose that $f(x)$ is a PUP of positive degree. Prove that if f has more than one root, then it has infinitely many distinct roots. [3 pts]

7. Suppose $f(x) = (a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c$ is a quadratic PUP that has exactly one root. Find a formula for this root in terms of a, b, and c. Express the formula using only the operations $\oplus$, $\odot$, parallel-universe exponentiation, parallel square roots, and parallel reciprocals. [5 pts]

Suppose $f(x) = (a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c$ is a quadratic PUP. A **factorization** of f is an ordered triple of real numbers (k, r, s) such that $r \leq s$, and

$$f(x) = k \odot (x \oplus r) \odot (x \oplus s),$$

for all real numbers x.

8. Prove that if (k, r, s) is a factorization of f, then:

   a. $k = a$, and [2 pts]

   b. $r \odot s = c \odot \widehat{a}$. [2 pts]

9. Prove that every quadratic PUP has exactly one factorization. [5 pts]

Solutions to 2017 Power Question

1. a. No. This equation would mean that $\min(\mathcal{O}, x) = x$ for all x. However, if $\mathcal{O}$ exists, then $\min(\mathcal{O}, \mathcal{O}+1) = \mathcal{O} \neq \mathcal{O}+1$. This is a contradiction, so no such $\mathcal{O}$ exists.

   b. $\mathcal{I} = 0$. This makes $\mathcal{I} \odot x = \mathcal{I} + x = 0 + x = x$ for all real numbers x.

2. Translating this into "normal" arithmetic, the left-hand side is $a + \min(b, c)$, and the right-hand side is $\min(a+b, a+c)$. If $b \leq c$, then both sides are equal to $a+b$, and if $b > c$, then both sides are equal to $a+c$. Thus this equation holds for all real numbers a, b, c.

3. a. The equation translates to $\min(2x, x+2, 1) = \min(x, 2)$. If $x < \dfrac{1}{2}$, the equation is equivalent to $2x = x$, which implies $x = 0$. If $x \geq \dfrac{1}{2}$, the equation is equivalent to $1 = \min(x, 2)$, which implies $x = 1$. These both satisfy the original equation, and there cannot be any other solutions. Thus the two solutions are $x = 0$ and $x = 1$.

   b. This equation translates to $\min(2x, -1) = \min(x, 1) + \min(x, -1)$. If $x \leq -1$, then both sides are equal to $2x$, so every $x \leq -1$ is a solution. If $-1 < x < -\frac{1}{2}$, then the equation becomes $2x = x - 1$, which has no solution on the interval. If $-\frac{1}{2} \leq x \leq 1$, then the equation is $-1 = x - 1$, so $x = 0$ is a solution. Finally, if $x > 1$, then the equation becomes $-1 = 0$, which is not true for any value of x. Thus the solutions to this equation are $x = 0$ and all $x \leq -1$.

4. Note that, translating into normal arithmetic, $a^{\odot b} = b \cdot a$. Each of the statements will be proven by translation.

   a. This equation translates to $n(x+y) = nx + ny$, which is just the normal distributive property of multiplication over addition.

   b. This translates to $n \cdot \min(x, y) = \min(nx, ny)$. If $x \leq y$, then both sides are equal to nx, and if $x > y$, then both sides are equal to ny. Either way, the equation holds for all real x and y and all integers n.

   c. The first equality holds by part (b). For the second, the left-hand side is $\min(2x, 2y)$, and the right-hand side is $\min(2x, x+y, 2y)$. But note that $x+y$ is the arithmetic mean of $2x$ and $2y$, so it must be greater than or equal to one of them and less than or equal to the other. Thus $\min(2x, x+y, 2y) = \min(2x, 2y)$, as desired.

5. a. This equation translates to $2x = a$, which has exactly one solution: $x = \sqrt[\odot]{a} = \frac{a}{2}$.

   b. This equation translates to $a + x = \mathcal{I} = 0$, which has exactly one solution: $x = \widehat{a} = -a$.

   c. As shown in parts (a) and (b), the left-hand side is equal to $\frac{1}{2}(-a)$, and the right-hand side is equal to $-\left(\frac{1}{2}a\right)$, and these are equal by associativity of (normal) multiplication.

6. a. Let $f(x) = (0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 1$. Then $f(x) = \min(2x, x, 1)$, which is $2x$ for $x \leq 0$, x for $0 < x < 1$, and 1 for $x \geq 1$. The only root of this PUP is $x = 0$.

b. The answer is yes. As an example, let $f(x) = (4 \odot x^{\odot 2}) \oplus (3 \odot x) \oplus (-1)$. Then $f(x) = \min(2x+4, x+3, -1)$, so $f(x) \leq -1$ for all x, which means f has no roots.

c. The answer is yes. As an example, let $f(x) = (0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 0$. Then $f(x) = \min(2x, x, 0) = 0$ for all $x \geq 0$, so all nonnegative real numbers are roots of f.

Note: A general quadratic PUP $(a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c$ has no roots if $c < 0$, infinitely many roots if $c = 0$, and one root if $c > 0$. A proof of this fact is in the solution to Problem 7.

d. Consider a general PUP, $f(x)$, as defined in the background to Problem 6. Then

$$f(x) = \min(nx + a_n, (n-1)x + a_{n-1}, \ldots, 2x + a_2, x + a_1, a_0).$$

This is the minimum of $n+1$ functions, all of whose graphs are lines of nonnegative slope. Suppose $x \leq y$. Then $kx + a_k \leq ky + a_k$ for each $k \geq 0$, and therefore $f(x) \leq f(y)$. Now suppose f has two different roots, so there are real numbers r_1, r_2 with $r_1 < r_2$ and $f(r_1) = f(r_2) = 0$. Now, for any z with $r_1 < z < r_2$, $f(r_1) \leq f(z) \leq f(r_2)$, so it follows that $f(z) = 0$. Thus every z in the interval $[r_1, r_2]$ is a root of f, hence f has infinitely many roots. As seen in parts (a) and (b), if f does not have infinitely many roots, then it can only have either one root or zero roots. This shows that f has either zero, one, or infinitely many roots.

7. *There are multiple ways to solve this problem; in particular, the root can be found by translating to normal arithmetic. However, a solution using almost exclusively parallel-universe arithmetic follows.*

Setting $f(x) = 0$ and translating to normal arithmetic, note that the given equation is equivalent to $f(x) = \min(2x + a, x + b, c) = 0$. First, note that if $x > \max\left(\frac{c-a}{2}, c - b\right)$, then $f(x) = c$, and $f(x) \leq c$ for all x. Thus if $c < 0$, then f has no roots, and if $c = 0$, then f has infinitely many roots. Hence if f has exactly one root, it follows that $c > 0$. Notice that in general, $\widehat{(a^{\odot n})} = (\widehat{a})^{\odot n}$. This is because this equation translates to $-(na) = n(-a)$, which holds by associativity of (normal) multiplication. Now let $x = \widehat{\sqrt[\circ]{a} \oplus b}$; this is the root of f. Substituting and using the result from the above paragraph yields

$$f(x) = a \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right)^{\odot 2} \oplus b \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right) \oplus c.$$

Form the parallel-universe equivalent of a common denominator:

$$f(x) = \left(a \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right)^{\odot 2}\right) \oplus \left(b \odot \left(\sqrt[\circ]{a} \oplus b\right) \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right)^{\odot 2}\right) \oplus \left(c \odot \left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2} \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right)^{\odot 2}\right).$$

Now use the distributive property, effectively combining like terms:

$$f(x) = \left[a \oplus b \odot \left(\sqrt[\circ]{a} \oplus b\right) \oplus c \odot \left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}\right] \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right)^{\odot 2}.$$

Using the results of Problems 2 and 4 yields:

$$f(x) = \left[a \oplus \left(b \odot \sqrt[\circ]{a}\right) \oplus b^{\odot 2} \oplus (c \odot a) \oplus \left(c \odot b^{\odot 2}\right)\right] \odot \left(\widehat{\sqrt[\circ]{a} \oplus b}\right)^{\odot 2}.$$

Now recall the assumption that $c > 0$, and therefore $a \oplus (c \odot a) = a$, and $b^{\odot 2} \oplus (c \odot b^{\odot 2}) = b^{\odot 2}$. Thus

$$f(x) = \left[a \oplus \left(b \odot \sqrt[\circ]{a}\right) \oplus b^{\odot 2}\right] \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}.$$

By Problem 4c, this becomes:

$$f(x) = \left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2} \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}} = 0,$$

as desired.

Other forms of the answer, such as $\sqrt[\circ]{\widehat{(a \oplus b^{\odot 2})}}$*, are also possible. However, answers like* $\sqrt[\circ]{\widehat{a}} \oplus \widehat{b}$ *are **not** valid, because Problem 4a does not hold if n is a negative integer.*

8. Before proving any part of this problem, let (k, r, s) be a factorization of f, and let

$$g(x) = k \odot (x \oplus r) \odot (x \oplus s),$$

so that $f(x) = g(x)$ for all x. By the result of Problem 2,

$$g(x) = (k \odot x^{\odot 2}) \oplus (k \odot (r \oplus s) \odot x) \oplus (k \odot r \odot s).$$

Also, note as in Problem 7 that $f(x) = \min(2x + a, x + b, c)$.

   a. If $x < \min\left(\frac{c-a}{2}, b - a\right)$, then $f(x) = 2x + a = a \odot x^{\odot 2}$. Similarly, if $x < r$, then $g(x) = k \odot x^{\odot 2}$. Choose a value of x small enough to satisfy both inequalities. Because $f(x) = g(x)$, it follows that $a \odot x^{\odot 2} = k \odot x^{\odot 2}$. Multiplying (in the parallel universe) by $\widehat{(x^{\odot 2})}$ yields $k = a$, as desired.

   b. As argued in Problem 7, if $x > \max\left(\frac{c-a}{2}, c - b\right)$, then $f(x) = c$. Similarly, if $x > s$, then $g(x) = k \odot r \odot s$. Choose x large enough to satisfy both inequalities. Because $f(x) = g(x)$, it follows that $c = k \odot r \odot s$. By part (a), $k = a$, and then by the definition of the parallel reciprocal, $r \odot s = c \odot \widehat{a}$.

9. As in Problem 8, let (k, r, s) be a factorization of a quadratic PUP $f(x)$, and let $g(x) = k \odot (x \oplus r) \odot (x \oplus s)$. Note by the distributive property (Problem 2), that

$$g(x) = (k \odot x^{\odot 2}) \oplus (k \odot r \odot x) \oplus (k \odot r \odot s)$$

because $r \leq s$ and thus $r \oplus s = r$. Furthermore, from Problems 8a and 8b, this becomes

$$g(x) = (a \odot x^{\odot 2}) \oplus (a \odot r \odot x) \oplus c.$$

Claim: If $2b < a + c$, then the only factorization of f is $a \odot \left(x \oplus (b - a)\right) \odot \left(x \oplus (c - b)\right)$ (where "$-$" denotes normal subtraction), and if $2b \geq a + c$, then the only factorization of f is $a \odot (x \oplus d) \odot (x \oplus d)$, where $d = \frac{c-a}{2}$.

To prove the claim, consider each case of the claim separately:

Case 1: Assume $2b < a + c$, so that $b - a < c - b$. Let $x = \frac{c-a}{2}$. Then

$$f(x) = \min(2x + a, x + b, c) = \min\left(c, \frac{c-a}{2} + b, c\right).$$

Because $2b < a + c$, it follows that $c - a < 2c - 2b$, and thus $x < c - b$. Then $x + b < c$, so $f(x) = x + b$.

Note also that

$$g(x) = \min(2x + a, x + a + r, c) = \min\left(c, \frac{c+a}{2} + r, c\right).$$

Because $f(x) = x + b < c$, conclude that $g(x) \neq c$, hence $g(x) = \frac{c+a}{2} + r$. Setting this equal to $f(x)$ yields $r = b - a$, and the result of Problem 8b gives $s = c - b$.

This is the only possible factorization of f, and it remains to show that it is a factorization. Recall that

$$g(x) = (a \odot x^{\odot 2}) \oplus (a \odot r \odot x) \oplus (a \odot r \odot s).$$

Using the definition of $\odot$, $a \odot r = b$ and $a \odot r \odot s = c$, so $g(x) = f(x)$ for all x, as desired.

Case 2: Now assume $2b \geq a + c$, so that $c - b \leq b - a$. First, note that for every x, either $x \leq b - a$ (and then $2x + a \leq x + b$), or $c - b \leq x$ (and then $c \leq x + b$). Thus

$$f(x) = \min(2x + a, x + b, c) = \min(2x + a, c).$$

Now consider the claim that $r \geq c - a - r$. If not, then there exists an x with $r < x < c - a - r$. Because $r < x$, it follows that $x + a + r < 2x + a$. Note that $x < c - a - r$ is equivalent to $x + a + r < c$. This implies that $g(x) = x + a + r$. And because $f(x)$ is equal to either $2x + a$ or c, it follows that $x = r$ or $x = c - a - r$, a contradiction. Therefore $r \geq c - a - r$, and rearranging, $r \geq \frac{c-a}{2}$.

Now consider s. By Problem 8b, $r + s = c - a$, and $r \leq s$ by assumption, so $2r \leq c - a$, and $r \leq \frac{c-a}{2}$. Thus the only possible factorization of f has $r = s = \frac{c-a}{2}$ (and $k = a$).

It remains to verify that this is a factorization of f. For convenience, let $d = \frac{c-a}{2}$. An earlier result showed that

$$f(x) = \min(2x + a, c) = (a \odot x^{\odot 2}) \oplus c.$$

Now apply the result of Problem 4b to g:

$$g(x) = a \odot (x \oplus d)^{\odot 2} = (a \odot x^{\odot 2}) \oplus (a \odot d^{\odot 2}) = (a \odot x^{\odot 2}) \oplus c = f(x),$$

as desired.

Authors' Note: Elle has in fact discovered the relatively new field of *tropical algebra*, which is currently an area of active mathematical research. For a further introduction to the field, including applications to algebraic geometry and computational biology, see the article "Tropical Mathematics" by David Speyer and Bernd Sturmfels, appearing in the June 2009 issue of *Mathematics Magazine*. This article was also invaluable to the authors as a source of inspiration and a few problems in this Power Question. It is available online at `https://math.berkeley.edu/~bernd/mathmag.pdf`.

2017 Relay Problems

R1-1. Compute the greatest integer b for which $\log_b(136!) - \log_b(135!) - \log_b(17)$ is an integer.

R1-2. Let $T = \text{TNYWR}$. Given that x and y are integers satisfying $x^2 - y^2 = T$, compute the least possible value of $x^2 + y^2$.

R1-3. Let $T = \text{TNYWR}$. Rectangle $SHAW$ has side lengths $SH = 24$ and $SW = T$. Point D lies on $\overline{AW}$ so that $\overline{HD} \perp \overline{SA}$. Point E is the intersection of $\overline{HD}$ and $\overline{SA}$. Compute DE.

R2-1. Compute the number of three-digit positive integers that are divisible by 11 and have middle digit 6.

R2-2. Let $T = \text{TNYWR}$. Compute the least positive integer N such that when a fair N-sided die whose faces are numbered consecutively from 1 through N is rolled once, the probability of rolling a factor of N is less than $\dfrac{1}{T}$.

R2-3. Let $T = \text{TNYWR}$. Given that $1 - r + r^3 - r^4 + r^6 - r^7 + \cdots = \dfrac{T^2}{1 + T + T^2}$, compute r.

2017 Relay Answers

R1-1. 8

R1-2. 10

R1-3. $\dfrac{125}{78}$

R2-1. 8

R2-2. 17

R2-3. $\dfrac{1}{17}$

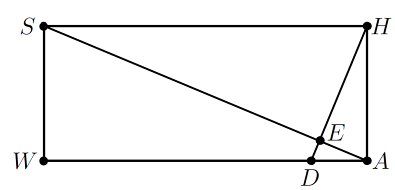
2017 Relay Solutions

R1-1. Using the property $\log_b x - \log_b y = \log_b \frac{x}{y}$, $\log_b(136!) - \log_b(135!) = \log_b\left(\frac{136!}{135!}\right) = \log_b(136)$ and $\log_b(136) - \log_b(17) = \log_b(136/17) = \log_b(8)$. Therefore the greatest integer b for which $\log_b(8)$ is an integer is $b = \mathbf{8}$.

R1-2. Note that $T = (x+y)(x-y)$. Let $m = x+y$ and $n = x-y$. Solving the system for x and y yields $(x, y) = \left(\frac{m+n}{2}, \frac{m-n}{2}\right)$. Therefore x and y are both integers if and only if m and n have the same parity. Thus it suffices to consider all factorizations of $T = mn$ such that m and n have the same parity and $m \geq n \geq 1$. (The reader can verify that the value of $x^2 + y^2$ does not change in the cases where $n = x+y$ and $m = x-y$ or where $-m = x+y$ and $-n = x-y$.) With $T = 8$, the only ordered pair of positive integers (m, n) of the same parity with $m \geq n$ and $mn = 8$ is $(4, 2)$, resulting in $(x, y) = (3, 1)$. Hence $x^2 + y^2 = 3^2 + 1^2 = \mathbf{10}$.

Alternate Solution: Noting that $x^2 + y^2 = x^2 - y^2 + 2y^2 = T + 2y^2$, if T is fixed, then $x^2 + y^2$ is minimized when y^2 is minimized. With $T = 8$, the integers $x = 3$ and $y = 1$ satisfy the given equation, so the least possible value of $x^2 + y^2$ is $\mathbf{10}$.

R1-3. Note that $\triangle SHE \sim \triangle ADE$ because $\angle HSE \cong \angle DAE$ and $\angle HES \cong \angle DEA$. Let $HE = a$ and $SE = b$. Then for some positive real number $k < 1$, it follows that $DE = ka$, $AE = kb$, and $AD = k \cdot SH = 24k$. Also note that $\triangle SHE \sim \triangle HAE \sim \triangle SAH$, hence $HE/SE = AE/HE = AH/SH$, which implies $a/b = kb/a = T/24$. Using $a/b = kb/a$, conclude that $k = a^2/b^2 = (T/24)^2$. Using $a/b = T/24$, conclude that $b = 24a/T$, and because $a^2 + b^2 = 24^2$ (by the Pythagorean Theorem), it follows that $a = \frac{24T}{\sqrt{T^2+24^2}}$. Hence $DE = ka = \frac{T^3}{24\sqrt{T^2+24^2}}$ $(\star)$. With $T = 10$, the denominator of the fraction in $(\star)$ simplifies to $24 \cdot 26$ and the numerator is $10^3 = 1000$. Because 1000 and 24 have a common factor of 8, the expression for DE simplifies to $\dfrac{125}{3 \cdot 26} = \mathbf{\dfrac{125}{78}}$.



R2-1. Represent the number as $\underline{A}\,\underline{6}\,\underline{B}$. Then $A - 6 + B$ must be a multiple of 11. This can only happen if $A + B = 6$ or $A + B = 17$. The first case has six possibilities $(A = 1, 2, 3, 4, 5, 6)$ and the latter case has two possibilities $(A = 8, 9)$ for a total of $\mathbf{8}$.

R2-2. Let $P(N)$ denote the probability of rolling a factor of N on a fair N-sided die. Then $P(1) = 1$, $P(2) = 1$, $P(3) = \frac{2}{3}$, $P(4) = \frac{3}{4}$, $P(5) = \frac{2}{5}$, and $P(6) = \frac{2}{3}$. More generally, $P(N) = \frac{2}{N}$ when N is prime and $P(N)$ is greater than $\frac{2}{N}$ when N is composite. Try the least prime N such that $\frac{2}{N} < \frac{1}{T}$, or $N > 2T$. As $T = 8$, try $N = 17$. It is straightforward to see that $P(17) = \frac{2}{17} < \frac{1}{8}$, and it can also easily be checked that for $1 \leq N \leq 16$, $P(N) \geq \frac{1}{T}$. Thus $N = \mathbf{17}$.

R2-3. The left-hand side of the given equation is $(1-r)+r^3(1-r)+r^6(1-r)+\cdots = \frac{1-r}{1-r^3} = \frac{1}{1+r+r^2}$ for $|r|<1$. The right-hand side of the given equation is $\frac{1}{\frac{1}{T^2}+\frac{1}{T}+1}$, so $r=\frac{1}{T}$ is a solution as long as $|T|>1$. More generally, $\frac{1}{1+r+r^2} = \frac{T^2}{1+T+T^2} \Rightarrow r^2+r+1 = \frac{1}{T^2}+\frac{1}{T}+1 \Rightarrow r^2+r-\frac{1}{T^2}-\frac{1}{T} = 0 \Rightarrow (r-\frac{1}{T})(r+1+\frac{1}{T}) = 0$ so r is either $\frac{1}{T}$ or $-1-\frac{1}{T}$. As $T=17$, the series only converges for $r = \mathbf{\dfrac{1}{17}}$.

2017 Tiebreaker Problems

TB-1. Compute the least positive N such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$

TB-2. Cube $ARMLKHJC$, with opposite faces $ARML$ and $HJCK$, is inscribed in a cone so that A is the vertex of the cone, edges $\overline{AR}, \overline{AL}, \overline{AH}$ lie on the surface of the cone, and vertex C, diagonally opposite A, is on the base of the cone. Given that $AR = 6$, compute the radius of the cone.

TB-3. Given that $\log_{b^3} a^2 + \log_{b^9} a^4 + \log_{b^{27}} a^8 + \cdots = 1$, compute $\log_b a$.

2017 Tiebreaker Answers

TB-1. 137

TB-2. $6\sqrt{6}$

TB-3. $\dfrac{1}{2}$

2017 Tiebreaker Solutions

TB-1. Because $f(x)$ is a quadratic polynomial, the graph of $y = f(x)$ is symmetric about some vertical line; for each output, there are no more than two corresponding inputs. Thus if $f(p) = f(q) = f(r) = f(s)$, then the variables p, q, r, s can take on at most two distinct values. Hence the set $\{f(1), f(5), f(7), f(11)\}$ contains at most two distinct integers. However, the same logic shows that the equation $f(p) = f(q) = f(r)$ has no solutions when p, q, r are required to be distinct. Hence it is impossible that three of the values $f(1), f(5), f(7), f(11)$ be the same. Thus the set $\{f(1), f(5), f(7), f(11)\}$ contains exactly two distinct integers, dividing the expressions $f(1), f(5), f(7), f(11)$ into two pairs of equal values. Because of the symmetry of the graph, there must be a unique value of h such that $f(x) = f(2h-x)$ for all x. It follows that $f(1) = f(11)$ and $f(5) = f(7)$, with $h = 6$. (Any other pairing yields a contradiction: for example, if $f(1) = f(7)$, then $h = 4$, but the equality of the other pair $f(5) = f(11)$ yields $h = 8$.) Hence $f(x)$ is of the form $f(x) = a(x-6)^2+k$, and $f(f(x)) = a\big(a(x-6)^2+k-6\big)^2+k$.

The following argument shows that N is minimal when $a = 1$. From the condition that $f(f(1)) = f(f(5)) = f(f(7)) = f(f(11))$, it follows that

$$a(f(1) - 6)^2 + k = a(f(5) - 6)^2 + k = a(f(7) - 6)^2 + k = a(f(11) - 6)^2 + k.$$

Simplifying yields the following equation:

$$\pm(f(1) - 6) = \pm(f(5) - 6) = \pm(f(7) - 6) = \pm(f(11) - 6).$$

Moreover, it follows from the above symmetry argument that $f(5) = f(7) \neq f(1) = f(11)$. Thus $f(1) - 6 = -(f(5) - 6)$. Because $f(1) = 25a + k$, and $f(5) = a + k$, it follows that $25a + k - 6 = -a - k + 6$, which is equivalent to $k = 6 - 13a$. (Notice that the equation $f(7) - 6 = -(f(11) - 6)$ adds no information and yields an equivalent equation.) Now, computing $N = f(f(5))$,

$$\begin{aligned} f(f(5)) &= f(a+k) \\ &= a \cdot (a+k-6)^2 + k \\ &= a \cdot (-12a)^2 + (6 - 13a) \\ &= 144a^3 - 13a + 6. \end{aligned}$$

So the problem reduces to finding the minimum positive value of the expression $144a^3-13a+6$ when a is an integer. First note that $a \neq 0$ because f is a quadratic polynomial. Next, if $a \leq -1$, then $a^3 \leq a$, so

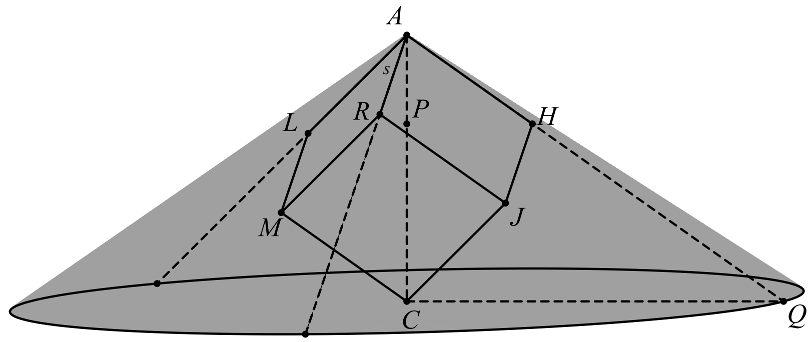
$$N = 144a^3 - 13a + 6 \leq 144a - 13a + 6 = 131a + 6 \leq -131 + 6 < 0,$$

which violates the condition that N be positive. Because a is an integer, no values between -1 and 1 need be considered. For $a \geq 1$, $a^3 \geq a$, so

$$N = 144a^3 - 13a + 6 \geq 144a - 13a + 6 = 131a + 6,$$

with equality precisely when $a = 1$. Thus, with $a = 1$, $N = 137$, and for any other positive value of a, N will be strictly larger. Hence the least positive value of N is **137.**

TB-2. The cone and cube are pictured in the diagram below.



First note that point C must be the center of the cone's base because of the rotational symmetry of vertices R, L, H about the axis of the cone. Thus AC is the height of the cone. Furthermore, plane RLH is parallel to the cone's base (again, because of the rotational symmetry of vertices R, L, H). Let P be the point where $\overline{AC}$ intersects plane RLH, and let $s = AR$. Because $\triangle RLH$ is equilateral with side length $s\sqrt{2}$, conclude that P is the centroid of $\triangle RLH$, and $PH = \frac{2}{3}\left(\frac{s\sqrt{2}}{2} \cdot \sqrt{3}\right) = \frac{s\sqrt{6}}{3}$. Use the Pythagorean Theorem on $\triangle APH$ to compute $AP = \frac{s\sqrt{3}}{3}$. Extend $\overrightarrow{AH}$ to meet the bottom of the cone at Q. Because $AC = s\sqrt{3}$ and $\triangle APH \sim \triangle ACQ$, it follows that $\frac{CQ}{PH} = \frac{AC}{AP} = 3$, hence $CQ = s\sqrt{6}$. Substitute $s = 6$ to obtain $CQ = \mathbf{6\sqrt{6}}$.

TB-3. Apply logarithm laws to the left-hand side of the given equation:

$$\begin{aligned}\log_{b^3} a^2 + \log_{b^9} a^4 + \log_{b^{27}} a^8 + \cdots &= \frac{2}{3}\log_b a + \frac{4}{9}\log_b a + \frac{8}{27}\log_b a + \cdots \\ &= (\log_b a)\left(\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots\right),\end{aligned}$$

which is an infinite geometric series with first term $\frac{2}{3}\log_b a$ and common ratio $\frac{2}{3}$. Hence the sum of the series is $2\log_b a = 1$. Therefore $\log_b a = \mathbf{\frac{1}{2}}$.

2017 Super Relay Problems

1. Compute the units digit of $17^1 + 17^2 + 17^3 + \cdots + 17^{16} + 17^{17}$.

2. Let $T = \text{TNYWR}$. Given that the polynomial $x^2 - 17x + TK$ can be factored over the integers and that K is an integer, compute the greatest possible value of K.

3. Let $T = \text{TNYWR}$ and let $K = 17 - T$. Let
$$P = \sqrt{K + \sqrt{K + \sqrt{K + \cdots}}} \qquad \text{and} \qquad J = 1 + \cfrac{K}{1 + \cfrac{K}{1 + \cfrac{K}{1 + \cdots}}}.$$
Compute $\lfloor P - J \rfloor$.

4. Let $T = \text{TNYWR}$. In $\triangle PJK$, each of $\text{m}\angle P$ and $\text{m}\angle J$ is an integral multiple of 17° and $\text{m}\angle P \geq \text{m}\angle J$. Let S be the number of triangles that satisfy these conditions, where no two of these triangles are similar to one another. Compute $S + T$.

5. Let $T = \text{TNYWR}$. Square $IJKL$ has area T. Diagonal KI is extended past I to point P so that $PK = 17\sqrt{2}$. Points E and O lie in the plane so that $EIOP$ is a square and E and L lie on the same side of $\overline{PK}$. Compute the perimeter of trapezoid $POLE$.

6. Let $T = \text{TNYWR}$. Donald and John each play the trumpet. They take turns (starting with Donald), where each person plays a note subject to the constraint that neither person can play the same note that was last played by the other person. There are a total of 17 different notes they can play and between them, they play a total of $\lfloor T + 1 \rfloor$ notes. Given that the number of melodies they can play can be expressed in the form $w \cdot x^y$, where w, x, and y are positive integers and w and x are relatively prime, compute the least possible value of $w + x + y$.

7. Let $T = \text{TNYWR}$. A triangle is similar to an $8-15-17$ triangle and one of its sides is T. Given that the perimeter of this triangle is an integer, compute the least possible perimeter this triangle can have.

8. Let J be the number you will receive from position 7 and let K be the number you will receive from position 9. Let A be the largest prime factor of J and let P be the largest prime factor of K. Consider the following system of equations:

$$x^3 + y = A$$
$$x^2 + 4y = P.$$

This system has three solutions, two of which are (x_1, y_1) and (x_2, y_2), where x_1 and x_2 are *nonreal*. Compute the value of $x_1 + x_2$.

9. Let T = TNYWR. Bryan has 1700 sheets of graph paper and is stuffing ARML team envelopes with 30 sheets in each envelope. Being exacting, it takes him 11 minutes and 20 seconds to count and stuff $\lfloor T - 3 \rfloor$ sheets of paper into one or more envelopes. He completely stuffs as many envelopes as he can, until he has fewer than 30 sheets remaining, at which point, he stops. Rounded to the nearest minute, compute the number of minutes it will take Bryan to complete this arduous task.

10. Let T = TNYWR. Iris uses voice recognition software to queue up her favorite songs. When she says a song title aloud, the probability that the software identifies the wrong song is $\frac{1}{100} \max\left(T, \frac{17}{T}\right)$. Given that Iris says 40 song titles from her favorite band, Iron Maiden, into her voice recognition software, compute the expected number of songs that the software correctly identifies.

11. Let T = TNYWR, let $K = T - 10$, and let $P = 2^{17}$. Compute

$$\left\lceil \log_P \left(2^0 + 2^1 + 2^2 + \cdots + 2^{K-1} + 2^K\right) \right\rceil.$$

12. Let T = TNYWR. The ages of Catherine, Charlie, and Elizabeth are integers. Charlie's age is $\lfloor 4T + 68 \rfloor$, and the sum of Catherine's and Elizabeth's ages is 100. Nine years ago, Elizabeth's age was a positive multiple of 17 and that number had a common factor (greater than 1) with Charlie's current age. Compute Elizabeth's current age.

13. Let T = TNYWR. The circle defined by $x^2 + y^2 = 17$ intersects line $\ell : y = -Tx + 3$ in two points (x_1, y_1) and (x_2, y_2). Let P be the product $\dfrac{1}{x_1} \cdot \dfrac{1}{x_2}$, let J be the sum $x_1 + x_2$, and let K be the slope of a line perpendicular to ℓ. Compute PJK.

14. Let T = TNYWR and let $P = \left\lfloor \sqrt{T} \right\rfloor$. The complex number $\dfrac{20}{i + P} + \dfrac{17}{i - P - 1}$ can be expressed in the form $J + Ki$, where J and K are real. Compute $J + K$.

15. Compute the number of ordered triples of integers (a, b, c) that are solutions to the equation $abc = 17$.

2017 Super Relay Answers

1. 7

2. 10

3. 0

4. 25

5. 54

6. 87

7. 232

8. $-\dfrac{11}{4}$

9. 544

10. 38

11. 5

12. 94

13. $-\dfrac{3}{4}$

14. -1

15. 12

2017 Super Relay Solutions

1. The units digit of 17^n is the same as the units digit of 7^n. The sequence of units digits of 7^n is readily found to be $7, 9, 3, 1, 7, 9, \ldots$, which is periodic with period 4. Moreover, the sum of any four consecutive terms of this sequence has the same units digit as $7 + 9 + 3 + 1 = 20$. Thus $17^1 + 17^2 + 17^3 + \cdots + 17^{16}$ has a units digit of 0, and the units digit of the given sum is the same as the units digit of 17^{17}, which is **7**.

2. Suppose that $x^2 - 17x + TK$ can be factored as $(x - r)(x - s)$. By Vieta's Formulas, $r + s = 17$ and $rs = TK$. Now consider different possibilities for the sign of T. If $T = 0$, then K is not uniquely determined. If $T < 0$, then $K = \frac{rs}{T} = \frac{r(17-r)}{T}$ can be written as a quadratic function of r with a positive leading coefficient, so K has no maximum value. If $T > 0$, then to determine the possible values of K, examine pairs of positive integers r and s that sum to 17 such that T divides rs. With $T = 7$, consider $r = 7, s = 10$, so that $K = \frac{rs}{T} = 10$, or $r = 14, s = 3$, so that $K = \frac{rs}{T} = 6$. Thus the greatest possible value of K is **10**.

3. To compute P, write $P = \sqrt{K + P}$. This is equivalent to $P^2 - P = K$. To compute J, write $J = 1 + \frac{K}{J}$. This is equivalent to $J^2 - J = K$. Thus P and J are both roots of the equation $x^2 - x - K = 0$. The roots of this equation are $x = \frac{1 \pm \sqrt{1+4K}}{2}$. If $K = 0$, then by inspection, $P = 0$ and $J = 1$. If $K > 1$, then $P = J = \frac{1+\sqrt{1+4K}}{2}$ because the other root is negative, yet $P > 0, J > 0$. With $T = 10$, $K = 17 - 10 = 7 > 0$. Thus $P - J = 0$, and the answer is **0**.

4. Let $p = \mathrm{m}\angle P = (17a)^\circ$ and $j = \mathrm{m}\angle J = (17b)^\circ$, where a and b are positive integers. Then $a \geq b$ and $2 \leq n \leq 10$, where $n = a + b$. By examining small values of n, conclude that the equation $a + b = n$ has $\lfloor \frac{n}{2} \rfloor$ solutions with $a \geq b$. Thus there are $2(1 + 2 + 3 + 4) + 5 = 25$ possible solutions, so $S = 25$. With $T = 0$, $S + T = \mathbf{25}$.

5. The side of square $IJKL$ is $\sqrt{T}$ and the diagonal's length is $\sqrt{2T}$. Thus the diagonal of square $EIOP$ has length $17\sqrt{2} - \sqrt{2T}$, hence the side of square $EIOP$ is $17 - \sqrt{T}$. Note that $OL = OI + IL = (17 - \sqrt{T}) + \sqrt{T} = 17$, $PO = EP = EI = 17 - \sqrt{T}$, and $LE = \sqrt{EI^2 + IL^2}$. Thus the perimeter of trapezoid $POLE$ is $51 - 2\sqrt{T} + \sqrt{2T - 34\sqrt{T} + 289}$. With $T = 25$, $\sqrt{T} = 5$. Thus square $EIOP$ has side length $17 - 5 = 12$, $\triangle LIE$ is a 5-12-13 triangle, and the perimeter of $POLE$ is **54**.

6. The first note can be played in 17 ways. Each note after the first can be played in 16 ways. Thus the total number of melodies that Donald and John can play is $17 \cdot (2^4)^{\lfloor T \rfloor}$. Thus $w = 17$. With $T = 54$, the possible values of x are numbers of the form 2^d where d is a divisor of $216 = 4 \cdot 54$. Some quick computations reveal that $x + y = 2^d + \frac{216}{d}$ is minimized when $d = 4$ so that $x = 16$ and $y = 54$. Thus the least possible value of $w + x + y$ is $17 + 16 + 54 = \mathbf{87}$.

7. Let the triangle have sides $8x, 15x, 17x$, where $x > 0$. Then the triangle's perimeter is $P = 40x$ and $T \in \{8x, 15x, 17x\}$. If $T = 8x$, then $P = 5(8x) = 5T$. If $T = 15x$, then $P = \frac{8T}{3}$. Finally, if $T = 17x$, then $P = \frac{40T}{17}$. Note that $\frac{40}{17} < \frac{8}{3} < 5$. With $T = 87$, note that 17 does not divide 87, hence P is not an integer. But because 3 divides 87, $P = \frac{8T}{3}$ is integer and is equal to **232**.

8. Eliminate y by multiplying the first equation by 4 and subtracting the second equation to obtain $4x^3 - x^2 - 4A + P = 0$. The sum of the three roots of this equation is $-\left(\frac{-1}{4}\right) = \frac{1}{4}$. Let $f(x) = 4x^3 - x^2 - 4A + P$. As long as A and P are real, the equation $f(x) = 0$ will have at least one real root (to see this, consider the graph of $y = f(x)$). The problem implies that two of the roots of the equation $f(x) = 0$ are nonreal and are equal to x_1 and x_2. To determine $x_1 + x_2$, it suffices to determine the real root r of $f(x) = 0$, and then $x_1 + x_2$ will be equal to $\frac{1}{4} - r$. With $J = 232 = 2^3 \cdot 29$ and $K = 544 = 2^5 \cdot 17$, it follows that $A = 29$, $P = 17$, and $f(x) = 4x^3 - x^2 - 99$. Note that $f(3) = 108 - 9 - 99 = 0$, hence $x = 3$ is the real root of $f(x) = 0$, and it can be verified that the quadratic factor of $f(x)$ (i.e., $4x^2 + 11x + 33$) has a negative discriminant, hence no real roots. Thus the answer is $\frac{1}{4} - 3 = -\mathbf{\frac{11}{4}}$.

9. Bryan can stuff $\left\lfloor \frac{1700}{30} \right\rfloor = 56$ envelopes, so he stuffs a total of $56 \cdot 30 = 1680$ sheets of paper. Note also that 11 minutes and 20 seconds is equivalent to $\frac{34}{3}$ minutes. Let S be the number of minutes Bryan needs to stuff the 1680 sheets. Then $\lfloor T - 3 \rfloor / \left(\frac{34}{3}\right) = 1680/S$. Thus $S = \frac{56 \cdot 30 \cdot 34}{3\lfloor T-3 \rfloor}$. With $T = 38$, $S = \frac{56 \cdot 30 \cdot 34}{3 \cdot 35} = 8 \cdot 2 \cdot 34 = \mathbf{544}$.

10. When a song title is read aloud, the probability that the software identifies the correct song is $1 - \frac{1}{100} \max\left(T, \frac{17}{T}\right)$. Thus the expected number of correctly identified songs is $40 - \max\left(\frac{2T}{5}, \frac{34}{5T}\right)$. With $T = 5$, $\max\left(\frac{2T}{5}, \frac{34}{5T}\right) = \max\left(2, \frac{34}{25}\right) = 2$. Thus the answer is $40 - 2 = \mathbf{38}$.

11. By the formula for the sum of the terms of a geometric series, it follows that $2^0 + 2^1 + 2^2 + \cdots + 2^{K-1} + 2^K = 2^{K+1} - 1$. Thus $\log_P\left(2^0 + 2^1 + 2^2 + \cdots + 2^{K-1} + 2^K\right) \lessapprox \log_{2^{17}} 2^{K+1} = \frac{K+1}{17}$. With $T = 94$, $K = 84$, and $\frac{K+1}{17} = 5$. Thus $\left\lceil \log_P\left(2^0 + 2^1 + 2^2 + \cdots + 2^{K-1} + 2^K\right)\right\rceil = \mathbf{5}$.

12. List the multiples of 17 that are less than 100: $17, 34, 51, 68, 85$. Thus Elizabeth's current age is one of $26, 43, 60, 77$, or 94. With $T = -\frac{3}{4}$, Charlie's current age is $65 = 5 \cdot 13$. The only possible age of Elizabeth nine years ago that shares a common factor greater than 1 with 65 is 85. Hence Elizabeth's current age is **94**.

13. Because the slope of ℓ is $-T$, it follows that $K = \frac{-1}{-T} = \frac{1}{T}$. Substitute $y = -Tx + 3$ into the equation $x^2 + y^2 = 17$ to obtain $x^2 + (-Tx + 3)^2 = 17$ or $(1 + T^2)x^2 - 6Tx - 8 = 0$. The two roots of this equation are x_1 and x_2. By Vieta's Formulas, $x_1 + x_2 = \frac{6T}{1+T^2}$ and $x_1 x_2 = \frac{-8}{1+T^2}$. Hence $P = \frac{1+T^2}{-8}$ and $PJ = \frac{6T}{-8}$. Finally, $PJK = \left(\frac{6T}{-8}\right) \cdot \frac{1}{T} = -\mathbf{\frac{3}{4}}$ (independent of T).

14. Note that $\frac{20}{i+P} = \frac{20}{i+P} \cdot \frac{-i+P}{-i+P} = \frac{-20i+20P}{P^2+1}$. Similarly, $\frac{17}{i-P-1} = \frac{17}{i-P-1} \cdot \frac{-i-P-1}{-i-P-1} = \frac{-17i-17(P+1)}{(P+1)^2+1}$. With $T = 12$, $P = \lfloor\sqrt{12}\rfloor = 3$. Thus the given expression is equal to $\frac{-20i+60}{10} + \frac{-17i-68}{17} = (-2i+6)+(-i-4) = 2-3i$. Hence $J = 2, K = -3$, and $J + K = \mathbf{-1}$.

15. If a, b, and c are all positive, then two of the variables must be 1 and the third variable must be 17. This gives 3 solutions. If not all of a, b, and c are positive, then exactly two variables must be negative. For each of the 3 positive solutions, there are 3 choices for which variables could be negative. Thus the answer is $3 + 3 \cdot 3 = \mathbf{12}$.

2018 Contest

Contents

2018 Team Problems

T-1. Compute the number of non-empty subsets S of $\{1, 2, 3, \ldots, 20\}$ for which $|S| \cdot \max\{S\} = 18$. (Note: $|S|$ is the number of elements of the set S.)

T-2. A class of 218 students takes a test. Each student's score is an integer from 0 to 100, inclusive. Compute the greatest possible difference between the mean and the median scores.

T-3. Regular hexagon $RANGES$ has side length 6. Pentagon $RANGE$ is revolved 360° about the line containing $\overline{RE}$ to obtain a solid. The volume of the solid is $k \cdot \pi$. Compute k.

T-4. A fair 12-sided die has faces numbered 1 through 12. The die is rolled twice, and the results of the two rolls are x and y, respectively. Given that $\tan(2\theta) = \frac{x}{y}$ for some θ with $0 < \theta < \frac{\pi}{2}$, compute the probability that $\tan\theta$ is rational.

T-5. The absolute values of the five complex roots of $z^5 - 5z^2 + 53 = 0$ all lie between the positive integers a and b, where $a < b$ and $b - a$ is minimal. Compute the ordered pair (a, b).

T-6. Let S be the set of points (x, y) whose coordinates satisfy the system of equations:

$$\lfloor x \rfloor \cdot \lceil y \rceil = 20$$
$$\lceil x \rceil \cdot \lfloor y \rfloor = 18.$$

Compute the least upper bound of the set of distances between points in S.

T-7. Compute the least integer $d > 0$ for which there exist distinct lattice points A, B, and C in the coordinate plane, each exactly $\sqrt{d}$ units from the origin, satisfying $\csc(\angle ABC) > 2018$.

T-8. Compute the number of unordered collections of three integer-area rectangles such that the three rectangles can be assembled without overlap to form one 3×5 rectangle. (For example, one such collection contains one 3×3 and two 1×3 rectangles, and another such collection contains one 3×3 and two 2×1.5 rectangles. The latter collection is equivalent to the collection of two 1.5×2 rectangles and one 3×3 rectangle.)

T-9. Let Γ be a circle with diameter $\overline{XY}$ and center O, and let γ be a circle with diameter $\overline{OY}$. Circle ω_1 passes through Y and intersects Γ and γ again at A and B, respectively. Circle ω_2 also passes through Y and intersects Γ and γ again at D and C, respectively. Given that $AB = 1$, $BC = 4$, $CD = 2$, and $AD = 7$, compute the sum of the areas of ω_1 and ω_2.

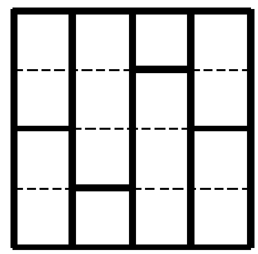
T-10. In the number puzzle below, clues are given for the four rows, each of which contains a four-digit number. Cells inside a region bounded by bold lines must all contain the same digit, and each of the eight regions contains a different digit. The variables in the clues are all positive integers. Complete the number puzzle.

1: $4^a + 13^b + 14^c$

2: $5^p + 13^q + 17^r$

3: $4^x + 5^y + 31^z$

4: the average of the other 3 rows



2018 Team Answers

T-1. 19

T-2. $\dfrac{5400}{109}$

T-3. $342\sqrt{3}$

T-4. $\dfrac{1}{18}$

T-5. $(2, 3)$

T-6. $2\sqrt{73}$

T-7. 2,038,181

T-8. 132

T-9. 20π

T-10.

5	1	9	7
5	1	0	7
4	1	0	2
4	8	0	2

2018 Team Solutions

T-1. Both $|S|$ and $\max\{S\}$ are integers between 1 and 20, so factor 18 and consider the possible ways to have $|S|$ and $\max\{S\}$ equal each factor. Then count the number of possible sets S that satisfy that condition.

$\boldsymbol{\lvert S \rvert}$	**max**$\boldsymbol{\{S\}}$	**Number of possible sets** $\boldsymbol{S}$
1	18	1: this can only be the set $S = \{18\}$.
2	9	8: this can be any set $S = \{a, 9\}$, where $1 \leq a \leq 8$.
3	6	$\binom{5}{2} = 10$: the greatest element must be 6, and there are $\binom{5}{2}$ ways of selecting the two other elements of S.
6	3	0: this is impossible because S cannot have 6 elements, the greatest of which is 3.
9	2	0: this is impossible because S cannot have 9 elements, the greatest of which is 2.
18	1	0: this is impossible because S cannot have 18 elements, the greatest of which is 1.

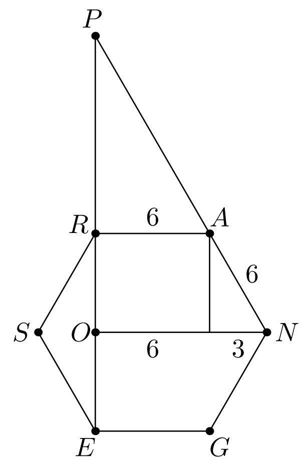
Thus the answer is $1 + 8 + 10 = \mathbf{19}$.

T-2. Intuitively, the maximum difference can be attained by forcing the median to be 0 by selecting just enough 0 scores, and then maximizing the mean by maximizing the remaining scores. Such a distribution would have 110 scores of 0 and 108 scores of 100, giving a median score of 0 and a mean score of $\frac{108 \cdot 100}{218} = \frac{5400}{109}$. This gives a difference of $\frac{5400}{109}$.

To prove that this is maximal, assume that the median score is m. Then, by definition, at least 109 scores must be at most m. Further, the remaining 109 scores are at most 100, so the mean μ satisfies $\mu \leq \frac{1}{218}(109m + 10900) = \frac{m+100}{2}$. Therefore $\mu - m \leq \frac{100-m}{2}$.

If $m = 0$, then the maximum mean is found as above. If $m \geq 1$, then $\mu - m \leq \frac{100-1}{2} = 49.5 < \frac{5400}{109}$, and the difference is greater than it is in the above case with $m = 0$. There only remains the possibility that $m = \frac{1}{2}$. For this to be the case, 109 scores must be 0 and one score must be 1. To maximize the mean, the remaining 108 scores must be 100. However, compared to the case of having 110 scores of 0 and 108 of 100, this increases the median by $\frac{1}{2}$ but increases the mean by only 1218, so the difference will certainly be smaller. Thus the answer is $\boldsymbol{\frac{5400}{109}}$.

T-3. First note that because $\overline{RA}$ and $\overline{EG}$ are parallel, the solid obtained will consist of two conic frustums, each with a base along the plane through $\overline{NS}$ and perpendicular to $\overline{ER}$. It is easier to compute the volume of one of these conic frustums at a time. Let O be the midpoint of $\overline{ER}$, and let $\overrightarrow{NA}$ and $\overrightarrow{ER}$ intersect at P. Then the conic frustum is a cone with radius ON and height OP, cut off by the plane containing $\overline{AR}$ and perpendicular to $\overline{ER}$.



Note that $ON = NS - OS = 12 - 3 = 9$, and $OP = OR + RP = 3\sqrt{3} + 6\sqrt{3} = 9\sqrt{3}$. Then the volume of one of the conic frustums is

$$\frac{1}{3}\pi\left(9^2\right)\left(9\sqrt{3}\right) - \frac{1}{3}\pi\left(6^2\right)\left(6\sqrt{3}\right) = 171\sqrt{3}\,\pi.$$

The volume of the entire solid is therefore $342\sqrt{3}\cdot\pi$, so $k = \mathbf{342\sqrt{3}}$.

T-4. Suppose $\tan 2\theta = \frac{x}{y}$. Then the identity for the tangent of a sum gives $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{x}{y}$, which is equivalent to $x - x\tan^2\theta = 2y\tan\theta$. Rearranging gives a quadratic equation in $\tan\theta$: $x\tan^2\theta + 2y\tan\theta - x = 0$. Thus $\tan\theta$ is rational if and only if the discriminant is a perfect square. The discriminant is $(2y)^2 - 4\cdot x\cdot(-x) = 4(x^2+y^2)$, which is a perfect square if and only if x^2+y^2 is a perfect square. The only Pythagorean triples (a,b,c) with $a,b \le 12$ are $(3,4,5),(6,8,10),(9,12,15)$, and $(5,12,13)$. Therefore the only (x,y) that give a rational value for $\tan\theta$ are $(3,4),(4,3),(6,8),(8,6),(9,12),(12,9),(5,12),(12,5)$. The desired probability is therefore $\frac{8}{12\cdot 12} = \mathbf{\frac{1}{18}}$.

T-5. First note that the question asks for the tightest possible integer bounds on the magnitude of a zero of the complex polynomial $z^5 - 5z^2 + 53$, essentially describing the smallest complex annulus with integer radii that contains the roots.

Now recall two important properties of the absolute value function: $|ab| = |a||b|$ and $|a+b| \le |a|+|b|$ for all complex numbers a and b. The latter also implies that $|a|-|b| \le |a-b|$. In order to have $z^5 - 5z^2 + 53 = 0$, it must be the case that $|z^5 - 5z^2| = 53$. This is enough to restrict the possible magnitude of z.

Suppose that $|z| > 3$. Then $|z^5 - 5z^2| = |z|^2|z^3 - 5| > 9|z^3 - 5| \geq 9(|z|^3 - 5) > 9(27 - 5) = 198 > 53$. So $|z| \leq 3$.

Suppose that $|z| < 2$. Then $|z^5 - 5z^2| = |z|^2|z^3 - 5| < 4|z^3 - 5| \leq 4(|z|^3 + |5|) < 4(8 + 5) = 51 < 53$. So $|z| \geq 2$.

Therefore $2 \leq |z| \leq 3$ for all roots z of $z^5 - 5z^2 + 53$. Because $a < b$, and a and b are integers, $b - a$ is at least 1, so this is the best possible bound. The answer is $\mathbf{(2, 3)}$.

T-6. First note that x and y cannot both be integers because otherwise, the given system would be inconsistent. Now suppose that exactly one of x or y is an integer. If x is an integer and $\lfloor y \rfloor = b$, then $\lceil y \rceil = b + 1$, and multiplying the two given equations and simplifying results in the equation $x^2 \cdot b(b + 1) = 360$ (†). Because $1 \leq x^2 < 360$, the only possible values of x^2 are the perfect square factors of 360, namely 1, 4, 9, and 36. Substituting in $x^2 = 1, 9$, and 36 into (†) results in non-integral solutions for b. But substituting $x^2 = 4$ into (†) results in $b^2 + b - 90 = (b + 10)(b - 9) = 0$, and $b = -10$ or $b = 9$. If $x = 2$, then $\lfloor y \rfloor = 9$ and $\lceil y \rceil = 10$, and this yields the solutions $(2, y)$, where $9 < y < 10$. On the other hand, if $x = -2$, then $\lfloor y \rfloor = -9$ and $\lceil y \rceil = -10$, which is impossible. By a similar argument, if x is not an integer and y is an integer, this results in the solutions $(x, -2)$, where $-10 < x < -9$. Thus if $\mathcal{I}_1$ is the open interval $(9, 10)$ and $\mathcal{I}_2$ is the open interval $(-10, -9)$, then the solutions (x, y) to the given system where exactly one of x and y is an integer are the ordered pairs belonging to the union of the two sets $\{2\} \times \mathcal{I}_1$* and $\mathcal{I}_2 \times \{-2\}$. Graphically, these represent segments of unit length (one vertical, one horizontal) that do not include the endpoints. For later reference, let $\mathcal{I}_x = \mathcal{I}_2 \times \{-2\}$ and let $\mathcal{I}_y = \{2\} \times \mathcal{I}_1$.

Now suppose that neither x nor y is an integer and that $\lfloor x \rfloor = a$ and $\lfloor y \rfloor = b$. Then $\lceil x \rceil = a + 1$, $\lceil y \rceil = b + 1$, and multiplying the two given equations and simplifying results in the equation $a(a + 1)b(b + 1) = 360$ (‡). By noting the factorization $360 = 3 \cdot 4 \cdot 5 \cdot 6$, the following ordered pairs (a, b) satisfy (‡):

$$(3, 5); \quad (-4, 5); \quad (-4, -6); \quad (3, -6); \quad (5, 3); \quad (5, -4); \quad (-6, -4); \quad (-6, 3).$$

However, only $(a, b) = (5, 3)$ and $(-4, -6)$ satisfy the given system:

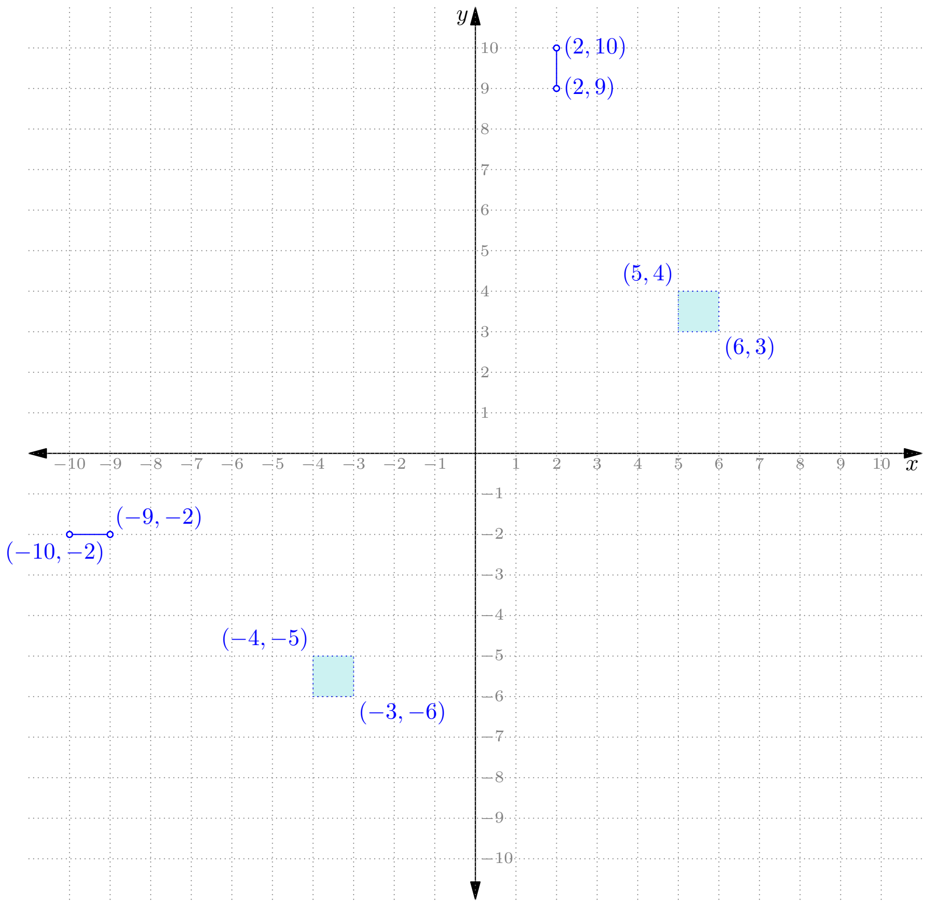
$$\begin{aligned} a(b + 1) &= 20 \\ (a + 1)b &= 18. \end{aligned}$$

Let $\mathcal{I}_3$ and $\mathcal{I}_4$ denote the open intervals $(5, 6)$ and $(3, 4)$, respectively and let $\mathcal{I}_5$ and $\mathcal{I}_6$ denote the open intervals $(-4, -3)$ and $(-6, -5)$, respectively. Then the solutions (x, y) to the given system in which neither x nor y is an integer are given by the union of the two regions $\mathcal{R}_1$ and $\mathcal{R}_2$, defined by

$$\mathcal{R}_1 = \mathcal{I}_3 \times \mathcal{I}_4 \quad \text{and} \quad \mathcal{R}_2 = \mathcal{I}_5 \times \mathcal{I}_6.$$

*Here, $\times$ denotes the Cartesian product. I.e., if P and Q are two sets, then $P \times Q = \{(p, q) \mid p \in P \text{ and } q \in Q\}$.

Each of $\mathcal{R}_1$ and $\mathcal{R}_2$ is the interior of a unit square ($\mathcal{R}_1$ lies in the first quadrant and $\mathcal{R}_2$ lies in the third quadrant). Also note that the solutions are symmetric about the line $y = -x$. A plot of the points of S is shown.



The answer to the problem is $\max\{\ell_1, \ell_2, \ell_3\}$, where:

- ℓ_1 is the least upper bound of the distances between a point in $\mathcal{I}_x$ and a point in $\mathcal{I}_y$;
- ℓ_2 is the least upper bound of the distances between a point in $\mathcal{R}_1$ and a point in $\mathcal{R}_2$;
- ℓ_3 is the least upper bound of the distances between a point in $\mathcal{I}_x \cup \mathcal{I}_y$ and a point in $\mathcal{R}_1 \cup \mathcal{R}_2$.

Compute ℓ_1 by taking the distance between the extremal boundary points of $\mathcal{I}_x$ and $\mathcal{I}_y$, namely $(-10, -2)$ and $(2, 10)$, respectively. This distance is $\sqrt{2 \cdot 12^2} = \sqrt{288}$.

Compute ℓ_2 by taking the distance between the extremal boundary points of $\mathcal{R}_1$ and $\mathcal{R}_2$, namely $(6, 4)$ and $(-4, -6)$, respectively. This distance is $\sqrt{2 \cdot 10^2} = \sqrt{200}$.

Compute ℓ_3 by taking the distance between the extremal boundary points of $\mathcal{I}_x$ and $\mathcal{R}_1$

(or of $\mathcal{I}_y$ and $\mathcal{R}_2$, owing to the symmetry), namely $(-10,-2)$ and $(6,4)$, respectively. This distance is $\sqrt{16^2+6^2}=\sqrt{292}$.

Thus the answer is $\max\{\sqrt{288},\sqrt{200},\sqrt{292}\}=\sqrt{292}=\mathbf{2\sqrt{73}}$.

T-7. The points A, B, and C lie on a circle with diameter $2\sqrt{d}$. Consequently, by the Extended Law of Sines,

$$2\sqrt{d}=\frac{AC}{\sin(\angle ABC)} \quad\to\quad \csc(\angle ABC)=\frac{2\sqrt{d}}{AC}.$$

The solution now proceeds in three cases.

- It is impossible that $AC=1$. Indeed, if $A=(x,y)$, then without loss of generality, assume $C=(x+1,y)$. Then $d=x^2+y^2=(x+1)^2+y^2$, which implies $2x+1=0$, contradiction.

- Suppose $AC=\sqrt{2}$. If $A=(x,y)$, then without loss of generality, assume $C=(x+1,y+1)$, by reflecting or rotating as necessary. Then $d=x^2+y^2=(x+1)^2+(y+1)^2$, hence $2x+1+2y+1=0$. It follows that $y=-(x+1)$ and d must be of the form $d=x^2+(x+1)^2$. In that case,

$$\csc(\angle ABC)=\frac{2\sqrt{d}}{\sqrt{2}}=\sqrt{2(x^2+(x+1)^2)}$$

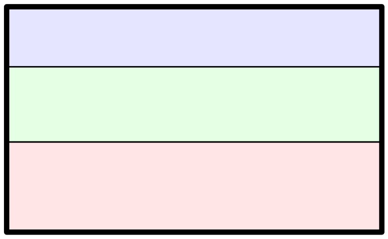
  and the least x for which this exceeds 2018 is $x=1009$, meaning $d=1009^2+1010^2$.

- Finally assume $AC\geq\sqrt{3}$. Then $2018<\frac{2\sqrt{d}}{AC}\leq\sqrt{\frac{4}{3}d}$, which would mean $d>2018^2\cdot\frac{3}{4}>1009^2+1010^2$. Thus all values of d achieved in this case are greater than the value of d in the second case.

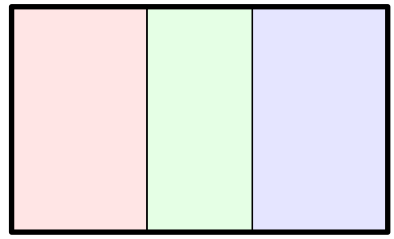
In conclusion, the least possible value of d is $1009^2+1010^2=\mathbf{2038181}$. This value of d is indeed possible, as the points $A=(1009,1010)$, $B=(-1009,-1010)$, and $C=(1010,1009)$ satisfy the conditions of the problem.

T-8. In order to enumerate the possible rectangles it is necessary to consider several cases. One possible approach is presented below.

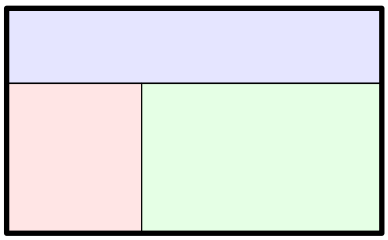
Consider the possible dissections of the given 3×5 rectangle. There are four shapes which can be achieved, as shown below.



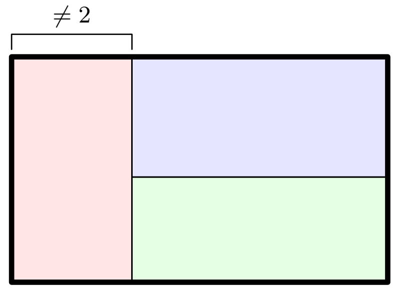
Case 1



Case 2



Case 3



Case 4

Begin by counting the number of unordered collections that arise from each case.

- In Case 1, a collection corresponds to a multiset $\{a, b, c\}$ satisfying $a + b + c = 15$ (as the three rectangles can be reordered freely). Assume, without loss of generality, that $a \leq b \leq c$. To eliminate the inequality constraints, let $x = a - 1$, $y = b - a$, $z = c - b$ denote nonnegative integers. Accordingly, $a = x + 1$, $b = x + y + 1$, $z = x + y + z + 1$, and the given equation now reads $x + 2y + 3z = 12$ or, equivalently, $2y + 3z \leq 12$. The number of solutions in nonnegative integers is then

$$\sum_{z=0}^{4} \left(1 + \left\lfloor \frac{12 - 3z}{2} \right\rfloor\right) = 7 + 5 + 4 + 2 + 1 = 19.$$

- Case 2 has 19 solutions by repeating the argument from Case 1 verbatim.
- In Case 3, denote by $15 - n$ the area of the top rectangle. The number of possible dissections of the lower rectangle is then exactly $\lfloor n/2 \rfloor$. Moreover, the top rectangle (because it has a side length of 5) is never congruent to either of the lower rectangles. Thus the number of solutions in this case is

$$\sum_{n=1}^{14} \left\lfloor \frac{n}{2} \right\rfloor = 49.$$

- In Case 4, assume the leftmost region does not have width 2. This ensures that Case 2 and Case 4 will be disjoint from each other, because the two rectangles on the right will not form a 3×3 square that could be rotated to obtain a dissection already counted in Case 2.

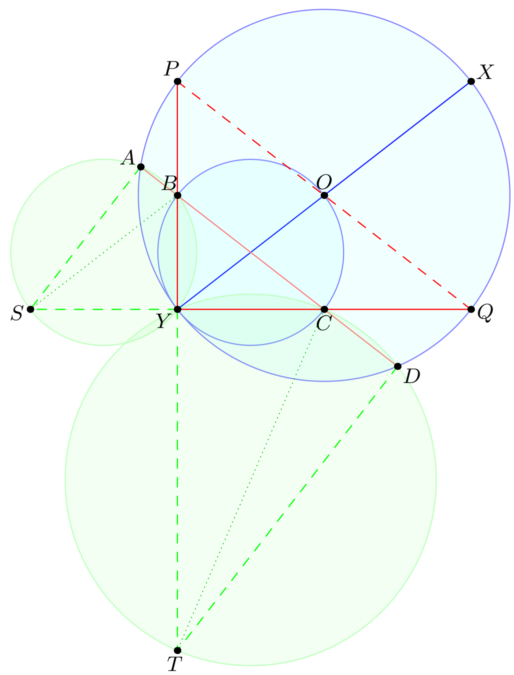
  In order to count the number of such configurations, denote by $15 - n$ the area of the left rectangle, where $15 - n \neq 3 \cdot 2$ or, equivalently, $n \neq 9$. Then repeating the logic of

Case 3 gives a count of

$$\sum_{\substack{n=1\\n\neq 9}}^{14} \left\lfloor \frac{n}{2} \right\rfloor = 49 - \left\lfloor \frac{9}{2} \right\rfloor = 45.$$

It has already been checked that Cases 3 and 4 are disjoint, and the other pairs of cases are seen to be mutually exclusive by comparing the number of sides of length 5 in the dissection. Consequently, the final answer is $19 + 19 + 49 + 45 = \mathbf{132}$.

T-9. Note that A, B, C, and D are collinear, in that order, because $AD = AB + BC + CD$. Extend $\overline{YB}$ and $\overline{YC}$ to meet Γ again at P and Q, respectively. Then $\overline{BC}$ is the midline of $\triangle YPQ$, by homothety. By power of a point, $AB \cdot BD = 1 \cdot (4 + 2) = PB \cdot BY = BY^2$, so $BY = \sqrt{6}$. Similarly, $CY = \sqrt{2 \cdot (4 + 1)} = \sqrt{10}$. In particular, $\triangle BYC$ is right.



Now let S and T be the antipodes of B and C on ω_1 and ω_2, respectively. It remains to evaluate BS and CT. First note that $m\angle BYS = 90°$ and $m\angle CAS = m\angle BAS = 90°$. Because $m\angle BYC = 90°$, it follows that points C, Y, and S are collinear and that $\triangle CYB \sim \triangle CAS$. Therefore

$$AS = \frac{YB}{YC} \cdot AC = \sqrt{\frac{6}{10}} \cdot 5 = \sqrt{15},$$

hence $BS = \sqrt{AS^2 + AB^2} = 4$. In the same fashion, with $\triangle BYC \sim \triangle BDT$,

$$DT = \frac{YC}{BY} \cdot BD = \sqrt{\frac{10}{6}} \cdot 6 = \sqrt{60},$$

hence $CT = \sqrt{DT^2 + CD^2} = 8$. So the sum of the areas of ω_1 and ω_2 is $\pi(2^2 + 4^2) = \mathbf{20\pi}$.

T-10. The following solution only relies on the clues for the third and fourth rows. Label the digits in the array as follows.

A	B	C	D
A	B	Y	D
W	B	Y	Z
W	X	Y	Z

The solution proceeds in three steps.

Step 1. The final row is the average of the first three rows, which implies

$$3(1000W + 100X + 10Y + Z) = 1000(2A + W) + 100(3B) + 10(C + 2Y) + (2D + Z).$$

This rearranges to

$$0 = 2000(A - W) + 300(B - X) + 10(C - Y) + 2(D - Z).$$

By comparing the magnitudes of $|2000(A-W)|$ to $|300(B-X)+10(C-Y)+2(D-Z)|$, it is impossible to have $|A - W| \geq 2$, hence $A - W = \pm 1$. A similar argument gives $B - X = \mp 7$, and then $C - Y = \pm 9$, $D - Z = \pm 5$, where the signs correspond. In particular, $\{C, Y\} = \{0, 9\}$.

Step 2. Next, the number in the third row satisfies

$$4^x + 5^y + 31^z \equiv 4^x + 6 \pmod{10}$$

and consequently either $Z = 0$ or $Z = 2$. However, one of C and Y must be 0, hence $\boxed{Z = 2}$. This can only occur if $\boxed{D = 7}$, which determines all the signs in the previous step: it follows that $A - W = -1$, $B - X = -7$, $C - Y = 9$, $D - Z = 5$. In particular, $\boxed{C = 9}$ and $\boxed{Y = 0}$.

Step 3. Because $B - X = -7$ and the digits 0 and 2 have already been used, it follows that $\boxed{B = 1}$ and $\boxed{X = 8}$. The third clue now reads

$$4^x + 5^y + 31^z = \underline{W}\ \underline{1}\ \underline{0}\ \underline{2}.$$

The mod 10 calculation in the previous step implies that x is even and in particular, $x > 1$. Taking the previous equation modulo 8 gives $5^y + 31^z \equiv 102 \equiv 6 \pmod{8}$, which can only occur if y is odd and z is even. This implies $z = 2$ (as $z \leq 2$). Therefore $4^x + 5^y \equiv 141 \pmod{1000}$ and because $x \leq 6$ is even, $y \leq 5$, and this forces $x = 2$ and $y \in \{3, 5\}$. The value $y = 3$ would give $W = 1$, which is not permitted because $B = 1$, so $y = 5$ and $\boxed{W = 4}$. Finally, $\boxed{A = 5}$.

The completed number puzzle reads as follows.

5	1	9	7
5	1	0	7
4	1	0	2
4	8	0	2

2018 Individual Problems

I-1. Compute the greatest prime factor of N, where

$$N = 2018 \cdot 517 + 517 \cdot 2812 + 2812 \cdot 666 + 666 \cdot 2018.$$

I-2. Compute the number of ordered pairs of positive integers (a, b) that satisfy

$$a^2 b^3 = 20^{18}.$$

I-3. Compute the number of positive three-digit multiples of 3 whose digits are distinct and nonzero.

I-4. In $\triangle ABC$, $\cos(A) = \dfrac{2}{3}$ and $\cos(B) = \dfrac{2}{7}$. Given that the perimeter of $\triangle ABC$ is 24, compute the area of $\triangle ABC$.

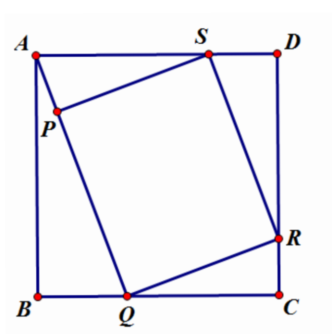
I-5. Compute the product of all positive values of x that satisfy

$$\lfloor x+1 \rfloor^{2x} - 19\lfloor x+1 \rfloor^{x} + 48 = 0.$$

I-6. Triangle ABC is inscribed in circle ω. The line containing the median from A meets ω again at M, the line containing the angle bisector of $\angle B$ meets ω again at R, and the line containing the altitude from C meets ω again at L. Given that quadrilateral $ARML$ is a rectangle, compute the degree measure of $\angle BAC$.

I-7. Let S be the set of points (x, y) whose coordinates satisfy $x^2 + y^2 \leq 36$ and $(\max\{x, y\})^2 \leq 27$. Compute the perimeter of S.

I-8. Rectangle $PQRS$ is drawn inside square $ABCD$, as shown. Given that $[APS] = 20$ and $[CQR] = 18$, compute $[ABCD]$.



I-9. Compute the least positive four-digit integer N for which N and $N + 2018$ contain a total of 8 distinct digits.

I-10. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(\alpha)),\ \operatorname{Arcsin}(\sin(2\alpha)),\ \operatorname{Arcsin}(\sin(7\alpha)),\ \operatorname{Arcsin}(\sin(t\alpha))$$

is a geometric progression for some α with $0 < \alpha < \frac{\pi}{2}$.

2018 Individual Answers

I-1. 23

I-2. 28

I-3. 180

I-4. $12\sqrt{5}$

I-5. 4

I-6. 36

I-7. $8\pi + 12$

I-8. 243

I-9. 1489

I-10. $9 - 4\sqrt{5}$

2018 Individual Solutions

I-1. Factor N as follows:

$$\begin{aligned}N &= 517 \cdot (2018 + 2812) + 666 \cdot (2812 + 2018) \\ &= (517 + 666) \cdot (2018 + 2812) \\ &= 1183 \cdot 4830 \\ &= (7 \cdot 169) \cdot (2 \cdot 5 \cdot 3 \cdot 161) \\ &= (7 \cdot 13^2) \cdot (2 \cdot 3 \cdot 5 \cdot 7 \cdot 23).\end{aligned}$$

Each of the integers 2, 3, 5, 7, 13, and 23 is prime. Thus the largest prime factor of N is **23**.

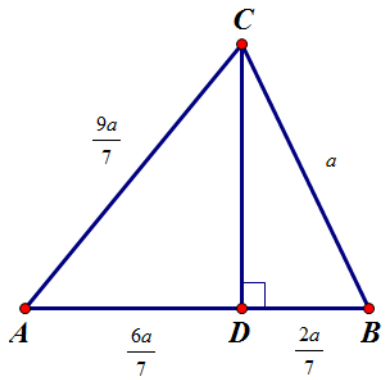
I-2. The prime factorization of 20^{18} is $2^{36} \cdot 5^{18}$, so a and b may only have prime factors of 2 and 5. Let $a = 2^x \cdot 5^y$. Then $b^3 = \frac{2^{36} \cdot 5^{18}}{a^2} = 2^{36-2x} \cdot 5^{18-2y}$. The value of b will be an integer if and only if both $36 - 2x$ and $18 - 2y$ are nonnegative multiples of 3. Therefore x and y must both be nonnegative multiples of 3. This means that $x \in \{0, 3, 6, 9, 12, 15, 18\}$ and $y \in \{0, 3, 6, 9\}$, so there are a total of $7 \cdot 4 = \mathbf{28}$ solution pairs.

I-3. If a number is divisible by 3, then the sum of its digits is divisible by 3. The digits must be equivalent to one of the following (modulo 3): 0, 0, 0; 1, 1, 1; 2, 2, 2; or 0, 1, 2. Note that there are 3 possible nonzero digits congruent to each of 0, 1, and 2 (modulo 3). Therefore the number of ways to choose the three digits (without regard to their order) is

$$3 \cdot \binom{3}{3} + \binom{3}{1} \cdot \binom{3}{1} \cdot \binom{3}{1} = 3 \cdot 1 + 3 \cdot 3 \cdot 3 = 30.$$

Having chosen the three digits, consider the number of ways the three digits can be ordered. There are $3! = 6$ ways to order the three digits, and this gives a total of $30 \cdot 6 = \mathbf{180}$ three-digit numbers.

I-4. Let D be the foot of the altitude from C. By the Law of Sines, $\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sqrt{5}/3}{3\sqrt{5}/7} = \frac{7}{9}$. Thus $b = \frac{9a}{7}$, $AD = b \cos A = \frac{6a}{7}$, and $BD = a \cos B = \frac{2a}{7}$. Because the perimeter of $\triangle ABC$ is 24, solve $a + \frac{9a}{7} + \frac{6a}{7} + \frac{2a}{7} = 24$ to obtain $a = 7$, which implies $b = 9$ and $c = 8$. By Heron's Formula, $[ABC] = \sqrt{12 \cdot 5 \cdot 3 \cdot 4} = \mathbf{12\sqrt{5}}$.



Alternatively, drop altitude $\overline{CD}$ as before, and then apply the definition of cosine for $\angle CAD$ in $\triangle CAD$ and for $\angle CBD$ in $\triangle CBD$. Thus $AD = 2x$ and $AC = 3x$ for some real x and $DB = 2y$ and $CB = 7y$ for some real y. By the Pythagorean Theorem, $(CD)^2 = 9x^2 - 4x^2 = 5x^2$ and $(CD)^2 = 49y^2 - 4y^2 = 45y^2$. This implies $5x^2 = 45y^2$ so $x = 3y$. Substituting, $AD = 6y$ and $AC = 9y$. Because the perimeter of $\triangle ABC$ is 24, $9y + 7y + 6y + 2y = 24 \rightarrow y = 1$, so $AB = 8$ and $CD = \sqrt{45y^2} = \sqrt{45} = 3\sqrt{5}$. Thus $[ABC] = \frac{1}{2} \cdot 8 \cdot (3\sqrt{5}) = 12\sqrt{5}$.

I-5. Introduce the substitution $y = \lfloor x+1 \rfloor^x$. Then the given equation becomes $y^2 - 19y + 48 = 0$, and the left-hand side of the equation in y factors as $(y-3)(y-16) = 0$. Therefore $\lfloor x+1 \rfloor^x = 3$ or $\lfloor x+1 \rfloor^x = 16$.

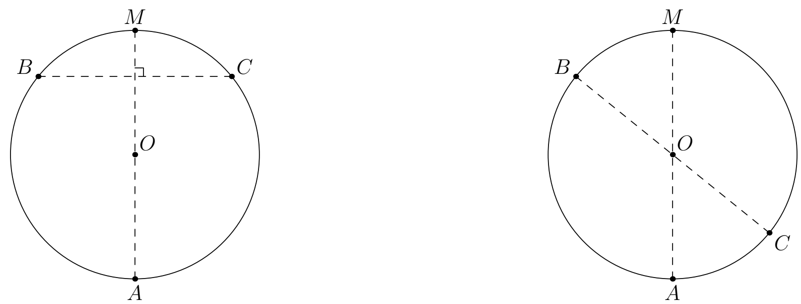
Case 1: $y = \lfloor x+1 \rfloor^x = 3$. Note that if $0 \leq x \leq 1$, then $\lfloor x+1 \rfloor^x \leq 2$ and if $x \geq 2$, then $\lfloor x+1 \rfloor^x \geq 9$. Therefore $1 < x < 2$, and hence $\lfloor x+1 \rfloor = 2$. The equation $2^x = 3$ has the solution $x = \log_2 3$, and indeed, $1 < \log_2 3 < 2$, which checks.

Case 2: $y = \lfloor x+1 \rfloor^x = 16$. Note that if $0 \leq x \leq 2$, then $\lfloor x+1 \rfloor^x \leq 9$ and if $x \geq 3$, then $\lfloor x+1 \rfloor^x \geq 64$. Therefore $2 < x < 3$, and hence $\lfloor x+1 \rfloor = 3$. The equation $3^x = 16$ has the solution $x = \log_3 16$, and indeed, $2 < \log_3 16 < 3$, which checks.

Use the change of base rule for logarithms to obtain the simplified product of the solutions:

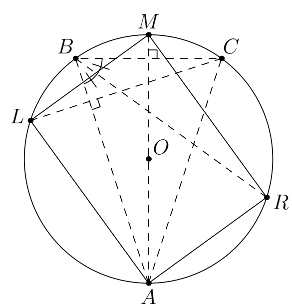
$$(\log_2 3)(\log_3 16) = \frac{\log 3}{\log 2} \cdot \frac{\log 16}{\log 3} = \frac{\log 16}{\log 2} = \log_2 16 = \mathbf{4}.$$

I-6. Because $ARML$ is a rectangle inscribed in a circle, its diagonal $\overline{AM}$ is a diameter of ω. The point B is on the circle and in one half-plane bounded by $\overleftrightarrow{AM}$. Because $\overline{AM}$ passes through the midpoint of side $\overline{BC}$, the point C is on the circle and in the opposite half-plane bounded by $\overleftrightarrow{AM}$ as B. Also, C must be the same distance from $\overline{AM}$ as B. Therefore either C is the reflection of B across line $\overline{AM}$ or C is the reflection of B across O, the center of ω, as shown below.



For $ARML$ to be a rectangle, it is sufficient that $\overline{LR}$ be a diameter of ω or, equivalently, that $m\angle LMR = 90°$. So consider the two cases for C, above, and determine if in either case it is possible to have $m\angle LMR = 90°$.

In the first case, $\triangle ABC$ is isosceles with $AB = AC$. Let $\beta = m\angle ABC = m\angle ACB$. Then $m\angle AMR = m\angle ABR = \frac{\beta}{2}$ and $m\angle AML = m\angle ACL = 180° - 90° - m\angle BAC = 2\beta - 90°$. Setting $\frac{\beta}{2} + 2\beta - 90° = 90°$ gives $\beta = 72°$. Then $m\angle BAC = 180° - 2\beta = 36°$.



In the second case, $\overline{BC}$ is a diameter of ω, so $m\angle BAC = 90°$. This means that L is coincident with A, which leads to a degenerate result. Thus the only possible value for $m\angle BAC$ is $\mathbf{36°}$.

Alternate Solution: Place the triangle in the complex plane so that ω coincides with the unit circle, and $a = 1$. (Here, a lowercase letter represents the complex coordinates of the point denoted by the corresponding uppercase letter.) Then $m = -1$ because diagonal $\overline{AM}$ of rectangle $ARML$ is a diameter of ω. Because the midpoint of $\overline{BC}$ has coordinates $\frac{b+c}{2}$ and because it lies on $\overline{AM}$ (which coincides with the real axis), it follows that $b + c \in \mathbb{R} \to b + c = \frac{1}{b} + \frac{1}{c} \to bc(b+c) = b + c$, and thus either $bc = 1$ or $b + c = 0$. The latter cannot be true, as it implies that $\angle BAC$ is a right angle, and thus $\ell = a$, $r = m = -1$, and then $c = a$, which results in a degenerate triangle. Hence $bc = 1$. Now, using some identities about the unit circle in the complex plane (which are straightforward to prove), conclude that $r^2 = ac$ (due to arc bisection) and $c\ell = -ab$ (from perpendicular chords). Expressing everything in terms of r:

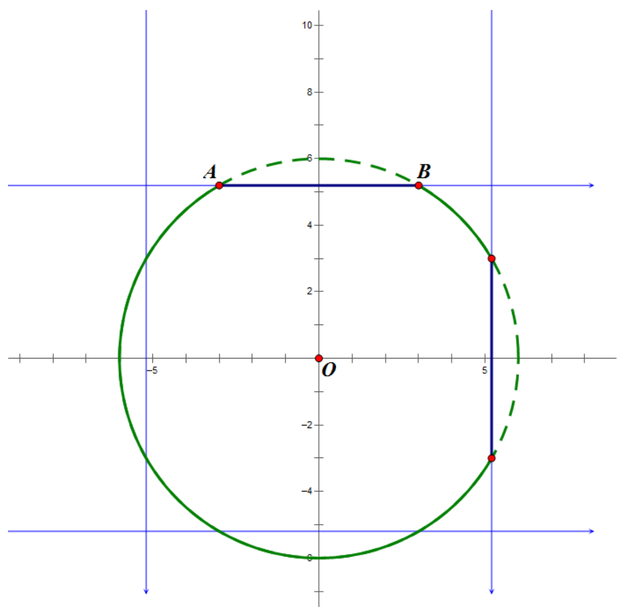
$$c = \frac{r^2}{a} = r^2, \quad b = \frac{1}{c} = r^{-2}, \quad \text{and} \quad \ell = -\frac{ab}{c} = -r^{-4}.$$

Because $ARML$ is a rectangle, its *other* diagonal, $\overline{RL}$, must also be a diameter of ω. Then $r = -\ell = r^{-4} \to r^5 = 1$, and so r is a fifth root of unity. It is impossible for r to be 1, as this would cause triangle ABC to be degenerate. Also, by replacing r with $\overline{r}$, note that the entire diagram is reflected across the real axis, and the measure of $\angle BAC$ does not change. Thus there are only two cases to consider: $r = e^{\frac{2\pi i}{5}}$ and $r = e^{\frac{4\pi i}{5}}$. Suppose that $r = e^{\frac{4\pi i}{5}}$. Then $c = e^{\frac{8\pi i}{5}}$ and $b = e^{\frac{2\pi i}{5}}$. But then B and L would lie on the same side of $\overline{AC}$, and $\overline{BL}$ would be the *external* angle bisector of $\angle B$. Thus $r = e^{\frac{2\pi i}{5}}$, which implies $b = e^{\frac{6\pi i}{5}}$ and $c = e^{\frac{4\pi i}{5}}$. Then minor arc BC measures $\frac{2\pi}{5}$ radians, which is 72°, and so the degree measure of $\angle BAC$ is half of that, or $\mathbf{36^\circ}$.

I-7. Let C be the region defined by $x^2+y^2 \leq 36$, and let R be the region defined by $\left(\max\{x,y\}\right)^2 \leq 27$, so $S = C \cap R$. If $(x,y) \in R$, then neither x nor y can be greater than $3\sqrt{3}$, and at least one of the coordinates must be between $-3\sqrt{3}$ and $3\sqrt{3}$. Thus if

$$R_1 = \left\{(x,y) \,:\, x \leq 3\sqrt{3},\ |y| \leq 3\sqrt{3}\right\} \text{ and}$$
$$R_2 = \left\{(x,y) \,:\, |x| \leq 3\sqrt{3},\ y \leq 3\sqrt{3}\right\},$$

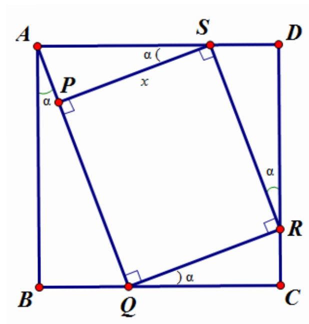
then $R = R_1 \cup R_2$. Consider the diagram below.



Most of circle C lies inside the central square $R_1 \cap R_2$, except for the segment cut off by the chord from $A(-3, 3\sqrt{3})$ to $B(3, 3\sqrt{3})$, and three other identical segments, none of which overlap. The segment on the left side of C lies inside R_1, and the bottom segment lies inside R_2. Thus S consists of the circle C with two such segments removed. If O is the origin, then $\triangle ABO$ is equilateral with side length 6, so the arc of the circle cut off by chord $\overline{AB}$ measures 60°. Thus removing each segment replaces a 60° arc of the circle with a chord of length 6, so the total perimeter of S, the region remaining, is

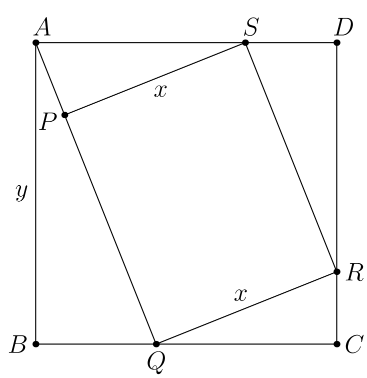
$$2\pi \cdot 6 - 2 \cdot \frac{1}{6}(2\pi \cdot 6) + 2 \cdot 6 = \mathbf{8\pi + 12}.$$

I-8. Notice first that triangles ABQ, QCR, SPA, and RDS are all similar. Let $PS = x$ and $m\angle ASP = \alpha$, as shown below.



Then $[APS] = \frac{PS \cdot AP}{2} = \frac{x \cdot x \tan\alpha}{2} = \frac{x^2 \tan\alpha}{2}$ and $[CQR] = \frac{CQ \cdot CR}{2} = \frac{x\cos\alpha \cdot x\cos\alpha\tan\alpha}{2} = \frac{x^2\tan\alpha}{2} \cdot \cos^2\alpha$. Hence $\frac{18}{20} = \cos^2\alpha \to \cos\alpha = \frac{3}{\sqrt{10}}$. Thus $\tan\alpha = \frac{1}{3}$. Now let $AB = y$. Then $BQ = \frac{y}{3}$, so $CQ = \frac{2y}{3}$ and $CR = \frac{2y}{9}$. Thus $[CQR] = \frac{2y^2}{27} = 18$, and so $[ABCD] = y^2 = \mathbf{243}$.

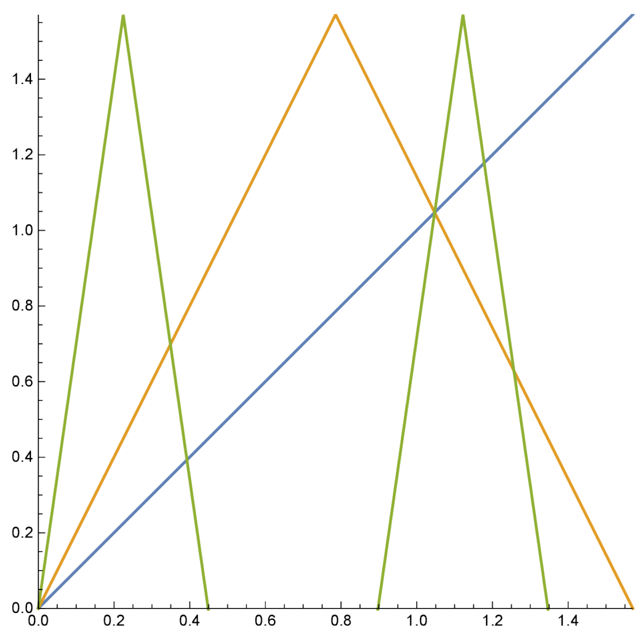
Alternate Solution: Because $[APS] = 20$, $[RCQ] = 18$, and $\triangle APS \sim \triangle RCQ$, conclude that $\dfrac{AS^2}{RQ^2} = \dfrac{20}{18}$, so $AS = \dfrac{\sqrt{10}}{3} \cdot RQ$. Let $RQ = x$. Then $RQ = PS = x$, thus $AS = \dfrac{\sqrt{10}}{3}x$, giving $AP^2 = \left(\dfrac{\sqrt{10}}{3}x\right)^2 - x^2 = \dfrac{x^2}{9}$, so $AP = \dfrac{x}{3}$. This implies $[APS] = \dfrac{1}{2} \cdot \dfrac{x}{3} \cdot x = 20$, so $x = 2\sqrt{30}$. Then $QC = \dfrac{3}{\sqrt{10}} \cdot 2\sqrt{30} = 6\sqrt{3}$. Now let $AB = y$. Because $\triangle ABQ \sim \triangle SPA$, it follows that $BQ = \dfrac{y}{3}$. From $y = BQ + QC$, conclude that $y = \dfrac{y}{3} + 6\sqrt{3} \to y = 9\sqrt{3}$. Thus the area of the square is $(9\sqrt{3})^2 = \mathbf{243}$.



I-9. Let $N = \underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ and $N + 2018 = \underline{E}\,\underline{F}\,\underline{G}\,\underline{H}$, where A, B, C, D, E, F, G, H represent distinct digits. To minimize N, first consider $A = 1$ and $B \neq 9$. Then $3018 < N + 2018 < 3918$, so $E = 3$. Because $B \neq F$, there must be a carry from the tens place to the hundreds place, and it follows that $C = 8$ or $C = 9$, and $B + 1 = F$. The least possible value for B is now 4, so assume $B = 4$ and $F = 5$. If there is no carry from the ones place to the tens place, then the only possibility for the ones digits is $D = 0$ and $H = 8$, but this requires $C = 9$ and $G = 0$ to achieve the carry into the hundreds place, which is a contradiction. Therefore there is a carry from the ones to the tens place. Because $G \neq 1$, it follows that $C = 8$ and $G = 0$. Finally, because $D + 8$ is the two-digit number $\underline{1}\,\underline{H}$, it must be that $D = H + 2$, and because $H \neq 0, 1, 3, 4, 5$ and $D \neq 4, 8$, the only possible values remaining are $D = 9$ and $H = 7$. Thus the least possible value of N is **1489**.

I-10. Let the ratio of consecutive terms in the sequence be r. If there exists an α for which the first three terms form a geometric progression with ratio r, the least t for which $\operatorname{Arcsin}\sin(t\alpha)$ continues the sequence will be $t = r^3$. This solution will focus on finding the least viable ratio r.

Note that as $2\alpha < \pi$, the first two terms of the progression are both positive, so the ratio r must also be positive. The graphs of $y = \operatorname{Arcsin}(\sin 2x)$ and $y = \operatorname{Arcsin}(\sin 7x)$ are both piecewise linear, the latter of which is positive only for $(0, \frac{\pi}{7}) \cup (\frac{2\pi}{7}, \frac{3\pi}{7})$, as seen below.



Thus consider only values of α in $(0, \frac{\pi}{7}) \cup (\frac{2\pi}{7}, \frac{3\pi}{7})$, because all terms in the progression must be positive. The function $\operatorname{Arcsin}(\sin 7x)$ is piecewise composed of four line segments in that domain. So, search along these segments to see if there exists an α for which a geometric progression results.

Along the first segment, no α exists, because the ratios between the first three terms are $\frac{2\alpha}{\alpha} = 2 \neq \frac{7\alpha}{2\alpha}$.

Along the second segment, $\text{Arcsin}(\sin 7\alpha) = \pi - 7\alpha$, and the ratio equality becomes $\frac{2\alpha}{\alpha} = \frac{\pi-7\alpha}{2\alpha} \to \alpha = \frac{\pi}{11}$. In this case, $r = 2$, and because $r^3\alpha = \frac{8\pi}{11} > \frac{\pi}{2}$, the geometric progression gets too large to be within the range of the Arcsin function.

Along the third segment, the three functions are equal to $x, \pi - 2x, 7x - 2\pi$, so the ratio equality is $\frac{\pi-2\alpha}{\alpha} = \frac{7\alpha-2\pi}{\pi-2\alpha} \to \alpha = \frac{\pi}{3}$. In this case, $r = 1$ and the least t is $r^3 = 1$.

Finally, along the fourth segment, the three functions are equal to $x, \pi - 2x, 3\pi - 7x$, and the ratio equality is $\frac{\pi-2\alpha}{\alpha} = \frac{3\pi-7\alpha}{\pi-2\alpha} \to \alpha = \frac{(7\pm\sqrt{5})\pi}{22}$. One can check that the smaller value of α is extraneous while the larger, $\alpha = \frac{(7+\sqrt{5})\pi}{22}$, is not, and so $r = \frac{\pi-2\alpha}{\alpha} = \frac{\pi}{\alpha} - 2 = \frac{3-\sqrt{5}}{2}$, which gives $r^3 = 9 - 4\sqrt{5}$, which is positive because $9 - 4\sqrt{5} = \sqrt{81} - \sqrt{80} > 0$. This is the least value of t among all cases, so $t = \mathbf{9 - 4\sqrt{5}}$.

Power Question 2018: Partitions

Binary Partitions

A *binary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 2, whose sum is n. Each of the integers in the n-tuple is called a *part* of the partition. Each binary partition of n has n parts. Let $p_2(n)$ denote the number of binary partitions of n. For example, $p_2(3) = 2$ because of the two ordered triples $(2, 1, 0)$ and $(1, 1, 1)$.

1. Compute $p_2(n)$ for $n = 4$, 5, 6, and 7. [4 pts]
2. a. Show that $p_2(n) \leq p_2(n+1)$ for all positive integers n. [3 pts]
   b. Is the inequality strict for sufficiently large n? Justify your answer. [3 pts]
3. a. Prove that if n is even and $n \geq 4$, then $p_2(n) = p_2\left(\frac{n}{2}\right) + p_2(n-2)$. [2 pts]
   b. Find the least $n > 1$ such that $p_2(n)$ is odd, or prove that no such n exists. [2 pts]

Partial Orderings

A *partial ordering* on a set S is a relation, usually denoted $\preceq$, such that all of the following conditions are true:

- $a \preceq a$ for all $a \in S$ **(reflexivity property)**,
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in S$ **(transitivity property)**, and
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in S$ **(antisymmetry property)**.

The word "partial" refers to the possibility that some two elements, a and b, may be *incomparable*, i.e., neither $a \preceq b$ nor $b \preceq a$ (these negative relations on $\preceq$ are sometimes written as $a \not\preceq b$ and $b \not\preceq a$, respectively). This Power Question largely defines and explores a partial ordering on the set of binary partitions of n.

The notation $a \prec b$ means that $a \preceq b$ and $a \neq b$. The symbols $\succ$ and $\succeq$ may also be used, and they are defined in the following way: $a \succ b$ means $b \prec a$, and $a \succeq b$ means $b \preceq a$.

If a and b are elements of S with $a \prec b$, and if there is no element $c \in S$ for which $a \prec c \prec b$, then it is said that b *covers* a.

Let a denote the binary partition $(a_1, a_2, \ldots, a_n)$. Similarly, let b denote the binary partition $(b_1, b_2, \ldots, b_n)$. In this Power Question, define $a \prec b$ if it is possible to obtain a from b by a sequence of replacing one 2^k by two 2^{k-1}s (and deleting a 0). For example,

$$(4, 1, 1, 1, 1, 0, 0, 0) \prec (4, 4, 0, 0, 0, 0, 0, 0)$$

because

$$(4,1,1,1,1,0,0,0) \prec (4,2,1,1,0,0,0,0) \prec (4,2,2,0,0,0,0,0) \prec (4,4,0,0,0,0,0,0).$$

This Power Question will use this partial ordering on binary partitions.

4. a. Show that the binary partitions of 5 are totally ordered; i.e., if p and p' are two different binary partitions of 5, then either $p \prec p'$ or $p' \prec p$. [2 pts]

   b. Show that the binary partitions of 8 are not totally ordered by dominance, i.e., find two binary partitions of 8—call them q and q'—such that $q \not\prec q'$ and $q' \not\prec q$. [3 pts]

5. a. Find the smallest binary partition of n, using this partial ordering. That is, find the binary partition p such that for all other binary partitions p', $p \prec p'$. [2 pts]

   b. Find the largest binary partition of n, using this partial ordering. That is, find the binary partition P such that for all other binary partitions P', $P \succ P'$. [3 pts]

Hasse Diagrams

Suppose that a set S has a partial ordering $\preceq$. Then a *Hasse diagram* can be used to display the covering relation in a graphical way. The Hasse diagram is a graph whose vertices are the elements of S and where edges are drawn between two elements x and y if $x \prec y$ or $y \prec x$ and if there is no element z for which $x \prec z$ and $z \prec y$ or for which $y \prec z$ and $z \prec x$. Also, y appears "above" x if $x \prec y$. Note that it is possible for more than one element to appear on the same level of a Hasse diagram. For example, the partially ordered set of divisors of 12, ordered by divisibility, is shown in Figure 1. In this diagram, the number 1 is said to be at Level 0 because it is the least divisor in the partial ordering shown. The numbers 2 and 3 are said to be on Level 1 because $2 \succ 1$ and $3 \succ 1$ and there is no number n such that $2 \succ n$ and $n \succ 1$ or $3 \succ n$ and $n \succ 1$. Other Levels are similarly defined. The number 12 is on Level 3.

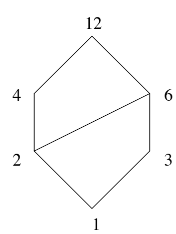


Figure 1

Hasse diagrams can be drawn to show the ordering of partitions such as the ones from Problems 4 and 5. For convenience, rather than labeling the vertices in the Hasse diagram with the partition itself, like $(8,0,0,0,0,0,0,0)$, it is common to label the vertex with its nonzero parts only, using exponents to indicate parts within the partition with multiplicity greater than 1. For example, the partition $(8,0,0,0,0,0,0,0)$ would be labeled 8, the partition $(4,2,2,0,0,0,0,0)$ would be labeled 42^2, and the partition $(2,2,1,1,1,1,0,0)$ would be labeled 2^21^4.

6. a. Draw the Hasse diagram for the binary partitions of 8. Label each vertex. [1 pt]

   b. List a path through the Hasse diagram for the binary partitions of 8 that begins at the bottom vertex (that is, the vertex at Level 0) and passes through every other vertex exactly once. Such a path is called a *Hamiltonian path*. [1 pt]

   c. Let n be an even integer with $n > 4$. Let S be the set of binary partitions of n. Let S_1 be the set of binary partitions of $\frac{n}{2}$. Let S_2 be the set of binary partitions of $n-2$. Prove that there is a bijection B (i.e., a one-to-one correspondence) from the set S to the set $S_1 \cup S_2$. Prove that this bijection B preserves order; that is, given that $p \prec p'$ for binary partitions $p, p' \in S$, then either $B(p) \prec B(p')$ or $B(p)$ and $B(p')$ are incomparable. [3 pts]

   d. Prove that for each positive integer n, the Hasse diagram of the binary partitions of n has a Hamiltonian path that begins with the vertex at Level 0. [5 pts]

Let $f_L(n)$ represent the number of elements at level L in the Hasse diagram of the binary partitions of n.

7. Prove that the value of $f_L(n)$ is the number of binary partitions of n that have $n - L$ nonzero parts. [3 pts]

8. Not all partitions are binary partitions. Some partitions have parts of the form $2^j - 1$ where j is a nonnegative integer. Such partitions will be called *s-partitions*, and their parts are written in nonincreasing order. Two s-partitions of 5 are $(3, 1, 1, 0, 0)$ and $(1, 1, 1, 1, 1)$.

   a. List two partitions of 7, one that is an s-partition and one that is neither an s-partition nor a binary partition. [2 pts]

   b. Prove that if $n \geq 2L$, the value of $f_L(n)$ is equal to the number of s-partitions of L. [3 pts]

Trinary Partitions

A *trinary partition* of a positive integer n is an ordered n-tuple of non-increasing integers, each of which is either 0 or a power of 3, whose sum is n. Let $p_3(n)$ denote the number of trinary partitions of n. For example, $p_3(4) = 2$ because of the two ordered quadruples $(3, 1, 0, 0)$ and $(1, 1, 1, 1)$.

As with binary partitions, one can define partial orderings for trinary partitions. Let c denote the trinary partition $(c_1, c_2, \ldots, c_n)$. Similarly, let d denote the trinary partition $(d_1, d_2, \ldots, d_n)$. Define $c \prec d$ if it is possible to obtain c from d by a sequence of replacing one 3^k by three 3^{k-1}s (and deleting two 0s).

9. Draw the Hasse diagram for the trinary partitions of 12. Label each vertex. [2 pts]

10. State a value of n less than 23 for which the Hasse diagram of the trinary partitions of n does **not** contain a Hamiltonian path. Prove your claim. (Recall that a Hamiltonian path is defined in Problem 6b.) [6 pts]

Solutions to 2018 Power Question

1. The values are as follows.
   The value of $p_2(4)$ is **4** from $(4,0,0,0)$, $(2,2,0,0)$, $(2,1,1,0)$, and $(1,1,1,1)$.
   The value of $p_2(5)$ is **4** from $(4,1,0,0,0)$, $(2,2,1,0,0)$, $(2,1,1,1,0)$, and $(1,1,1,1,1)$.
   The value of $p_2(6)$ is **6** from $(4,2,0,0,0,0)$, $(4,1,1,0,0,0)$, $(2,2,2,0,0,0)$, $(2,2,1,1,0,0)$, $(2,1,1,1,1,0)$, and $(1,1,1,1,1,1)$.
   The value of $p_2(7)$ is **6** from $(4,2,1,0,0,0,0)$, $(4,1,1,1,0,0,0)$, $(2,2,2,1,0,0,0)$, $(2,2,1,1,1,0,0)$, $(2,1,1,1,1,1,0)$, and $(1,1,1,1,1,1,1)$.

2. a. A binary partition for n can be changed into a binary partition for $n+1$ by inserting a 1 immediately following the rightmost nonzero entry. For example, $(2,2,1,1,0,0)$ is a binary partition of 6, and this can be changed into $(2,2,1,1,1,0,0)$, which is a binary partition of 7. Similarly, $(1,1,1,1)$ becomes $(1,1,1,1,1)$, and these are binary partitions of 4 and 5, respectively. Different binary partitions of n become different binary partitions of $n+1$ in this way. Because every binary partition of n maps to a binary partition of $n+1$, there are at least as many binary partitions of $n+1$ as of n.

   b. The answer to the question is **no**. If n is even, then $n+1$ is odd. Because a binary partition of an odd number must contain at least one 1, the correspondence described in the solution to Problem 2a is one-to-one. To establish this, given a binary partition of $n+1$, delete one 1 to obtain a binary partition of n. Thus $p_2(n+1) = p_2(n)$ for any even n. However, note that if n is odd, then $p_2(n+1)$ is strictly greater than $p_2(n)$ because in addition to the binary partitions of $n+1$ that can be obtained by inserting a 1 into a binary partition of n, there are binary partitions of $n+1$ consisting of all even numbers.

3. a. Let n be an even integer. Given a binary partition of n, either all of its parts are even or there is at least one 1. If all parts are even, then all of the positive parts must occur in the first $\frac{n}{2}$ elements of the n-tuple (otherwise, their sum would exceed n). Dividing all the parts by 2 yields an n-tuple of powers of 2 and 0s that sum to $\frac{n}{2}$. Note that the last $\frac{n}{2}$ elements of this n-tuple must all be 0s and truncating them results in an $\left(\frac{n}{2}\right)$-tuple which is a binary partition of $\frac{n}{2}$. Conversely, for any binary partition of $\frac{n}{2}$, by doubling each element and appending $\frac{n}{2}$ zeros at the end results in an n-tuple which is a binary partition of n. Hence $p_2\left(\frac{n}{2}\right)$ counts the number of binary partitions of n in which all the parts are even. If there is at least one 1, then the assumption that n is even means that there are at least two 1s, and deleting them produces a binary partition of $n-2$. Conversely, for each binary partition of $n-2$, appending two 1s (right before the 0s, if there are any, otherwise, append them to the end of the $(n-2)$-tuple) produces an n-tuple that is a binary partition of n. Hence $p_2(n-2)$ counts the number of binary partitions of n with at least one 1, and thus the desired recursion follows.

   b. **There is no such n.** Proceed by cases. If n is odd, then $p_2(n) = p_2(n-1)$ because all binary partitions of an odd number include at least one 1, and removing that 1 gives a bijection to partitions of $n-1$. If n is even, then $p_2(n) = p_2\left(\frac{n}{2}\right) + p_2(n-2)$ by Problem 3a. Because $p_2(2) = 2$ and $p_2(4) = 4$, it follows inductively that $p_2(n)$ is even for all $n > 1$.

4. a. The four binary partitions of 5 are ordered in the following way:

$$(1,1,1,1,1) \prec (2,1,1,1,0) \prec (2,2,1,0,0) \prec (4,1,0,0,0).$$

This is a total ordering.

b. Consider these two binary partitions of 8: $(4,1,1,1,1,0,0,0)$ and $(2,2,2,2,0,0,0,0)$. The 4 can break down into two 2s, but that is the only way to generate 2s, and that does not generate four 2s, so it is not the case that $(2,2,2,2,0,0,0,0) \prec (4,1,1,1,1,0,0,0)$. Likewise, it is not possible for a partition with no 4s to be related by $\prec$ to a partition with 4s, so it is not the case that $(4,1,1,1,1,0,0,0) \prec (2,2,2,2,0,0,0,0)$.

Similarly, the following two pairs of partitions are not comparable:

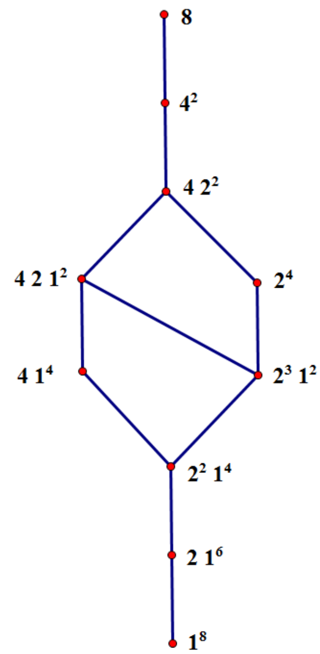
$$(4,1,1,1,1,0,0,0) \text{ and } (2,2,2,1,1,0,0,0),$$

$$(4,2,1,1,0,0,0,0) \text{ and } (2,2,2,2,0,0,0,0).$$

5. a. The smallest binary partition of n is the partition with n 1s. That is, it is the partition $(1,1,\ldots,1)$ with n copies of 1 in the partition. If a partition has any part that is not equal to 0 or 1, then that part is a positive power of 2 that may be replaced by lesser powers of 2. This process can continue until all of the powers of 2 are 1s, but the process cannot continue beyond that because a partition must consist only of integers. Thus every binary partition is greater than the one consisting of all 1s. Also note that the partition $(1,1,\ldots,1)$ is not larger than or incomparable with any other partition. Thus $(1,1,\ldots,1)$ is the smallest binary partition.

b. The largest binary partition of n is the one that essentially gives the base-2 equivalent of n. That is, if the base-2 representation of n has 1s in the 2^a, 2^b, 2^c places, and so on, with $a > b > c > \cdots$, then the largest binary partition of n is $(2^a, 2^b, 2^c, \ldots, 0, 0, 0)$ (with enough 0s to finish the partition). If a partition contains two copies of a number other than 0, they may be combined to form a greater power of 2. Thus, starting with any partition, continue combining powers of 2 until the partition contains no repeated numbers. Because no further combinations are allowed at this point, by the definition, there can be no partition larger than a partition with no repeated numbers. *A priori* there could be other partitions that are not comparable to such a partition, however. But because such a partition expresses n as a sum of powers of 2, with no repeated nonzero parts, it is essentially the base-2 representation of n, which is unique. So because any partition may undergo the process of combining like powers of 2 until there are no repeats, every binary partition is less than the unique partition which expresses n in binary.

6. a. A Hasse diagram for the binary partitions of 8 is shown below.



b. The following sequence of vertices is a Hamiltonian path:

$$1^8, 21^6, 2^21^4, 41^4, 421^2, 2^31^2, 2^4, 42^2, 4^2, 8.$$

This path begins at the vertex at Level 0, then visits each vertex exactly once.

c. The bijection was effectively established in the solution to 3a. Now consider whether $B(p) \prec B(p')$ for p and p' which are different binary partitions of n and for which $p \prec p'$. Suppose first that both p and p' have no 1s at all. Then dividing all parts by 2 preserves the covering relation. This is because p can be produced by swapping out powers of 2 in p' for lesser powers of 2, and dividing all parts by 2 does not change this covering. Now suppose that p and p' both have at least two 1s. Then removing two 1s from each partition preserves the covering relation. This is because removing two 1s from the partition in no way changes the "swap-outs" of powers of 2 for lesser powers of 2. Now suppose that one of p or p' has two 1s and the other doesn't. Then $B(p)$ is incomparable with $B(p')$, because $B(p) \in S_1$ and $B(p') \in S_2$ or vice versa, and these are sets of partitions of different numbers because $n - 2 > \frac{n}{2}$ when $n > 4$. The relation $\prec$ is not defined between partitions of different numbers, so $B(p)$ and $B(p')$ are not comparable.

d. As in the solutions to Problems 2 and 3, it is only necessary to consider the problem for even n. This is because a Hasse diagram for partitions of $2r+1$ is the same as that for the partitions of $2r$ with an extra 1 in every partition. So let $n = 2r = a \cdot 2^k$, where a is an odd integer and k is positive. The proof proceeds by strong induction to prove that there is a Hamiltonian path through the Hasse diagram for partitions of n starting at 1^n and ending at $(2^k)^a$. To get started, the path $1^2 \to 2$ is clearly a Hamiltonian path through the vertices of the Hasse diagram for partitions of 2, and the path $1^4 \to 21^2 \to 2^2 \to 4$ is a Hamiltonian path through the vertices of the Hasse diagram for partitions of 4. These establish the base case.

The idea of the proof will be to use the bijection from part c. In essence, the partitions of n will be split into those that are matched with partitions of $n-2$ and those that are partitions of $n/2$, and the Hamiltonian paths through each of these sets of smaller partitions will be joined into a single path through the partitions of n.

The path starts at 1^n. Consider this as $1^{n-2}1^2$. That is, group the first $n-2$ 1s together and the last two 1s together. From the induction hypothesis, there is a Hamiltonian path through the partitions of $n-2$ starting with 1^{n-2}. By adjoining two 1s to each of these, there is a Hamiltonian path through the partitions of n that have two or more 1s in them. This path ends at $(2^\ell)^c 1^2$, where $n-2 = 2(r-1) = c \cdot 2^\ell$, c is an odd integer, and ℓ is positive. The manner in which the path will be continued through the partitions of n with no 1s in them is dependent on the parity of r.

First consider the case in which r is even. Then $r-1$ is odd. In this case, $c = r-1$ and $\ell = 1$. Thus the path in the previous paragraph ended at $2^c 1^2$. The next step in the path will be to $2^{c+1} = 2^r$. Now by induction, there is a Hamiltonian path through the partitions of $n/2 = r$ starting at 1^r and ending at $(2^{k-1})^a$. Double every entry of each of these partitions. This gives a Hamiltonian path through the partitions of n that have no 1s in them, starting at 2^r and ending at $(2^k)^a$. Attaching this to the end of the path from 1^n to 2^r already obtained results in the desired Hamiltonian path from 1^n to $(2^k)^a$.

In the case where r is odd, $r-1$ is even. In this case, $a = r$ and $k = 1$, while $n-2 = 2(r-1)$ must be a multiple of 4. So the path through the partitions of n containing 1s ended at $(2^\ell)^c 1^2$ for some $\ell \geq 2$. The next step will be to the partition $(2^\ell)^c 2$. Now $r = n/2$ is odd, so every binary partition of r must contain at least one 1. Note that these are the binary partitions of $r-1$ with a 1 adjoined, and by induction, there is a Hamiltonian path through them starting at 1^r and ending at $(2^{\ell-1})^c 1$. Double each entry of each of these, and run through this path *backward* to obtain a path starting at $(2^\ell)^c 2$ and ending at 2^r. Concatenating this with the path from 1^n to $(2^\ell)^c 2$ previously obtained will yield the desired path from 1^n to $2^r = (2^k)^a$.

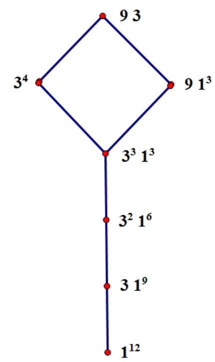
7. Proceed by induction on L. The only element for which $L = 0$ is the smallest binary partition of n, which indeed has n 1s by the solution to Problem 5, and so this partition has $n-L = n-0$ parts. Moving up via a covering relation, when two 2^ks are replaced by one 2^{k+1}, it is true that one nonzero part is lost. Assuming an element at level L has $n-L$ nonzero parts, an element at level $L+1$ will have $n-L-1 = n-(L+1)$ parts. This establishes the inductive step.

8. a. There are several s-partitions of 7. They include $(7,0,0,0,0,0,0)$, $(3,3,1,0,0,0,0)$, $(3,1,1,1,1,0,0)$, and $(1,1,1,1,1,1,1)$. Other partitions of 7 are not s-partitions; some of those are also not binary. They include $(6,1,0,0,0,0,0)$, $(5,2,0,0,0,0,0)$, $(5,1,1,0,0,0,0)$, $(3,2,2,0,0,0,0)$, and $(3,2,1,1,0,0,0)$.

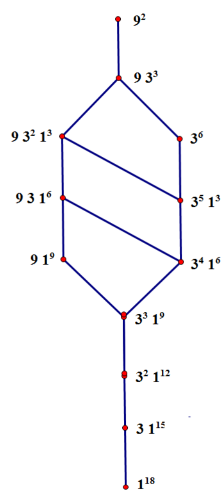
b. Construct a bijection. Given a binary partition of n at level L, subtract 1 from each nonzero part. Hence each part 2^j is replaced by $2^j - 1$, leaving an s-partition of $n-(n-L)=L$.

For the inverse map, consider an s-partition of L, with parts of the form $2^j - 1$ for some positive integer j. This cannot have more than L parts, and that is achievable only with all parts equal to 1. Consider the partition to end in enough 0s to bring the total number of parts up to $n-L$; this is possible because $n-L = n-2L+L \geq 2L-2L+L = L$. Now add 1 to each part. The result is a binary partition of $L+n-L=n$ with $n-L$ parts, which puts it at level L by Problem 7.

9. A Hasse diagram for the trinary partitions of 12 is shown below.



10. The three possible answers are **18** or **19** or **20**. A Hasse diagram for the trinary partitions of 18 is shown below. The answer will be established when it is shown that there is no Hamiltonian path through the Hasse diagram. Note that all Hasse diagrams for the trinary partitions of 18 are isomorphic to the one in the figure. Note also that the Hasse diagram for the trinary partitions of 19 is isomorphic to the Hasse diagram for the trinary partitions of 18; a bijection can be established by adding a 1 after the rightmost nonzero part of any trinary partition of 18. Similarly, the Hasse diagram for the trinary partitions of 20 is isomorphic to the Hasse diagram for the trinary partitions of 18.



Suppose there is a Hamiltonian path through the Hasse diagram. Because the Hasse diagram has only one edge coming from 1^{18} and only one edge coming from 9^2, the Hamiltonian path must begin and end at those vertices in some order. Without loss of generality, assume the path begins at 1^{18} and ends at 9^2.

The path must be of the form $1^{18}, 31^{15}, 3^21^{12}, 3^31^9, \ldots, 9^2$. The first choice comes when deciding if the fifth vertex in the path is 91^9 or 3^41^6. If the choice is to make 3^41^6 the fifth vertex in the path, then it is not possible to enter and leave 91^9 because the edge $\{3^31^9, 91^9\}$ cannot be used for entry or for exit; both would require revisiting 3^31^9. So the fifth vertex in the path must be 91^9, and thus the sixth vertex in the path is 931^6.

The next choice comes when considering the seventh vertex in the path. If the seventh vertex is 93^21^3, then it will be impossible to visit 3^41^6 and also 9^2. If the seventh vertex is 3^41^6, then the eighth vertex in the path is 3^51^3, and either choice from that vertex (93^21^3 or 3^6) leads to a contradiction because to visit the other requires a path that prevents visiting 9^2. This completes the proof that the Hasse diagram of the trinary partitions of 18 does not contain a Hamiltonian path.

Checking other possible values of n that are multiples of 3, there is a Hamiltonian path through the Hasse diagrams for each one, as the following argument demonstrates.

- A Hamiltonian path through the Hasse diagram for $n = 3$ is $1^3, 3$.
- A Hamiltonian path through the Hasse diagram for $n = 6$ is $1^6, 31^3, 3^2$.
- A Hamiltonian path through the Hasse diagram for $n = 9$ is $1^9, 31^6, 3^21^3, 3^3, 9$.
- A Hamiltonian path through the Hasse diagram for $n = 12$ is $1^{12}, 31^9, 3^21^6, 3^31^3, 91^3, 93, 3^4$.
- A Hamiltonian path through the Hasse diagram for $n = 15$ is

$$1^{15}, 31^{12}, 3^21^9, 3^31^6, 91^6, 931^3, 93^2, 3^5, 3^41^3.$$

- A Hamiltonian path through the Hasse diagram for $n = 21$ is

$$1^{21}, 31^{18}, 3^21^{15}, 3^31^{12}, 91^{12}, 931^9, 3^41^9, 3^51^6, 93^21^6, 93^31^3, 93^4, 9^23, 9^21^3.$$

Note that there are Hamiltonian paths through the Hasse diagram for both $n = 1$ (the path is just the vertex 1) and $n = 2$ (the path is $1^2, 2$). Note also that if k is a positive integer, the Hasse diagrams for the trinary partitions of $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$ are all isomorphic, in the same way that the Hasse diagrams for the trinary partitions of 18, 19, and 20 are isomorphic.

Thus the only values of n less than 23 for which the Hasse diagram of trinary partitions does not contain a Hamiltonian path are 18 or 19 or 20.

2018 Relay Problems

R1-1. The number $N = 327763$ can be expressed as

$$N = 51^3 + 58^3 = 30^3 + 67^3.$$

Compute the least prime factor of N.

R1-2. Let $T = $ TNYWR. Rectangle $ABCD$ has area $T + 5$. Point M is the midpoint of $\overline{AB}$ and point N is a trisection point on $\overline{CD}$. The segment $\overline{MN}$ divides rectangle $ABCD$ into two trapezoids. Compute the area of the <u>larger</u> of these two trapezoids.

R1-3. Let $T = $ TNYWR. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = \sqrt[4]{2}, a_2 = \sqrt{2}$, and for $n \geq 3$, $a_n = a_{n-1}a_{n-2}$. Compute the least value of k such that a_k is an integer multiple of $2^{\lfloor T \rfloor}$.

R2-1. Given that A and B are integers such that $3A + 2B = 218$ and $|A - B|$ is as small as possible, compute $A + B$.

R2-2. Let $T = $ TNYWR. Compute the number of lattice points in the interior of the triangle with vertices $(0, T)$, $(T, 0)$, and $(0, 0)$.

R2-3. Let $T = $ TNYWR. The integers a and b are chosen from the set $\{1, 2, 3, \ldots, T\}$, with replacement. Compute the number of ordered pairs (a, b) for which

$$a + b = 4036.$$

2018 Relay Answers

R1-1. 31

R1-2. 21

R1-3. 11

R2-1. 87

R2-2. 3655

R2-3. 3275

2018 Relay Solutions

R1-1. Note that the factorization $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ implies that $a+b$ is a divisor of $a^3 + b^3$. Thus $51 + 58 = 109$ and $30 + 67 = 97$ are each factors of N. From long division, $N = 31 \cdot 97 \cdot 109$, and each of the three factors in the product is prime, hence the answer is **31**.

Alternate Solution: Note that $N = 327 \cdot 1000 + 763$ and that $327 = 3 \cdot 109$ and $763 = 7 \cdot 109$. Thus $N = 109(3 \cdot 1000 + 7)$. As in the first solution, conclude that 97 must be a factor of 3007 and, using long division, the other factor is 31.

Note: Numbers that can be represented as the sum of two positive cubes in two different ways are examples of *taxicab numbers*. The least taxicab number is 1729 ($1729 = 10^3 + 9^3 = 12^3 + 1^3$). The origin of the name "taxicab number" comes from an anecdote involving a visit between mathematicians G. H. Hardy and S. Ramanujan. Hardy related that the number of the taxi he took to visit Ramanujan was 1729, and he remarked that the number seemed to be rather dull. Ramanujan countered by stating the fact discussed above.

R1-2. Let $AB = CD = 6x$ and $BC = AD = y$. The smaller trapezoid has bases of lengths $2x$ and $3x$, so its area is $\frac{5}{2}xy$. The larger trapezoid has bases of lengths $3x$ and $4x$, so its area is $\frac{7}{2}xy$. Because $6xy = T + 5$, the area of the larger trapezoid is $\frac{7}{12}(T+5)$. With $T = 31$, the desired area is **21**.

R1-3. Make a table consisting of the first few values of the sequence $a_1, a_2, a_3, \ldots$.

n	1	2	3	4	5	6
a_n	$2^{1/4}$	$2^{2/4}$	$2^{3/4}$	$2^{5/4}$	$2^{8/4}$	$2^{13/4}$

Note that a_n is of the general form $2^{F_{n+1}/4}$, where F_n denotes the n^{th} Fibonacci number. The desired value of k has the property that F_{k+1} is a multiple of 4 and is the least such value satisfying $F_{k+1} \geq 4\lfloor T \rfloor$. With $T = 21$, the least Fibonacci number that is both a multiple of 4 and which is greater than or equal to $4 \cdot 21 = 84$ is $F_{12} = 144$, hence $k = \mathbf{11}$.

R2-1. Note that $|A - B| \neq 0$ because with $A = B$, $3A + 2B$ is a multiple of 5, but 218 is not a multiple of 5. Suppose there exist integers A and B such that $|A - B| = 1$. If $B = A + 1$, then the given equation becomes $5A + 2 = 218$, but this does not result in an integral value for A. On the other hand, if $A = B + 1$, then the given equation becomes $5B + 3 = 218$, and this yields the solution $(A, B) = (44, 43)$, hence $A + B = \mathbf{87}$.

R2-2. The following solution assumes that T is a positive integer. (As an exercise, the reader might want to consider how the solution would change if T were a negative integer or if T were not an integer.) It is easy to verify that if $T = 1$ or $T = 2$, then the answer is 0. So suppose that $T \geq 3$. For a fixed value of N with $1 \leq N \leq T-2$, the points $(1, N), (2, N), \ldots, (T-N-1, N)$

all lie in the interior of the triangle. This is a total of $T-N-1$ points. Summing over the possible values of N yields $(T-2)+(T-3)+\cdots+1 = \frac{(T-2)(T-1)}{2}$. With $T = 87$, the answer is $\mathbf{3655}$.

Alternatively, one can apply Pick's Theorem, which states that for a polygon whose vertices are all lattice points, $K = I + \frac{B}{2} - 1$, where K is the area of the polygon, I is the number of lattice points in its interior, and B is the number of lattice points on its boundary. For the given triangle, $K = \frac{1}{2}T^2$ and $B = (T+1) + T + (T-1) = 3T$. Hence $I = \frac{T^2-3T+2}{2}$, as established above.

R2-3. The answer is the number of ordered pairs of integers $(x, 4036-x)$ for which x and $4036-x$ are both in the set $\{1, 2, 3, \ldots, T\}$. If $T < \frac{4036}{2} = 2018$, then the answer is 0 because the greatest possible sum would be achieved with the ordered pair (T, T), but $2T < 4036$. If $2018 \leq T < 4036$, the ordered pairs are $(4036-T, T)$, $(4037-T, T-1), \ldots, (T, 4036-T)$; there are $T-(4036-T)+1 = 2T-4035$ of these ordered pairs. Finally, if $T \geq 4036$, then the answer is 4035 because each of the ordered pairs $(1, 4035), (2, 4034), \ldots, (4035, 1)$ has the sum of its coordinates equal to 4036. Because $T = 3655$, the desired number of ordered pairs is therefore $2 \cdot 3655 - 4035 = \mathbf{3275}$.

2018 Tiebreaker Problems

TB-1. The increasing infinite arithmetic sequence of integers $x_1, x_2, x_3, \ldots$ contains the terms 17! and 18!. Compute the greatest integer X for which $X!$ must also appear in the sequence.

TB-2. Let $\mathcal{S}_1 = \{1, 2, 3, 4\}$, $\mathcal{S}_2 = \{3, 4, 5, 6\}$, and $\mathcal{S}_3 = \{6, 7, 8\}$. Compute the number of sets $\mathcal{H}$ that satisfy both of the following properties:

- $\mathcal{H} \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$;
- each of $\mathcal{H} \cap \mathcal{S}_1$, $\mathcal{H} \cap \mathcal{S}_2$, and $\mathcal{H} \cap \mathcal{S}_3$ is nonempty.

TB-3. Tom writes down the integers from 1 to 100, inclusive, on a chalkboard and then erases every number that contains a prime digit. Compute the sum of the *digits* that remain on the chalkboard.

2018 Tiebreaker Answers

TB-1. 32

TB-2. 201

TB-3. 337

2018 Tiebreaker Solutions

TB-1. Because the sequence contains 17! and 18!, its common difference must divide $M = 18! - 17! = 17 \cdot 17!$. Note that the set of integers appearing in all such sequences forms an arithmetic sequence with first term 17! and common difference M. The problem is thus equivalent to finding the greatest integer X such that $X! = 18! + kM$ for some integer k. Divide both sides of the previous equation by 17!, to obtain $X(X-1)(X-2)\cdot \cdots \cdot 19 \cdot 18 = 18 + 17k$, or in other words,

$$X(X-1)(X-2)\cdot \cdots \cdot 19 \cdot 18 \equiv 18 \pmod{17}. \qquad (*)$$

Note that modulo 17, the left-hand side of $(*)$ is equal to $(X-17)!$, hence the problem reduces to finding the greatest solution to $(X-17)! \equiv 1 \pmod{17}$. There is no such solution with $X > 33$, because then $(X-17)!$ is divisible by 17. Also, if $X = 33$, then by Wilson's Theorem, $16! \equiv -1 \pmod{17}$ because 17 is prime. However, $16! = 15! \cdot 16 \equiv 15! \cdot (-1) \equiv -1 \pmod{17}$, hence $15! \equiv 1 \pmod{17}$. Thus the desired value of X is **32**.

TB-2. There are a total of 2^8 subsets of $\{1,2,3,4,5,6,7,8\}$. Proceed indirectly and consider counting the sets $\mathcal{H}$ such that at least one of $\mathcal{H} \cap \mathcal{S}_1$, $\mathcal{H} \cap \mathcal{S}_2$, and $\mathcal{H} \cap \mathcal{S}_3$ is empty. Let N_a be the number of subsets $\mathcal{H}$ of $\{1,2,3,4,5,6,7,8\}$ in which $\mathcal{H} \cap \mathcal{S}_a$ is empty. Define $N_{a,b}$ and $N_{a,b,c}$ similarly. Then, by the Inclusion-Exclusion Principle, the answer is

$$2^8 - \big(N_1 + N_2 + N_3 - (N_{1,2} + N_{1,3} + N_{2,3}) + N_{1,2,3}\big).$$

Because $|\mathcal{S}_1| = 4$, it follows that $N_1 = 2^{8-4} = 16$. Similarly, $N_2 = 16$ and $N_3 = 32$. There are 6 elements in $\mathcal{S}_1 \cup \mathcal{S}_2$, thus $N_{1,2} = 2^{8-6} = 4$. Similarly, $N_{1,3} = 2$ and $N_{2,3} = 4$. Finally, $N_{1,2,3} = 1$ because $\emptyset$ is the only subset of $\{1,2,3,4,5,6,7,8\}$ that has an empty intersection with each of $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. Thus the answer is $256 - \big(16 + 16 + 32 - (4+2+4) + 1\big) = \mathbf{201}$.

Alternate Solution: Define a set $\mathcal{H}$ with the desired properties to be a *hitting set*. Count the number of hitting sets $\mathcal{H}$ according to the possible sizes of the set $\mathcal{H}$.

$\|\mathcal{H}\|$	**Number of hitting sets $\mathcal{H}$**
$0, 1$	0: A set $\mathcal{H}$ with $\|\mathcal{H}\| \leq 1$ cannot have a non-empty intersection with each of the three sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ because there is no element common to all three of these sets.
2	8: Note that $\|\mathcal{H} \cap \mathcal{S}_3\| = 1$. If $6 \in \mathcal{H}$, then because $\mathcal{H} \cap \mathcal{S}_1 \neq \emptyset$, There are 4 possible elements of $\mathcal{S}_1$ that could belong to $\mathcal{H}$. If $6 \notin \mathcal{H}$, then there are 2 choices of the element of $\mathcal{S}_3$ that could belong to $\mathcal{H}$ (7 or 8) and there are two choices of the element of $\mathcal{S}_1 \cap \mathcal{S}_2$ that could belong to $\mathcal{H}$ (3 or 4). Thus if $6 \notin \mathcal{H}$, then there are $2 \cdot 2 = 4$ possible hitting sets $\mathcal{H}$, for a total of $4 + 4 = 8$ hitting sets of size 2.
3	$\binom{8}{3} - 18 = 38$: Note that every subset $\mathcal{H}$ of $\{1,2,3,4,5,6,7,8\}$ with $\|\mathcal{H}\| = 3$ will have a non-empty intersection with at least two of the sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. There are 4 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_1$; these are the $\binom{4}{3}$ subsets of $\{5,6,7,8\}$ of size 3. There are 4 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_2$; these are the $\binom{4}{3}$ subsets of $\{1,2,7,8\}$ of size 3. Finally, there are 10 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_3$; these are the $\binom{5}{3}$ subsets of $\{1,2,3,4,5\}$ of size 3. Thus there are $\binom{8}{3} - (4 + 4 + 10) = 38$ hitting sets of size 3.
4	$\binom{8}{4} - 7 = 63$: Note that every subset $\mathcal{H}$ of $\{1,2,3,4,5,6,7,8\}$ with $\|\mathcal{H}\| = 4$ will have a non-empty intersection with at least two of the sets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. There is 1 set $\mathcal{H}$ that has an empty intersection with $\mathcal{S}_1$: $\{5,6,7,8\}$. There is 1 set $\mathcal{H}$ that has an empty intersection with $\mathcal{S}_2$: $\{1,2,7,8\}$. Finally, there are 5 sets $\mathcal{H}$, each of which has an empty intersection with $\mathcal{S}_3$: $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$, $\{1,3,4,5\}$, and $\{2,3,4,5\}$. Thus there are $\binom{8}{4} - (1 + 1 + 5) = 63$ hitting sets of size 4.
5	$\binom{8}{5} - 1 = 55$: Any set $\mathcal{H}$ with $\|\mathcal{H}\| = 5$ will intersect each of $\mathcal{S}_1$ and $\mathcal{S}_2$ in at least one element. The only set of size 5 that does not intersect $\mathcal{S}_3$ is $\{1,2,3,4,5\}$, hence there are $\binom{8}{5} - 1 = 55$ hitting sets of size 5.
6	$\binom{8}{6} = 28$: Because each of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ has at least 3 elements, a 6-element hitting set will have a non-empty intersection with each of $\mathcal{S}_1, \mathcal{S}_2$, and $\mathcal{S}_3$.
7	$\binom{8}{7} = 8$: Because each of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ has at least 3 elements, a 7-element hitting set will have a non-empty intersection with each of $\mathcal{S}_1, \mathcal{S}_2$, and $\mathcal{S}_3$.
8	1: The only possible set is $\{1,2,3,4,5,6,7,8\}$, which clearly works.

From the above analysis, the total number of hitting sets is

$$0 + 0 + 8 + 38 + 63 + 55 + 28 + 8 + 1 = \mathbf{201}.$$

Note: As in the alternate solution, a set $\mathcal{H}$ with the property that it has a non-empty intersection with each set in a collection of sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ is known as a *hitting set*. Intuitively, this name makes sense because $\mathcal{H}$ "hits" (i.e., intersects) each set $\mathcal{S}_i$ so that $|\mathcal{H} \cap \mathcal{S}_i| \geq 1$. The following general problem is believed to be computationally hard in the sense that there are no known efficient algorithms for answering it.

> *Given a collection of n sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ and a positive integer k, does there exist a hitting set $\mathcal{H}$ with $|\mathcal{H}| \leq k$?*

TB-3. Every one- or two-digit integer can be represented as $\underline{T}\,\underline{U}$ where $0 \leq T, U \leq 9$ (and for one-digit integers, $T = 0$). The prime digits are 2, 3, 5, and 7, so Tom will erase the 40 numbers of the form $\underline{2}\,\underline{U}$, $\underline{3}\,\underline{U}$, $\underline{5}\,\underline{U}$, and $\underline{7}\,\underline{U}$. After that, consider a fixed value of T, not equal to 2, 3, 5, or 7. Tom will erase the numbers $\underline{T}\,\underline{2}$, $\underline{T}\,\underline{3}$, $\underline{T}\,\underline{5}$, and $\underline{T}\,\underline{7}$. Among the six numbers remaining with tens digit T, the sum of their units digits is $0 + 1 + 4 + 6 + 8 + 9 = 28$, and the sum of their tens digits is $6T$. There are six such values of T, so the sum of all the digits of the remaining one- and two-digit integers is $6 \cdot 28 + 6(0 + 1 + 4 + 6 + 8 + 9) = 12 \cdot 28 = 336$. This sum includes the "number" 00, but the zeros do not affect the sum. Finally, because the number 100 remains, the answer is $336 + 1 = \mathbf{337}$.

2018 Super Relay Problems

1. Given that a, b, and c are positive integers such that $a + bc = 20$ and $a + b = 18$, compute the least possible value of abc.

2. Let T = TNYWR. In convex quadrilateral $LEOS$, $LE = EO$, $LS = SO$, and $\overline{ES}$ and $\overline{LO}$ intersect in point J. Given that $EJ = 24$, $JS = 60$, and $EO = T$, compute LS.

3. Let T = TNYWR. For each integer n, let $f(n)$ be the remainder when $n^2 - 1$ is divided by 8. Compute $f(T) + f(T + 1) + f(T + 2) + f(T + 3)$.

4. Let T = TNYWR. Compute the shortest distance between the lines $y = x + T$ and $y = x - T$.

5. Let T = TNYWR. Compute the value of x for which $\sqrt{x} + \sqrt{x + 80} = T$.

6. Let T = TNYWR. Jane rolls a fair T-sided die whose faces are numbered from 1 to T inclusive. The probability that she rolls a multiple of either 2 or 3 is the same as the probability that she rolls a multiple of either 4, 5, 7, p, q, or r, where p, q, and r are prime numbers and $7 < p < q < r$. Compute r.

7. Let T = TNYWR, and let $K = T + 3$. Compute

$$\log_4 2^1 + \log_4 2^2 + \log_4 2^3 + \cdots + \log_4 2^{K-1} + \log_4 2^K.$$

8. Let A be the number you will receive from position 7 and let B be the number you will receive from position 9. A circular track has a perimeter of A meters. Jasmine and Richard start running clockwise around the track at the same time and from the same starting position on the track. Jasmine runs at a constant speed and takes 16 seconds to run A meters. Richard also runs at a constant speed and takes 160 seconds to run B meters. When Jasmine and Richard first meet again at the starting point, Jasmine will have run m laps and Richard will have run n laps. Compute $20m + 18n$.

9. Let T = TNYWR. In the increasing arithmetic sequence $a_1, a_2, a_3, \ldots$, the common difference between consecutive terms is 4, and $a_1 = T$. Compute the value of n for which $a_n = 2018$.

10. Let T = TNYWR. Given that $f(x) = -x^2 + Tx + c$ for some real number c and $f(20) = 18$, compute the maximum value of $f(x)$, where x ranges over the real numbers.

11. Let T = TNYWR, and let P be the greatest prime factor of T. Given that a sphere has a surface area of $(P + 3)\pi$, compute the radius of the sphere.

12. Let $T = \text{TNYWR}$, and let $K = \dfrac{1}{T}$. The complex numbers a and b are the solutions to the equation $x^2 + 20x - K = 0$, and the complex numbers c and d are the solutions to the equation $x^2 - 18x + K - 4 = 0$. Compute $a^2 + b^2 + c^2 + d^2$.

13. Let $T = \text{TNYWR}$. A group of $T + 60$ students compete at an ARML site. At that site, a total of 50 students got at least one of individual questions 9 and 10 correct, 48 students got question 9 correct, and an astonishing 22 students got question 10 correct! Compute the probability that a randomly selected student from this group got *both* questions 9 and 10 correct.

14. Let $T = \text{TNYWR}$. In $\triangle DES$, $\sin D = \dfrac{1}{T}$ and $\sin E = \dfrac{1}{20}$. Given that $DS = 18$, compute ES.

15. Compute $\left|2018^3 - 3 \cdot 2018^2 \cdot 2017 + 3 \cdot 2018 \cdot 2017^2 - 2017^3\right|$.

2018 Super Relay Answers

1. 51
2. 75
3. 10
4. $10\sqrt{2}$
5. 18
6. 17
7. 105
8. 136
9. 420
10. 342
11. 4
12. 732
13. $\dfrac{1}{21}$
14. 360
15. 1

2018 Super Relay Solutions

1. Subtract the second equation from the first equation and factor to obtain $b(c-1) = 2$. Hence either $(b, c) = (2, 2)$ or $(b, c) = (1, 3)$. In the former case, $a = 18 - 2 = 16$, and in the latter case, $a = 18 - 1 = 17$. The value of abc is thus minimized when $a = 17$, $b = 1$, and $c = 3$, and $abc = \mathbf{51}$.

2. Note that quadrilateral $LEOS$ is a kite, hence $\overline{ES} \perp \overline{LO}$. By the Pythagorean Theorem, $JO^2 = T^2 - 24^2$ and $LS^2 = OS^2 = JO^2 + JS^2 = T^2 - 24^2 + 60^2 = T^2 + 3024$, hence $LS = \sqrt{T^2 + 3024}$. With $T = 51$, $LS = \mathbf{75}$. (Note: The calculation can be simplified by noting that once a value of T is received, $\triangle EJO$ is similar to an 8−15−17 triangle, and $\triangle OJS$ is similar to a 3−4−5 triangle.)

3. Note that exactly two elements of $\{T, T+1, T+2, T+3\}$ are even, and the other two elements of the set are odd. Similarly, exactly two elements of $\{T^2-1, (T+1)^2-1, (T+2)^2-1, (T+3)^2-1\}$ are even, and the other two elements of the set are odd. Without loss of generality, assume that $T = 4k$, for some integer k. Then $f(T) = 16k^2-1 \equiv 7 \pmod 8$, $f(T+1) = (4k)(4k+2) \equiv 0 \pmod 8$, $f(T+2) = 16k^2+8k+3 \equiv 3 \pmod 8$, and $f(T+3) = (4k+2)(4k+4) \equiv 0 \pmod 8$. Hence the answer is $7 + 0 + 3 + 0 = \mathbf{10}$ (independent of T).

4. Let $A = (0, T)$, $B = (0, -T)$, and let C be the foot of the perpendicular from A to the line $y = x - T$. Note that the two lines are parallel and, because their slopes are 1, they make an angle of inclination of 45° with the x-axis. Hence $m\angle ABC = m\angle BAC = 45°$. Because $AB = |2T|$, the distance between the two lines, AC, is equal to $|T\sqrt{2}|$. With $T = 10$, the answer is therefore $\mathbf{10\sqrt{2}}$.

5. Rewrite the given equation as $\sqrt{x+80} = T - \sqrt{x}$ and square each side to obtain $x + 80 = T^2 - 2T\sqrt{x} + x$. Thus $2T\sqrt{x} = T^2 - 80$, hence $x = \left(\frac{T^2-80}{2T}\right)^2$. With $T = 10\sqrt{2}$, the value of x is $\mathbf{18}$, which checks.

6. Equivalently, the number of outcomes that are multiples of 2 or 3 should equal the number of outcomes that are multiples of 4, 5, 7, p, q, or r. There are $\lfloor T/2 \rfloor$ multiples of 2 and $\lfloor T/3 \rfloor$ multiples of 3. Each of these counts multiples of 2 *and* 3, so by the Inclusion-Exclusion Principle, the number of multiples of 2 or 3 is $\lfloor T/2 \rfloor + \lfloor T/3 \rfloor - \lfloor T/6 \rfloor$. Similarly, the number of outcomes that are multiples of 4, 5, 7, p, q, or r is bounded above by $\lfloor T/4 \rfloor + \lfloor T/5 \rfloor + \lfloor T/7 \rfloor + \lfloor T/p \rfloor + \lfloor T/q \rfloor + \lfloor T/r \rfloor$. Depending on the value of T, this count may need to be adjusted in the event that an outcome is a multiple of at least two of the numbers 4, 5, 7, p, q, and r. However, if $T < 20$, this will not happen. With $T = 18$, the number of multiples of 2 or 3 is $9 + 6 - 3 = 12$, and the number of multiples of 4, 5, 7, p, q, or r is $4 + 3 + 2 + 1 + 1 + 1 = 12$. The only possible triple (p, q, r) is $(11, 13, 17)$, hence $r = \mathbf{17}$.

7. The given expression is equal to

$$\begin{aligned}\log_4\left(2^1\cdot 2^2\cdot\cdots\cdot 2^{K-1}\cdot 2^K\right) &= \log_4\left(2^{1+2+\cdots+(K-1)+K}\right)\\ &= \log_4\left(\left(4^{1/2}\right)^{K(K+1)/2}\right)\\ &= K(K+1)/4\\ &= (T+3)(T+4)/4.\end{aligned}$$

With $T=17$, the answer is **105**.

8. Note that Jasmine takes 16 seconds to run a lap around the track, whereas Richard takes $\frac{160}{B}\cdot A$ seconds to run a lap around the track. Let $\ell=\operatorname{lcm}(16,160A/B)$. Then Jasmine and Richard will first meet again at the starting point after ℓ seconds have elapsed. At that time, Jasmine will have run a total of $\frac{\ell}{16}$ laps and Richard will have run a total of $\frac{\ell}{160A/B}$ laps. With $A=105$ and $B=420$, it follows that $\ell=80$ and therefore $m=\frac{80}{16}=5$ and $n=\frac{80}{40}=2$. Therefore $20m+18n=100+36=\mathbf{136}$.

9. The desired value of n satisfies the equation $T+4(n-1)=2018$, thus $n=\dfrac{2022-T}{4}$. With $T=342$, this yields $n=\mathbf{420}$.

10. Plug in $x=20$ to obtain $18=f(20)=-400+20T+c$, hence $c=418-20T$. Complete the square in the quadratic polynomial to obtain $f(x)=-\left(x-\frac{T}{2}\right)^2+\frac{T^2}{4}+418-20T$. When x ranges over the real numbers, f will be maximized when $x=\frac{T}{2}$ because $-\left(x-\frac{T}{2}\right)^2$ is negative except when $x=\frac{T}{2}$. Thus the maximum value of f is $\frac{T^2}{4}+418-20T$. With $T=4$, this yields the answer of **342**.

11. Let r be the radius of the sphere. Then $4\pi r^2=(P+3)\pi$, so $r=\dfrac{\sqrt{P+3}}{2}$. With $T=732=2\cdot 3^2\cdot 61$, $P=61$, and the answer is $\frac{\sqrt{64}}{2}=\mathbf{4}$.

12. By Vieta's Formulas, $a+b=-20$, $ab=-K$, $c+d=18$, $cd=K-4$. Note also that $a^2+b^2+c^2+d^2=(a+b)^2+(c+d)^2-2ab-2cd=(-20)^2+18^2-2(-K)-2(K-4)=400+324+2K-2K+8=\mathbf{732}$ (independent of K and T).

13. Let N be the number of students who got both questions 9 and 10 correct. By the Inclusion-Exclusion Principle, it follows that $50=48+22-N$, so $N=20$. Thus the desired probability is $\dfrac{20}{T+60}$. With $T=360$, this gives an answer of $\mathbf{\frac{1}{21}}$.

14. By the Law of Sines, $\dfrac{ES}{\sin D}=\dfrac{DS}{\sin E}$, so $ES=DS\cdot\dfrac{\sin D}{\sin E}=18\cdot\dfrac{20}{T}=\dfrac{360}{T}$. Because $T=1$, the answer is **360** (and $\angle D$ is a right angle).

15. Let $a=2018$ and $b=2017$. Then the given expression equals $|a^3-3a^2b+3ab^2-b^3|=|(a-b)^3|$. Because $a-b=1$, the answer is therefore $|1^3|=\mathbf{1}$.

2019 Contest

Contents

2019 Team Problems

T-1. The points $(1, 2, 3)$ and $(3, 3, 2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

T-2. Eight students attend a Harper Valley ARML practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

T-3. Compute the least positive value of t such that

$$\operatorname{Arcsin}(\sin(t)),\ \operatorname{Arccos}(\cos(t)),\ \operatorname{Arctan}(\tan(t))$$

form (in some order) a three-term arithmetic progression with a nonzero common difference.

T-4. In non-right triangle ABC, distinct points P, Q, R, and S lie on $\overline{BC}$ in that order so that $\angle BAP \cong \angle PAQ \cong \angle QAR \cong \angle RAS \cong \angle SAC$. Given that the angles of $\triangle ABC$ are congruent to the angles of $\triangle APQ$ in some order of correspondence, compute $\mathrm{m}\angle B$ in degrees.

T-5. Consider the system of equations

$$\begin{aligned}
\log_4 x + \log_8(yz) &= 2\\
\log_4 y + \log_8(xz) &= 4\\
\log_4 z + \log_8(xy) &= 5.
\end{aligned}$$

Given that xyz can be expressed in the form 2^k, compute k.

T-6. A complex number z is selected uniformly at random so that $|z| = 1$. Compute the probability that z and z^{2019} both lie in Quadrant II in the complex plane.

T-7. Compute the least positive integer n such that the sum of the digits of n is five times the sum of the digits of $(n + 2019)$.

T-8. Compute the greatest real number K for which the graphs of

$$(|x| - 5)^2 + (|y| - 5)^2 = K \quad \text{and} \quad (x - 1)^2 + (y + 1)^2 = 37$$

have exactly two intersection points.

T-9. To *morph* a sequence means to replace two terms a and b with $a+1$ and $b-1$ if and only if $a+1 < b-1$, and such an operation is referred to as a morph. Compute the least number of morphs needed to transform the sequence $1^2, 2^2, 3^2, \ldots, 10^2$ into an arithmetic progression.

T-10. Triangle ABC is inscribed in circle ω. The tangents to ω at B and C meet at point T. The tangent to ω at A intersects the perpendicular bisector of $\overline{AT}$ at point P. Given that $AB = 14$, $AC = 30$, and $BC = 40$, compute $[PBC]$.

2019 Team Answers

T-1. 216

T-2. 12

T-3. $\dfrac{3\pi}{4}$

T-4. $\dfrac{45}{2}$

T-5. $\dfrac{66}{7}$

T-6. $\dfrac{505}{8076}$

T-7. 7986

T-8. 29

T-9. 56

T-10. $\dfrac{800}{3}$

2019 Team Solutions

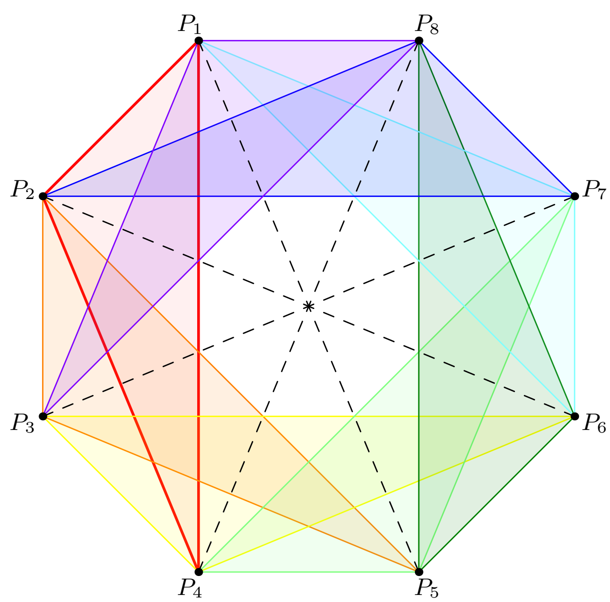
T-1. The distance between points $A(1,2,3)$ and $B(3,3,2)$ is $AB = \sqrt{(3-1)^2+(3-2)^2+(2-3)^2} = \sqrt{6}$. Denote by s the side length of the cube. Consider three possibilities.

- If $\overline{AB}$ is an edge of the cube, then $AB = s$, so one possibility is $s_1 = \sqrt{6}$.
- If $\overline{AB}$ is a face diagonal of the cube, then $AB = s\sqrt{2}$, so another possibility is $s_2 = \sqrt{3}$.
- If $\overline{AB}$ is a space diagonal of the cube, then $AB = s\sqrt{3}$, so the last possibility is $s_3 = \sqrt{2}$.

The answer is then $s_1^3 s_2^3 s_3^3 = (s_1 s_2 s_3)^3 = 6^3 = \mathbf{216}$.

T-2. The answer is 12. To give an example in which 12 selfies is possible, consider regular octagon $P_1P_2P_3P_4P_5P_6P_7P_8$. Each vertex of the octagon represents a student and each of the diagonals and sides of the octagon represents a pair of students. Construct eight triangles $P_1P_2P_4$, $P_2P_3P_5$, $P_3P_4P_6$, $\ldots$, $P_8P_1P_3$. Each of the segments in the forms of $\overline{P_iP_{i+1}}$, $\overline{P_iP_{i+2}}$, $\overline{P_iP_{i+3}}$ appears exactly once in these eight triangles. Taking 8 three-person selfies (namely $\{P_1,P_2,P_4\}$, $\{P_2,P_3,P_5\}$, $\ldots$, $\{P_8,P_1,P_3\}$) and 4 two-person selfies (namely $\{P_1,P_5\}$, $\{P_2,P_6\}$, $\{P_3,P_7\}$, $\{P_4,P_8\}$) gives a total of 12 selfies, completing the desired task.

A diagram of this construction is shown below. Each of the eight triangles is a different color, and each of the two-person selfies is represented by a dotted diameter.



It remains to show fewer than 12 selfies is impossible. Assume that the students took x three-person selfies and y two-person selfies. Each three-person selfie counts 3 pairs of student appearances (in a selfie), and each two-person selfie counts 1 pair of student appearances (in a selfie). Together, these selfies count $3x+y$ pairs of student appearances. There are $\binom{8}{2} = 28$

pairs of student appearances. Hence $3x + y = 28$. The number of selfies is $x + y = 28 - 2x$, so it is enough to show that $x \le 8$.

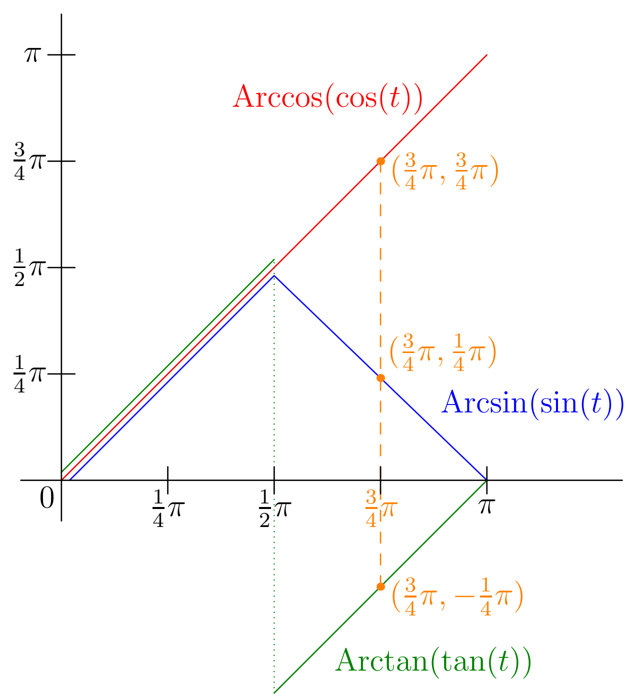
Assume for contradiction there are $x \ge 9$ three-person selfies; then there are at least $3 \cdot 9 = 27$ (individual) student appearances on these selfies. Because there are 8 students, some student s_1 had at least $\lceil 27/8 \rceil$ appearances; that is, s_1 appeared in at least 4 of these three-person selfies. There are $2 \cdot 4 = 8$ (individual) student appearances other than s_1 on these 4 selfies. Because there are only 7 students besides s_1, some other student s_2 had at least $\lceil 8/7 \rceil$ (individual) appearances on these 4 selfies; that is, s_2 appeared (with s_1) in at least 2 of these 4 three-person selfies, violating the condition that each pair of the students appears in exactly one selfie. Thus the answer is **12**.

T-3. For $0 \le t < \pi/2$, all three values are t, so the desired t does not lie in this interval.

For $\pi/2 < t < \pi$,

$$\begin{aligned} \operatorname{Arcsin}(\sin(t)) &= \pi - t \in (0, \pi/2) \\ \operatorname{Arccos}(\cos(t)) &= t \qquad \in (\pi/2, \pi) \\ \operatorname{Arctan}(\tan(t)) &= t - \pi \in (-\pi/2, 0)\,. \end{aligned}$$

A graph of all three functions is shown below.



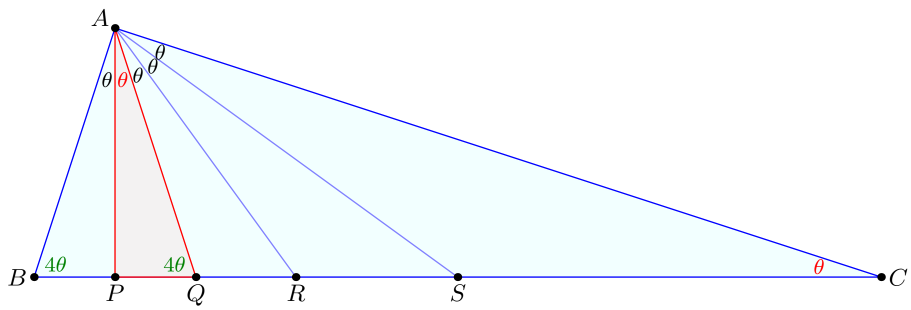
Thus if the three numbers are to form an arithmetic progression, they should satisfy

$$t - \pi < \pi - t < t.$$

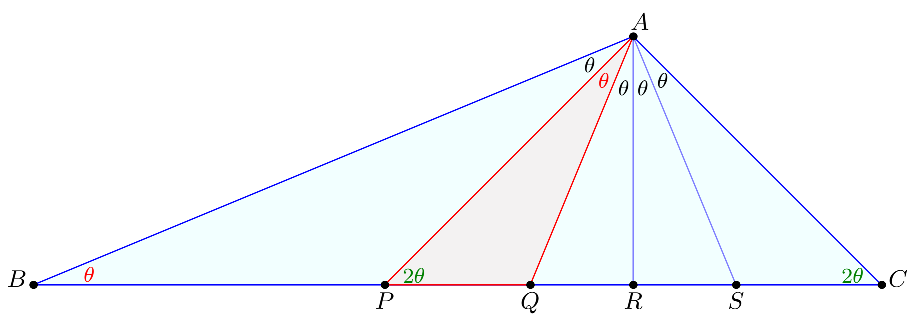
The three numbers will be in arithmetic progression if and only if $t+(t-\pi)=2(\pi-t)$, which implies $t=\boldsymbol{\frac{3\pi}{4}}$. Note that if $t=\frac{3\pi}{4}$, the arithmetic progression is $-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$, as required.

T-4. Let $\theta=\frac{1}{5}\mathrm{m}\angle A$. Because $\mathrm{m}\angle PAQ=\theta<5\theta=\mathrm{m}\angle A$, it follows that either $\mathrm{m}\angle B=\theta$ or $\mathrm{m}\angle C=\theta$. Thus there are two cases to consider.

If $\mathrm{m}\angle C=\theta$, then it follows that $\mathrm{m}\angle AQP=\mathrm{m}\angle QAC+\mathrm{m}\angle ACB=4\theta$, and hence $\mathrm{m}\angle B=4\theta$. So $\triangle ABC$ has angles of measures 5θ, 4θ, θ, and thus $\theta=18°$. However, this implies $\mathrm{m}\angle A=5\theta=90°$, which is not the case.



If instead $\mathrm{m}\angle B=\theta$, then it follows that $\mathrm{m}\angle APQ=\mathrm{m}\angle BAP+\mathrm{m}\angle ABP=2\theta$, and hence $\mathrm{m}\angle C=2\theta$. So $\triangle ABC$ has angles of measures 5θ, 2θ, θ, and thus $\theta=22.5°$. Hence $\mathrm{m}\angle B=\theta=\mathbf{22.5°}$.



T-5. Note that for $n>0$, $\log_4 n=\log_{64} n^3$ and $\log_8 n=\log_{64} n^2$. Adding together the three given equations and using both the preceding facts and properties of logarithms yields

$$\begin{aligned} &\log_4(xyz)+\log_8(x^2y^2z^2)=11\\ \Longrightarrow\ &\log_{64}(xyz)^3+\log_{64}(xyz)^4=11\\ \Longrightarrow\ &\log_{64}(xyz)^7=11\\ \Longrightarrow\ &7\log_{64}(xyz)=11. \end{aligned}$$

The last equation is equivalent to $xyz=64^{11/7}=2^{66/7}$, hence the desired value of k is $\boldsymbol{\frac{66}{7}}$.

T-6. For convenience, let $\alpha = \pi/4038$. Denote by

$$0 \le \theta < 2\pi = 8076\alpha$$

the complex argument of z, selected uniformly at random from the interval $[0, 2\pi)$. Then z itself lies in Quadrant II if and only if

$$2019\alpha = \frac{\pi}{2} < \theta < \pi = 4038\alpha.$$

On the other hand, z^{2019} has argument 2019θ, and hence it lies in Quadrant II if and only if there is some integer k with

$$\begin{aligned} &\frac{\pi}{2} + 2k\pi < 2019\theta < \pi + 2k\pi \\ \iff &(4k+1)\cdot\frac{\pi}{2} < 2019\theta < (4k+2)\cdot\frac{\pi}{2} \\ \iff &(4k+1)\alpha < \theta < (4k+2)\alpha. \end{aligned}$$

Because it is also true that $2019\alpha < \theta < 4038\alpha$, the set of θ that satisfies the conditions of the problem is the union of intervals:

$$(2021\alpha, 2022\alpha) \cup (2025\alpha, 2026\alpha) \cup \cdots \cup (4037\alpha, 4038\alpha).$$

There are 505 such intervals, the j^{th} interval consisting of $(4j+2017)\alpha < \theta < (4j+2018)\alpha$. Each interval has length α, so the sum of the intervals has length 505α. Thus the final answer is

$$\frac{505\alpha}{2\pi} = \frac{505}{2\cdot 4038} = \frac{\mathbf{505}}{\mathbf{8076}}.$$

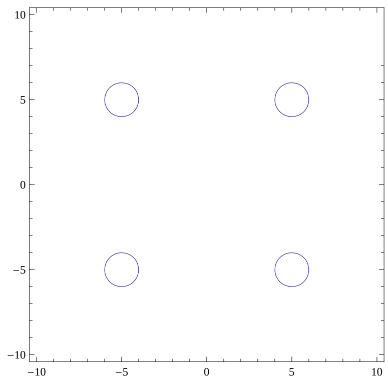
T-7. Let $S(n)$ denote the sum of the digits of n, so that solving the problem is equivalent to solving $S(n) = 5S(n+2019)$. Using the fact that $S(n) \equiv n \pmod 9$ for all n, it follows that

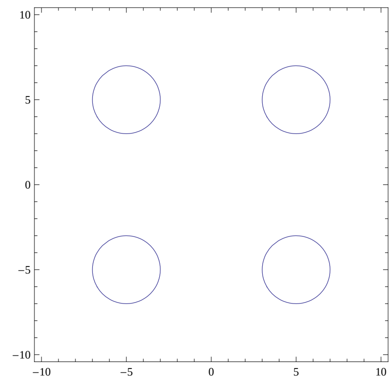
$$\begin{aligned} n &\equiv 5(n+2019) \equiv 5(n+3) \pmod 9 \\ 4n &\equiv -15 \pmod 9 \\ n &\equiv 3 \pmod 9. \end{aligned}$$

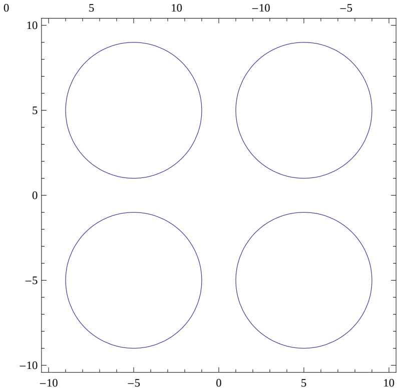
Then $S(n+2019) \equiv 6 \pmod 9$. In particular, $S(n+2019) \ge 6$ and $S(n) \ge 5\cdot 6 = 30$. The latter inequality implies $n \ge 3999$, which then gives $n + 2019 \ge 6018$. Thus if $n + 2019$ were a four-digit number, then $S(n+2019) \ge 7$. Moreover, $S(n+2019)$ can only be 7, because otherwise, $S(n) = 5S(n+2019) \ge 40$, which is impossible (if n has four digits, then $S(n)$ can be no greater than 36). So if $n+2019$ were a four-digit number, then $S(n+2019) = 7$ and $S(n) = 35$. But this would imply that the digits of n are 8, 9, 9, 9 in some order, contradicting the assumption that $n + 2019$ is a four-digit number. On the other hand, if $n + 2019$ were a five-digit number such that $S(n+2019) \ge 6$, then the least such value of $n + 2019$ is 10005, and indeed, this works because it corresponds to $n = \mathbf{7986}$, the least possible value of n.

T-8. The graph of the second equation is simply the circle of radius $\sqrt{37}$ centered at $(1, -1)$. The first graph is more interesting, and its behavior depends on K.

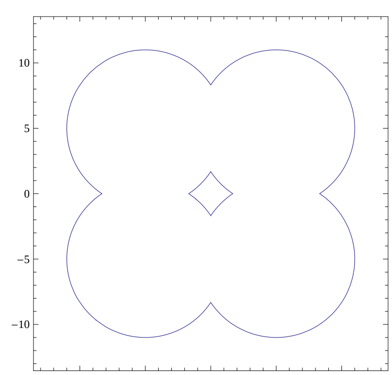
- For small values of K, the first equation determines a set of four circles of radius $\sqrt{K}$ with centers at $(5, 5), (5, -5), (-5, 5)$, and $(-5, -5)$. Shown below are versions with $K = 1$, $K = 4$, and $K = 16$.



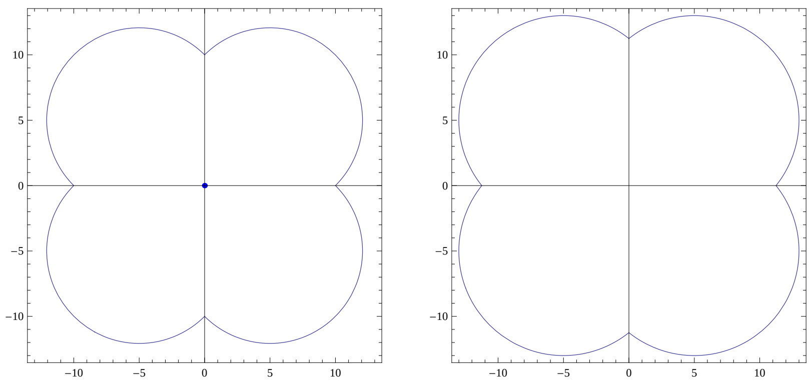




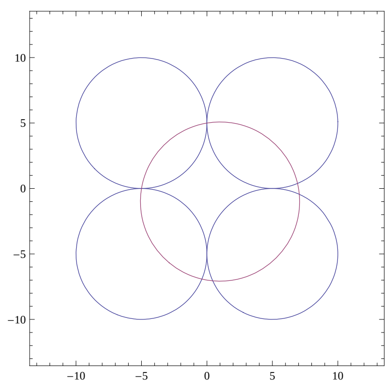
- However, when $K > 25$, the graph no longer consists of four circles! As an example, for $K = 36$, the value $x = 5$ gives $(|y| - 5)^2 = 36$; hence $|y| = -1$ or $|y| = 6$. The first option is impossible; the graph ends up "losing" the portions of the upper-right circle that would cross the x- or y-axes compared to the graph for $(x - 5)^2 + (y - 5)^2 = 36$. The graph for $K = 36$ is shown below.



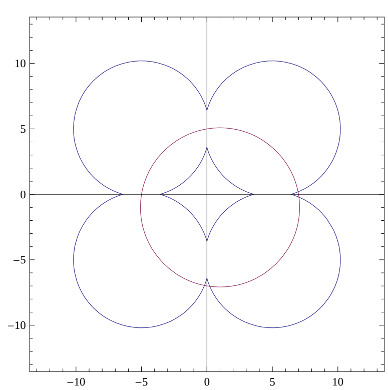
- As K continues to increase, the "interior" part of the curve continues to shrink, until at $K = 50$, it simply comprises the origin, and for $K > 50$, it does not exist. As examples, the graphs with $K = 50$ and $K = 64$ are shown below.



Overlay the graph of the circle of radius $\sqrt{37}$ centered at $(1, -1)$ with the given graphs. When $K = 25$, this looks like the following graph.



Note that the two graphs intersect at $(0, 5)$ and $(-5, 0)$, as well as four more points (two points near the positive x-axis and two points near the negative y-axis). When K is slightly greater than 25, this drops to four intersection points. The graph for $K = 27$ is shown below.



Thus for the greatest K for which there are exactly two intersection points, those two intersection points should be along the positive x- and negative y-axes. If the intersection point on the positive x-axis is at $(h, 0)$, then $(h-1)^2 + (0+1)^2 = 37$ and $(h-5)^2 + (0-5)^2 = K$. Thus $h = 7$ and $K = \mathbf{29}$.

T-9. Call the original sequence of ten squares $T = (1^2, 2^2, \ldots, 10^2)$. A *morphed sequence* is one that can be obtained by morphing T a finite number of times.

This solution is divided into three steps. In the first step, a characterization of the possible final morphed sequences is given. In the second step, a lower bound on the number of steps is given, and in the third step, it is shown that this bound can be achieved.

Step 1. Note the following.

- The sum of the elements of T is $1^2 + 2^2 + \cdots + 10^2 = 385$, and morphs are sum-preserving. So any morphed sequence has sum 385 and a mean of 38.5.
- The sequence T has positive integer terms, and morphs preserve this property. Thus any morphed sequence has positive integer terms.
- The sequence T is strictly increasing, and morphs preserve this property. Thus any morphed sequence is strictly increasing.

Now if the morphed sequence is an arithmetic progression, it follows from the above three observations that it must have the form

$$(38.5 - 4.5d,\ 38.5 - 3.5d,\ \ldots,\ 38.5 + 4.5d)$$

where d is an odd positive integer satisfying $38.5 - 4.5d > 0$. Therefore the only possible values of d are 7, 5, 3, 1; thus there are at most four possibilities for the morphed sequence, shown in the table below. Denote these four sequences by A, B, C, D.

	T	1	4	9	16	25	36	49	64	81	100
$d = 7:$	A	7	14	21	28	35	42	49	56	63	70
$d = 5:$	B	16	21	26	31	36	41	46	51	56	61
$d = 3:$	C	25	28	31	34	37	40	43	46	49	52
$d = 1:$	D	34	35	36	37	38	39	40	41	42	43

Step 2. Given any two sequences $X = (x_1, \ldots, x_{10})$ and $Y = (y_1, \ldots, y_{10})$ with $\sum_{i=1}^{10} x_i = \sum_{i=1}^{10} y_i = 385$, define the *taxicab distance*

$$\rho(X, Y) = \sum_{i=1}^{10} |x_i - y_i|.$$

Observe that if X' is a morph of X, then $\rho(X', Y) \geq \rho(X, Y) - 2$. Therefore the number of morphs required to transform T into some sequence Z is at least $\frac{1}{2}\rho(T, Z)$. Now

$$\frac{1}{2}\rho(T, A) = \frac{1}{2}\sum_{i=1}^{10} \left|i^2 - 7i\right| = 56$$

and also $\rho(T, A) < \min\left(\rho(T, B), \rho(T, C), \rho(T, D)\right)$. Thus at least 56 morphs are needed to obtain sequence A (and more morphs would be required to obtain any of sequences B, C, or D).

Step 3. To conclude, it remains to verify that one can make 56 morphs and arrive from T to A. One of many possible constructions is given below.

T	1	4	9	16	25	36	49	64	81	100
6 morphs	1	4	9	16	25	42	49	58	81	100
2 morphs	1	4	9	16	27	42	49	56	81	100
8 morphs	1	4	9	16	35	42	49	56	73	100
10 morphs	1	4	9	26	35	42	49	56	63	100
2 morphs	1	4	9	28	35	42	49	56	63	98
12 morphs	1	4	21	28	35	42	49	56	63	86
10 morphs	1	14	21	28	35	42	49	56	63	76
6 morphs	7	14	21	28	35	42	49	56	63	70

Therefore the least number of morphs needed to transform T into an arithmetic progression is **56**.

Remark: For step 3, one may prove more generally that any sequence of 56 morphs works as long as both of the following conditions hold:

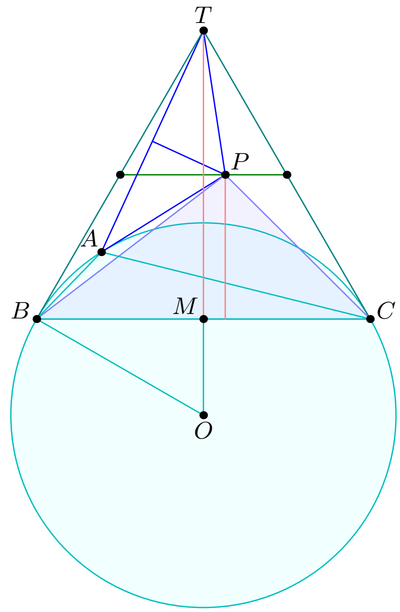
- each morph increases one of the first six elements and decreases one of the last three elements, and
- at all times, the i^{th} term is at most $7i$ for $i \le 6$, and at least $7i$ for $i \ge 8$.

T-10. To begin, denote by R the radius of ω. The semiperimeter of triangle ABC is 42, and then applying Heron's formula yields

$$[ABC] = \frac{14 \cdot 30 \cdot 40}{4R} = \sqrt{42 \cdot 28 \cdot 12 \cdot 2} = 168$$

from which it follows that $R = \dfrac{14 \cdot 30 \cdot 40}{4 \cdot 168} = 25.$

Now consider the point circle with radius zero centered at T in tandem with the circle ω. Because $PA = PT$, it follows that P lies on the radical axis of these circles. Moreover, the midpoints of $\overline{TB}$ and $\overline{TC}$ lie on this radical axis as well. Thus P lies on the midline of $\triangle TBC$ that is parallel to $\overline{BC}$.



To finish, let O be the center of ω and let M be the midpoint of $\overline{BC}$. By considering right triangle TBO with altitude $\overline{BM}$, it follows that $MT \cdot MO = MB^2$, but also $MO = \sqrt{OB^2 - MB^2} = \sqrt{25^2 - 20^2} = 15$, so

$$MT = \frac{MB^2}{MO} = \frac{400}{15} = \frac{80}{3}.$$

Thus the distance from P to $\overline{BC}$ is $\frac{1}{2}MT = \frac{40}{3}$. Finally,

$$[PBC] = \frac{1}{2} \cdot \frac{40}{3} \cdot BC = \mathbf{\frac{800}{3}}.$$

2019 Individual Problems

I-1. In rectangle $PAUL$, point D is the midpoint of $\overline{UL}$ and points E and F lie on $\overline{PL}$ and $\overline{PA}$, respectively, so that $\frac{PE}{EL} = \frac{3}{2}$ and $\frac{PF}{FA} = 2$. Given that $PA = 36$ and $PL = 25$, compute the area of pentagon $AUDEF$.

I-2. Rectangle $ARML$ has length 125 and width 8. The rectangle is divided into 1000 squares of area 1 by drawing in gridlines parallel to the sides of $ARML$. Diagonal $\overline{AM}$ passes through the interior of exactly n of the 1000 unit squares. Compute n.

I-3. Compute the least integer $n > 1$ such that the product of all positive divisors of n equals n^4.

I-4. Each of the six faces of a cube is randomly colored red or blue with equal probability. Compute the probability that no three faces of the same color share a common vertex.

I-5. Scalene triangle ABC has perimeter 2019 and integer side lengths. The angle bisector from C meets $\overline{AB}$ at D so that $AD = 229$. Given that AC and AD are relatively prime, compute BC.

I-6. Given that a and b are positive and

$$\lfloor 20 - a \rfloor = \lfloor 19 - b \rfloor = \lfloor ab \rfloor,$$

compute the least upper bound of the set of possible values of $a + b$.

I-7. Compute the number of five-digit integers $\underline{M}\,\underline{A}\,\underline{R}\,\underline{T}\,\underline{Y}$, with all digits distinct, such that $M > A > R$ and $R < T < Y$.

I-8. In parallelogram $ARML$, points P and Q are the midpoints of sides $\overline{RM}$ and $\overline{AL}$, respectively. Point X lies on segment $\overline{PQ}$, and $PX = 3$, $RX = 4$, and $PR = 5$. Point I lies on segment $\overline{RX}$ so that $IA = IL$. Compute the maximum possible value of $\frac{[PQR]}{[LIP]}$.

I-9. Given that a, b, c, and d are positive integers such that

$$a! \cdot b! \cdot c! = d! \quad \text{and} \quad a + b + c + d = 37,$$

compute the product $abcd$.

I-10. Compute the value of

$$\sin(6^\circ) \cdot \sin(12^\circ) \cdot \sin(24^\circ) \cdot \sin(42^\circ) + \sin(12^\circ) \cdot \sin(24^\circ) \cdot \sin(42^\circ).$$

.

2019 Individual Answers

I-1. 630

I-2. 132

I-3. 24

I-4. $\dfrac{9}{32}$

I-5. 888

I-6. $\dfrac{41}{5}$

I-7. 1512

I-8. $\dfrac{4}{3}$

I-9. 2240

I-10. $\dfrac{1}{16}$

2019 Individual Solutions

I-1. For convenience, let $PA = 3x$ and let $PL = 5y$. Then the given equations involving ratios of segment lengths imply that $PE = 3y$, $EL = 2y$, $PF = 2x$, and $FA = x$. Then $[PAUL] = (3x)(5y) = 15xy$ and

$$\begin{aligned}[AUDEF] &= [PAUL] - [PEF] - [ELD] \\ &= 15xy - \frac{1}{2}(3y)(2x) - \frac{1}{2}(2y)\left(\frac{3x}{2}\right) \\ &= 15xy - 3xy - \frac{3xy}{2} \\ &= \frac{21xy}{2}.\end{aligned}$$

Because $15xy = 36 \cdot 25$, it follows that $3xy = 36 \cdot 5 = 180$ and that

$$\frac{21xy}{2} = \frac{7}{2}(3xy) = \frac{7}{2} \cdot 180 = \mathbf{630}.$$

I-2. Notice that 125 and 8 are relatively prime. Examining rectangles of size $a \times b$ where a and b are small and relatively prime suggests an answer of $a + b - 1$. To see that this is the case, note that other than the endpoints, the diagonal does not pass through any vertex of any unit square. After the first square, it must enter each subsequent square via a vertical or horizontal side. By continuity, the total number of these sides is the sum of the $a - 1$ interior vertical lines and $b-1$ interior horizontal lines. The diagonal passes through $(a-1)+(b-1) = a+b-2$ additional squares, so the total is $a + b - 1$. Because 125 and 8 are relatively prime, it follows that $N = 125 + 8 - 1 = \mathbf{132}$.

Remark: As an exercise, the reader is encouraged to show that the answer for general a and b is $a + b - \gcd(a, b)$.

I-3. Note that every factor pair d and $\frac{n}{d}$ have product n. For the product of all such divisor pairs to equal n^4, there must be exactly 4 divisor pairs, or 8 positive integer divisors. A number has 8 positive integer divisors if it is of the form a^3b^1 or a^7 where a and b are distinct primes. The prime factorization a^3b^1 $(a \neq b)$ provides a set of divisors each of which has 4 options for using a (a^0, a^1, a^2, a^3) and an independent 2 options for using b (b^0, b^1). Using the least values $(a, b) = (2, 3)$, $a^3b^1 = 24$. If instead the prime factorization is a^7 (having divisors $a^0, a^1, a^2, \ldots, a^7$), the least answer would be $2^7 = 128$. Thus the answer is $\mathbf{24}$.

I-4. There are $2^6 = 64$ colorings of the cube. Let r be the number of faces that are colored red. Define a *monochromatic vertex* to be a vertex of the cube for which the three faces meeting there have the same color. It is clear that a coloring without a monochromatic vertex is only possible in the cases $2 \le r \le 4$. If $r = 2$ or $r = 4$, the only colorings that do not have a monochromatic vertex occur when two opposing faces are colored with the minority color (red in the $r = 2$ case, blue in the $r = 4$ case). Because there are 3 pairs of opposite faces of a cube, there are 3 colorings without a monochromatic vertex if $r = 2$ and another 3 such colorings if $r = 4$. For the $r = 3$ colorings, of which there are 20, the only cases in which there *are* monochromatic vertices occur when opposing faces are monochromatic, but in different colors. There are $2^3 = 8$ such colorings, leaving $20 - 8 = 12$ colorings that do not have a monochromatic vertex. Therefore $3 + 3 + 12 = 18$ of the 64 colorings have no monochromatic vertex, and the answer is $\mathbf{\dfrac{9}{32}}$.

I-5. Let $BC = a, AC = b, AB = c$. Also, let $AD = e$ and $BD = f$. Then $a + b + e + f = 2019$, the values a, b, and $e + f$ are integers, and by the Angle Bisector Theorem, $\dfrac{e}{f} = \dfrac{b}{a}$. So $b = \dfrac{ae}{f} = \dfrac{229a}{f}$. Because 229 is prime and $\gcd(b, e) = 1$, conclude that f must be an integer multiple of 229. So let $f = 229x$ for some integer x. Then $a = b \cdot x$ and $a + b + c = 2019$ implies $2019 = bx + b + 229 + 229x = (b + 229)(1 + x)$. Because $2019 = 673 \cdot 3$, it follows that $b = 444$ and $x = 2$, from which $BC = a = \mathbf{888}$.

I-6. Let the common value of the three expressions in the given equation be N. Maximizing $a + b$ involves making at least one of a and b somewhat large, which makes the first two expressions for N small. So, to maximize $a + b$, look for the least possible value of N. One can show that $N = 14$ is not possible because that would require $a > 5$ and $b > 4$, which implies $ab > 20$. But $N = 15$ is possible by setting $a = 4 + x, b = 3 + y$, where $0 < x, y \le 1$. The goal is to find the least upper bound for $x + y$ given $15 \le (4 + x)(3 + y) < 16 \Rightarrow 3 \le 3(x + y) + y + xy < 4$. This is equivalent to seeking the maximum value of $x + y$ given $3(x + y) + y + xy \le 4$. By inspection, if $x = 1$ and $y = \frac{1}{5}$, then $3(x + y) + y + xy = 4 \le 4$. This is in fact optimal. To see this, consider that because $3x + 3y + y + xy \le 4$, it follows that $y \le \dfrac{4 - 3x}{x + 4}$, and so $x + y \le x + \dfrac{4 - 3x}{x + 4} \le \dfrac{x^2 + x + 4}{x + 4}$, which is increasing on $0 \le x \le 1$. Thus the maximum for $x + y$ is attained when $x = 1$. Hence the least upper bound for $a + b$ is $5 + (3 + \frac{1}{5}) = \mathbf{\dfrac{41}{5}}$.

I-7. There are $\binom{10}{5} = 252$ ways to choose the values of the digits M, A, R, T, Y, without restrictions. Because R is fixed as the least of the digits and because $T < Y$, it suffices to find the number of ways to choose M and A. Once M and A are chosen, the other three digits are uniquely determined. There are $\binom{4}{2} = 6$ ways to select M, A. Thus the number of five-digit integers of the type described is $252 \cdot 6 = \mathbf{1512}$.

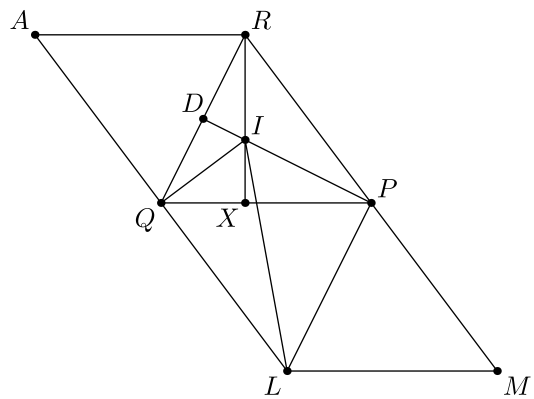
I-8. Because $AI = LI$ and $AQ = LQ$, line IQ is the perpendicular bisector of $\overline{AL}$. Because $ARML$ is a parallelogram, $\overline{QI} \perp \overline{RP}$. Note also that $m\angle RXP = 90°$. Thus I is the orthocenter of triangle PQR, from which it follows that $\overleftrightarrow{PI} \perp \overline{RQ}$ and $\overline{PI} \perp \overline{PL}$ (because $PRQL$

is a parallelogram). Extend $\overline{PI}$ through I to meet $\overline{RQ}$ at D. Then $2[PQR] = RQ \cdot PD$ and $2[LIP] = PI \cdot PL = PI \cdot RQ$. Hence the problem is equivalent to determining the maximum value of PD/PI.

Set $\text{m}\angle RPD = \text{m}\angle RPI = \alpha$ and $\text{m}\angle IPX = \beta$, and note that $PD = PR\cos\alpha = 5\cos\alpha$ and $PI = PX/\cos\beta = 3/\cos\beta$. It follows that

$$\frac{PD}{PI} = \frac{5\cos\alpha\cos\beta}{3} = \frac{5\big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\big)}{6} \le \frac{5(3/5+1)}{6} = \mathbf{\frac{4}{3}},$$

with equality when $\alpha = \beta$.



I-9. Without loss of generality, assume $a \le b \le c < d$. Note that d cannot be prime, as none of $a!$, $b!$, or $c!$ would have it as a factor. If $d = p+1$ for some prime p, then $c = p$ and $a!b! = p+1$. The least possible values of $a!b!$ are $1, 2, 4, 6, 24, 36, 48, 120, 144, 240$, so the case where $d = p+1$ is impossible. If $d \ge 21$, then $a+b+c \le 16$ and it is impossible to find values of a and b such that $a! \cdot b! = \dfrac{d!}{c!}$. If $d = 16$, either $a!b! = 16$ or $a!b! = 16\cdot 15$ or $a!b! = 16\cdot 15\cdot 14$. Comparing to the list above, the only possible value $a!b!$ on the list is $16 \cdot 15 = 240$ and so $(a, b, c, d) = (2, 5, 14, 16)$ and $abcd = \mathbf{2240}$.

I-10. Let $S = (1 + \sin 6°)(\sin 12° \sin 24° \sin 42°)$. It follows from a sum-to-product identity that $1+\sin 6° = \sin 90° + \sin 6° = 2\sin 48° \cos 42°$. Because the sine of an angle is the cosine of its complement, it follows that

$$S = (2\sin 48° \cos 42°)(\sin 12° \sin 24° \sin 42°) = 2(\sin 48°)^2(\sin 12° \sin 24° \cos 48°).$$

By the double-angle formula, this means $S = \sin 12° \sin 24° \sin 48° \sin 96°$. By a product-to-sum identity,

$$\sin 12° \sin 48° = \frac{\cos 36° - \cos 60°}{2} = \frac{\sqrt{5}-1}{8} \qquad (1)$$

and

$$\sin 24° \sin 96° = \frac{\cos 72° - \cos 120°}{2} = \frac{\sqrt{5}+1}{8}. \qquad (2)$$

Multiply the expressions on the right-hand sides of (1) and (2) to obtain $\mathbf{\dfrac{1}{16}}$.

Note: The following is an alternate approach for computing $\sin 12° \sin 24° \sin 48° \sin 96°$. It relies on the identity $\sin(5x) = 16\sin^5 x - 20\sin^3 x + 5\sin x$. Note that $\sin 12°$, $\sin 24°$, $\sin(-48°)$, $\sin 96°$, and $\sin(-60°)$ all satisfy $\sin(5x) = \frac{\sqrt{3}}{2}$, and these are all different values. Therefore these five values are the solutions of the equation $16s^5 - 20s^3 + 5s - \frac{\sqrt{3}}{2} = 0$.

Using Vieta's formulas, it therefore follows that

$$\begin{aligned} &(\sin 12°)(\sin 24°)(\sin(-48°))(\sin 96°)(\sin(-60°)) \\ &= (\sin 12°)(\sin 24°)(\sin 48°)(\sin 96°)(\sin 60°) \\ &= \frac{\sqrt{3}/2}{16}. \end{aligned}$$

Thus $(\sin 12°)(\sin 24°)(\sin 48°)(\sin 96°) = \mathbf{\dfrac{1}{16}}$.

Note that this alternate approach used an identity expressing $\sin(5x)$ in terms of $\sin x$. This can be developed using the addition identity for the sine function as well as the well-known identities:

$$\begin{aligned} \sin(2x) &= 2(\sin x)(\cos x), &\qquad \cos(2x) &= 1 - 2\sin^2 x, \\ \sin(3x) &= 3\sin x - 4\sin^3 x, &\qquad \cos(3x) &= 4\cos^3 x - 3\cos x. \end{aligned}$$

Thus

$$\begin{aligned} \sin(5x) &= \sin(3x + 2x) \\ &= (\sin 3x)(\cos 2x) + (\sin 2x)(\cos 3x) \\ &= (3\sin x - 4\sin^3 x)(1 - 2\sin^2 x) + (2\sin x\cos x)(4\cos^3 x - 3\cos x) \\ &= (3\sin x - 4\sin^3 x)(1 - 2\sin^2 x) + (2\sin x)(4\cos^4 x - 3\cos^2 x) \\ &= (3\sin x - 4\sin^3 x)(1 - 2\sin^2 x) + (2\sin x)\big(4(1 - \sin^2 x)^2 - 3(1 - \sin^2 x)\big). \end{aligned}$$

The right-hand side of the last line is expressed in terms of $\sin x$, and when it is simplified, it yields $\sin(5x) = 16\sin^5 x - 20\sin^3 x + 5\sin x$.

Power Question 2019: Elizabeth's Escape

Elizabeth is in an "escape room" puzzle. She is in a room with one door which is locked at the start of the puzzle. The room contains n light switches, each of which is initially off. Each minute, she must flip exactly k different light switches (to "flip" a switch means to turn it on if it is currently off, and off if it is currently on). At the end of each minute, if all of the switches are on, then the door unlocks and Elizabeth escapes from the room.

Let $E(n, k)$ be the minimum number of minutes required for Elizabeth to escape, for positive integers n, k with $k \leq n$. For example, $E(2, 1) = 2$ because Elizabeth cannot escape in one minute (there are two switches and one must be flipped every minute) but she can escape in two minutes (by flipping Switch 1 in the first minute and Switch 2 in the second minute). Define $E(n, k) = \infty$ if the puzzle is impossible to solve (that is, if it is impossible to have all switches on at the end of any minute).

For convenience, assume the n light switches are numbered 1 through n.

1. Compute the following.
   a. $E(6, 1)$ [1 pt]
   b. $E(6, 2)$ [1 pt]
   c. $E(7, 3)$ [1 pt]
   d. $E(9, 5)$ [1 pt]
2. Find the following in terms of n.
   a. $E(n, 2)$ for positive even integers n [2 pts]
   b. $E(n, 3)$ for values of n of the form $n = 3a + 2$ where a is a positive integer [2 pts]
   c. $E(n, n - 2)$ for $n \geq 5$ [2 pts]
3. Find an integer value of k with $1 < k < 2019$ such that $E(2019, k) = \infty$. [3 pts]
4. a. Show that if $n + k$ is even and $\frac{n}{2} < k < n$, then $E(n, k) = 3$. [3 pts]
   b. Show that if n is even and k is odd, then $E(n, k) = E(n, n - k)$. [3 pts]
5. Find the following.
   a. $E(2020, 1993)$ [3 pts]
   b. $E(2001, 501)$ [3 pts]
6. a. Show that if n and k are both even and $k \leq \frac{n}{2}$, then $E(n, k) = \lceil \frac{n}{k} \rceil$. [3 pts]
   b. Prove that if k is odd and $k \leq \frac{n}{2}$, then either $E(n, k) = \lceil \frac{n}{k} \rceil$ or $E(n, k) = \lceil \frac{n}{k} \rceil + 1$. [4 pts]
7. Find all ordered pairs (n, k) for which $E(n, k) = 3$. [3 pts]

One might guess that in most cases, $E(n,k) \approx \frac{n}{k}$. In light of this guess, define the *inefficiency* of the ordered pair (n,k), denoted $I(n,k)$, as

$$I(n,k) = E(n,k) - \frac{n}{k}$$

if $E(n,k) \neq \infty$. If $E(n,k) = \infty$, then by convention, $I(n,k)$ is undefined.

8. a. Compute $I(6,3)$. [1 pt]

   b. Compute $I(5,3)$. [1 pt]

   c. Find positive integers n and k for which $I(n,k) = \frac{15}{8}$. [2 pts]

   d. Prove that for any integer $x > 2$, there exists an ordered pair (n,k) for which $I(n,k) > x$. [3 pts]

9. Let S be the set of values of $I(n,k)$ for all n,k for which $k < \frac{n}{2}$ and $I(n,k)$ is defined. Find the least upper bound of S. Prove that your answer is correct. [4 pts]

10. Find two distinct non-integral positive rational numbers that are not the inefficiency of any ordered pair. That is, find positive rational numbers q_1 and q_2 with $q_1 \neq q_2$ such that neither q_1 nor q_2 is an integer and such that neither q_1 nor q_2 is $I(n,k)$ for any integers n and k. Prove that your answers are correct. [4 pts]

Solutions to 2019 Power Question

First, notice that a light switch is on if it has been flipped an odd number of times, and off if it has been flipped an even number of times.

The notation $\{a_1, a_2, \ldots, a_k\}$ will be used to denote the set of k switches flipped in any given minute.

1. a. $E(6,1) = 6$. Note that at least six minutes are required because exactly one switch is flipped each minute. By flipping all six switches (in any order) in the first six minutes, the door will open in six minutes.

   b. $E(6,2) = 3$. The sequence $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ will allow Elizabeth to escape the room in three minutes. It is not possible to escape the room in fewer than three minutes because every switch must be flipped, and that requires at least $\frac{6}{2} = 3$ minutes.

   c. $E(7,3) = 3$. First, note that $E(7,3) \geq 3$, because after only two minutes, it is impossible to flip each switch at least once. It is possible to escape in three minutes with the sequence $\{1,2,3\}$, $\{1,4,5\}$, and $\{1,6,7\}$.

   d. $E(9,5) = 3$. Notice that $E(9,5) \neq 1$ because each switch must be flipped at least once, and only five switches can be flipped in one minute. Also notice that $E(9,5) \neq 2$ because after two minutes, there have been 10 flips, but in order to escape the room, each switch must be flipped at least once, and this requires 9 of the 10 flips. However, the tenth flip of a switch returns one of the nine switches to the off position, so it is not possible for Elizabeth to escape in two minutes. In three minutes, however, Elizabeth can escape with the sequence $\{1,2,3,4,5\}$, $\{1,2,3,6,7\}$, $\{1,2,3,8,9\}$.

2. a. If n is even, then $E(n,2) = \frac{n}{2}$. This is the minimum number of minutes required to flip each switch at least once, and Elizabeth can clearly escape in $\frac{n}{2}$ minutes by flipping each switch *exactly* once.

   b. If $n = 3a+2$ ($a \geq 1$), then $E(n,3) = a+2$. The minimum number of minutes required to flip each switch once is $a+1$, but as in Problem 1d, this leaves exactly one "extra flip", so some switch must be flipped exactly twice. However, in $a + 2 = \frac{n+4}{3}$ minutes, Elizabeth can escape by starting with the sequence $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, and flipping each remaining switch exactly once.

   c. If $n \geq 5$, then $E(n, n-2) = 3$. Note that Elizabeth cannot flip every switch in one minute, and after two minutes, some switch (in fact, many switches) must be flipped exactly twice. However, Elizabeth can escape in three minutes using the sequence $\{1,4,5,\ldots,n\}$, $\{2,4,5,\ldots,n\}$, $\{3,4,5,\ldots,n\}$.

3. The answer is that k can be any even number between 2 and 2018 inclusive. If Elizabeth escapes, each switch must have been flipped an odd number of times. Because there are 2019 switches, the total number of flips must be odd. However, if an even number of flips are performed each minute, then the total number of flips cannot be odd. Therefore the puzzle is impossible, and $E(2019, k) = \infty$.

Alternate Solution: As in Problem 2a, consider the number of lights that are on after each minute. Flipping an even number of switches either leaves this number unchanged, or increases or decreases it by an even number. Because the puzzle starts with 0 lights on, and 2019 is odd, it is impossible to have 2019 lights on at the end of any minute. Thus $E(2019, k) = \infty$.

4. a. First, because $k < n$, not all switches can be flipped in one minute, and so $E(n, k) \neq 1$. Second, because $k > \frac{n}{2}$, some switches must be flipped twice in the first two minutes, and so $E(n, k) \neq 2$.

   Here is a strategy by which Elizabeth can escape in three minutes. Note that because $n+k$ is even, it follows that $3k-n = (k+n)+2(k-n)$ is also even. Then let $3k-n = 2b$. Note that $b > 0$ (because $k > \frac{n}{2}$), and $b < k$ (because $k < n$).

   Elizabeth's strategy is to flip switches 1 through b three times each, and the remaining $n-b$ switches once each. This is possible because $n-b = (3k-2b)-b = 3(k-b)$ is divisible by 3, and $b + \frac{n-b}{3} = b + (k-b) = k$. Therefore, each minute, Elizabeth can flip switches 1 through b, and one-third of the $3(k-b)$ switches from $b+1$ through n. This allows her to escape in three minutes, as desired.

   b. Because n is even, and because each switch must be flipped an odd number of times in order to escape, the total number of flips is even. Because k must be odd, $E(n, k)$ must be even. To show this, consider the case where $E(n, k)$ is odd. If $E(n, k)$ is odd, then an odd number of flips happen an odd number of times, resulting in an odd number of total flips. This is a contradiction because n is even.

   Call a switch "non-flipped" in any given minute if it is not among the switches flipped in that minute. Because $E(n, k)$ (i.e., the total number of minutes) is even, and each switch is flipped an odd number of times, each switch must also be non-flipped an odd number of times. Therefore any sequence of flips that solves the "(n, k) puzzle" can be made into a sequence of flips that solves the "$(n, n-k)$" puzzle by interchanging flips and non-flips. These sequences last for the same number of minutes, and therefore $E(n, k) = E(n, n-k)$.

5. a. $E(2020, 1993) = 76$. By the result of Problem 4, conclude that $E(2020, 1993) = E(2020, 27)$. Compute the latter instead. Because $\frac{2020}{27} > 74$, it will require at least 75 minutes to flip each switch once. Furthermore, $E(2020, 27) \geq 76$ because the solution to Problem 4 implies that $E(2020, 27)$ is even.

   To solve the puzzle in exactly 76 minutes, use the following strategy. For the first 33 minutes, flip switch 1, along with the first 26 switches that have not yet been flipped. The end result is that lights 1 through $26 \cdot 33 + 1 = 859$ are on, and the remaining 1161 lights are off. Note that $1161 = 27 \cdot 43$, so it takes 43 minutes to flip each remaining switch exactly once, for a total of 76 minutes, as desired.

   b. $E(2001, 501) = 5$. First, note that three minutes is not enough time to flip each switch once. In four minutes, Elizabeth can flip each switch once, but has three flips left over. Because there are an odd number of leftover flips to distribute among the 2001 switches, some switch must get an odd number of leftover flips, and thus an even number of total flips. Thus $E(2001, 501) > 4$.

To solve the puzzle in five minutes, Elizabeth can flip the following sets of switches:

- in the first minute, $\{1, 2, 3, \ldots, 501\}$;
- in the second minute, $\{1, 2, 3, \ldots, 102\}$ and $\{502, 503, 504, \ldots, 900\}$;
- in the third minute, $\{1, 2, 3, \ldots, 102\}$ and $\{901, 902, 903, \ldots, 1299\}$;
- in the fourth minute, $\{1, 2, 3, \ldots, 100\}$ and $\{1300, 1301, 1302, \ldots, 1700\}$;
- in the fifth minute, $\{1, 2, 3, \ldots, 100\}$ and $\{1701, 1702, 1703, \ldots, 2001\}$.

This results in switches $1, 2, 3, \ldots, 100$ being flipped five times, switches 101 and 102 being flipped three times, and the remaining switches being flipped exactly once, so that all the lights are on at the end of the fifth minute.

6. a. First, if k divides n, then $E(n, k) = \frac{n}{k} = \lceil \frac{n}{k} \rceil$. Assume then that k does not divide n. Then let $r = \lceil \frac{n}{k} \rceil$, which implies $(r-1)k < n < rk$.

Because $(r-1)k < n$, it follows that $r-1$ minutes are not enough to flip each switch once, so $E(n, k) \geq r$.

Because n and k are even, it follows that rk and $rk - n$ are also even. Then $rk - n = 2b$ for some integer $b \geq 1$, and note that $b < k$ because $rk - k < n$. Then the following strategy turns all of the lights on in r minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k - b$ switches that have not yet been flipped. This leaves $b + 3(k - b)$ lights on, and the rest off. Note that $b + 3(k - b) = 3k - 2b = 3k - (rk - n) = n - k(r - 3)$.
- In each of the remaining $r - 3$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r - 3)$ lights, and Elizabeth escapes the room after r minutes.

b. First, if k divides n, then $E(n, k) = \frac{n}{k} = \lceil \frac{n}{k} \rceil$. Assume then that k does not divide n. Then let $r = \lceil \frac{n}{k} \rceil$, which implies $(r-1)k < n < rk$.

Because $(r-1)k < n$, it follows that $r-1$ minutes are not enough to flip each switch once, so $E(n, k) \geq r$.

Exactly one of $rk - n$ and $(r+1)k - n$ is even. There are two cases.

Case 1: Suppose $rk - n = 2b$ for some integer $b \geq 1$. As in the solution to Problem 6a, $b < k$. Then the following strategy turns all of the lights on in r minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k-b$ switches that have not yet been flipped. This leaves $b+3(k-b)$ lights on, and the rest off. Note that $b+3(k-b) = 3k-2b = 3k-(rk-n) = n-k(r-3)$.
- In each of the remaining $r-3$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r-3)$ lights, and Elizabeth escapes the room after r minutes.

Case 2: Suppose $(r+1)k-n = 2b$ for some integer $b \geq 1$. As in the solution to Problem 6a, $b < k$. Then the following strategy turns all of the lights on in $r+1$ minutes.

- In each of the first three minutes, flip switches 1 through b, along with the next $k-b$ switches that have not yet been flipped. This leaves $b+3(k-b)$ lights on, and the rest off. Note that $b+3(k-b) = 3k-2b = 3k-(rk+k-n) = n-k(r-2)$.
- In each of the remaining $r-2$ minutes, flip the first k switches that have never been flipped before.

This strategy turns on the remaining $k(r-2)$ lights, and Elizabeth escapes the room after $r+1$ minutes.

7. Consider the parity of n and of k. If n is odd and k is even, then as shown in Problem 3, $E(n,k) = \infty \neq 3$. If n is even and k is odd, then as noted in the solution to Problem 5, $E(n,k)$ is even, so $E(n,k) \neq 3$. If n and k are either both even or both odd and $\frac{n}{3} \leq k < n$, then the argument in Problem 4 shows that $E(n,k) = 3$.

   If $k < \frac{n}{3}$, then three minutes is not enough time to flip each switch at least once, so $E(n,k) > 3$.

   If $k = n$, then $E(n,k) = 1$.

   Therefore $E(n,k) = 3$ if and only if $n+k$ is even (that is, n and k have the same parity) and $\frac{n}{3} \leq k < n$.

8. a. $I(6,3) = 0$. By definition, $I(6,3) = E(6,3) - \frac{6}{3}$. Because $3 \mid 6$, $E(6,3) = \frac{6}{3} = 2$, and so $I(6,3) = 2-2 = 0$.

   b. $I(5,3) = \frac{4}{3}$. By definition, $I(5,3) = E(5,3) - \frac{5}{3}$. By Problem 2b, $E(5,3) = E(3\cdot 1+2, 3) = 1+2 = 3$, and so $I(5,3) = 3 - \frac{5}{3} = \frac{4}{3}$.

   c. One such pair is $(n,k) = (18,16)$. If $I(n,k) = \frac{15}{8}$, then $E(n,k) - \frac{n}{k} = \frac{15}{8}$. Note that $E(n,k) > 2$ because $k < n$.
   Suppose $E(n,k) = 3$. Then $\frac{n}{k} = \frac{9}{8}$. From Problem 7, $E(n,k) = 3$ if and only if n and k have the same parity and $\frac{n}{3} \leq k < n$, so let $n = 18$ and $k = 16$. Then, $I(18,16) = E(18,16) - \frac{18}{16} = 3 - \frac{9}{8} = \frac{15}{8}$, as desired.

   d. Let $n = 2x$ and $k = 2x-1$ for some positive integer x. Then n is even and k is odd, so Problem 4 applies, and $E(n,k) = E(n, n-k) = E(n,1) = 2x$. Therefore

$I(n,k) = 2x - \frac{2x}{2x-1}$. Because $x > 2$, it follows that $\frac{2x}{2x-1} < 2 < x$, so $I(n,k) > x$, as desired.

9. The least upper bound of S is 2. First, note that Problems 6a and 6b together show that if $k < \frac{n}{2}$, then $E(n,k)$ is either $\lceil \frac{n}{k} \rceil$ or $\lceil \frac{n}{k} \rceil + 1$ (or ∞ in the case where k is even and n is odd). Therefore $I(n,k) \leq 2$ because $E(n,k)$ is at most $\lceil \frac{n}{k} \rceil + 1$ and $\lceil \frac{n}{k} \rceil + 1 - \frac{n}{k} \leq 2$.

   Now it will be shown that $I(n,k)$ can be arbitrarily close to 2. To do this, use values of n and k that are both odd, such that $\lceil \frac{n}{k} \rceil$ is even. Specifically, let $n = 3k+2$ for k odd; then $\lceil \frac{n}{k} \rceil = 4$. Therefore $E(n,k) = 5$ by Problem 6b, and

$$I(n,k) = 5 - \frac{3k+2}{k} = 2 - \frac{2}{k}.$$

   Letting k be a large odd integer, this gives an ordered pair (n,k) with $I(n,k)$ arbitrarily close to 2, as desired.

   Combining these two claims shows that the least upper bound of S is 2.

10. It should be noted that integer answers are not possible because no positive integer can be the inefficiency of an ordered pair (n,k). This is because if $I(n,k) = E(n,k) - \frac{n}{k}$ is an integer, then $\frac{n}{k}$ is an integer so $E(n,k) = \frac{n}{k}$ and $I(n,k) = 0$.

    To find non-integral rational numbers that cannot be the inefficiency of any ordered pair, start with a simple lemma.

    Lemma: Given positive integers n, k, and r, $E(rn, rk) \leq E(n,k)$.

    Proof: If n is odd and k is even, then $E(n,k) = \infty$ and the lemma is trivially true. Otherwise there is a sequence of $E(n,k)$ sets of switches to flip which will result in escape. Partition the rn switches into r groups of n switches, and apply the strategy within each of the r groups, which uses rk flips per minute. This will achieve escape in $E(n,k)$ minutes, so $E(rn,rk)$ is at most $E(n,k)$. □

    The values $E(n,k)$ and $E(rk,nk)$ can certainly equal one other, as when n is a multiple of k. The inequality can also be strict. For example, $E(4,3) = E(4,1) = 4$ but $E(8,6) = 3$ are obtained from values from earlier work.

    One implication of the above lemma is that $I(n,k) \geq I(rn,rk)$.

    Now all rational numbers q with $0 < q < 1$ are the inefficiency of some ordered pair (n,k). To show this, let $r = 3 - q = \frac{a}{b}$ in lowest terms. Then $r > 2$ so $a > 2b$, and by Problem 6a, $E(2a,2b) = \lceil \frac{a}{b} \rceil = 3$ and $I(2a,2b) = 3 - \frac{a}{b} = q$.

    Next, all rational numbers q with $1 < q < 2$ are the inefficiency of some ordered pair (n,k). Again, let $3 - q = \frac{a}{b}$ in lowest terms. The fact that $1 < q < 2$ implies that $1 < \frac{a}{b} < 2$. So by Problem 4a, $E(2a, 2b) = 3$ and $I(2a, 2b) = 3 - \frac{a}{b} = q$.

If $\frac{n}{k} = 2$, then $I(n, k) = 0$ as noted above, while if $\frac{n}{k} > 2$, Problem 6b concludes that $E(n, k) = \lceil \frac{n}{k} \rceil$ or $E(n, k) = \lceil \frac{n}{k} \rceil + 1$. In either case, $I(n, k) \leq \lceil \frac{n}{k} \rceil + 1 - \frac{n}{k} < 2$. As all nonnegative rational numbers less than 2 (except the excluded integer 1) have already been shown to be inefficiencies of ordered pairs, the cases $\frac{n}{k} \geq 2$ can now be excluded from consideration.

Consider ordered pairs (n, k) with $1 < \frac{n}{k} < 2$. If $n + k$ is even, Problem 4a implies that $E(n, k) = 3$ so $I(n, k) = 3 - \frac{n}{k} < 2$ and once again, all such numbers are already known to be inefficiencies. Of course if n is odd and k is even, then $I(n, k)$ is undefined.

Combining these results shows that a rational number will not be the inefficiency of some ordered pair simply whenever it is greater than 2 and avoids being the inefficiency of any ordered pair (n, k) where $1 < \frac{n}{k} < 2$, n is even, and k is odd.

Let $\frac{a}{b} > 2$ be a non-integral rational number in lowest terms. If $\frac{a}{b}$ is the inefficiency of some ordered pair (n, k), then $\frac{a}{b} = E(n, k) - \frac{n}{k}$. Because $E(n, k)$ is an integer, it follows that $\frac{a}{b} + \frac{n}{k}$ is also an integer. Because $1 < \frac{n}{k} < 2$, there is exactly one integer that it can be.

To find non-integral rational numbers that are not the inefficiency of any ordered pair (n, k), choose a rational number $\frac{a}{b}$ in such a way that the only possible choice for $\frac{n}{k}$, in lowest terms, leads to $I(n, k) < \frac{a}{b}$. Then because $I(rn, rk) \leq I(n, k)$, no choice of n and k can possibly lead to $I(n, k) = \frac{a}{b}$.

One such choice is $\frac{a}{b} = \frac{11}{3}$. The only integer between $\frac{11}{3} + 1$ and $\frac{11}{3} + 2$ is $\frac{11}{3} + \frac{4}{3} = 5$. Choosing $n = 4$ and $k = 3$ satisfies the conditions that n is even, k is odd, and $1 < \frac{n}{k} < 2$, and also leads to $I(4, 3) = E(4, 3) - \frac{4}{3} = E(4, 1) - \frac{4}{3} = 4 - \frac{4}{3} = \frac{8}{3} < \frac{11}{3}$. Thus no ordered pair (n, k) in the ratio $\frac{n}{k} = \frac{4}{3}$ can yield an inefficiency of $\frac{11}{3}$.

Another choice that will work is $\frac{a}{b} = \frac{17}{5}$. The only $\frac{n}{k}$ that could yield such an inefficiency would be $\frac{8}{5}$ (again, note that n is even, k is odd, and $1 < \frac{n}{k} < 2$). Then $I(8, 5) = E(8, 5) - \frac{8}{5} = 4 - \frac{8}{5} = \frac{12}{5} < \frac{17}{5}$, and this is a similar contradiction.

Note that there are many other non-integral rational numbers that are not the inefficiency of any ordered pair. They just aren't covered by the method outlined above and require different techniques. For instance, $\frac{5}{2}$ is such a number. For such an ordered pair (n, k) would require $\frac{n}{k}$ to be an odd multiple of $\frac{1}{2}$. Now n cannot be odd with k even; in that case, $I(n, k)$ is not defined. So both n and k are even. But then previous work shows that $I(n, k) < 2$, so $\frac{5}{2}$ is not an inefficiency of any ordered pair (n, k).

2019 Relay Problems

R1-1. Let $a = 19$, $b = 20$, and $c = 21$. Compute

$$\frac{a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}{a + b + c}.$$

R1-2. Let $T = \text{TNYWR}$. Lydia is a professional swimmer and can swim one-fifth of a lap of a pool in an impressive 20.19 seconds, and she swims at a constant rate. Rounded to the nearest integer, compute the number of <u>minutes</u> required for Lydia to swim T laps.

R1-3. Let $T = \text{TNYWR}$. In $\triangle ABC$, $\mathrm{m}\angle C = 90^\circ$ and $AC = BC = \sqrt{T-3}$. Circles O and P each have radius r and lie inside $\triangle ABC$. Circle O is tangent to $\overline{AC}$ and $\overline{BC}$. Circle P is externally tangent to circle O and to $\overline{AB}$. Given that points C, O, and P are collinear, compute r.

R2-1. Given that $p = 6.6 \times 10^{-27}$, then $\sqrt{p} = a \times 10^b$, where $1 \leq a < 10$ and b is an integer. Compute $10a + b$ rounded to the nearest integer.

R2-2. Let $T = \text{TNYWR}$. A group of children and adults go to a rodeo. A child's admission ticket costs \$5, and an adult's admission ticket costs more than \$5. The total admission cost for the group is $\$10 \cdot T$. If the number of adults in the group were to increase by 20%, then the total cost would increase by 10%. Compute the number of children in the group.

R2-3. Let $T = \text{TNYWR}$. Rectangles $FAKE$ and $FUNK$ lie in the same plane. Given that $EF = T$, $AF = \frac{4T}{3}$, and $UF = \frac{12}{5}$, compute the area of the intersection of the two rectangles.

2019 Relay Answers

R1-1. 60

R1-2. 101

R1-3. $3 - \sqrt{2}$

R2-1. 67

R2-2. 67

R2-3. 262

2019 Relay Solutions

R1-1. Note that the numerator of the given expression factors as $(a+b+c)^2$, hence the expression to be computed equals $a+b+c = 19+20+21 = \mathbf{60}$.

R1-2. Lydia swims a lap in $5 \cdot 20.19 = 100.95$ seconds. The number of minutes required for Lydia to swim T laps is therefore $100.95 \cdot T/60$. With $T = 60$, the desired number of minutes, rounded to the nearest integer, is **101**.

R1-3. Let A' and B' be the respective feet of the perpendiculars from O to $\overline{AC}$ and $\overline{BC}$. Let H be the foot of the altitude from C to $\overline{AB}$. Because $\triangle ABC$ is isosceles, it follows that $A'OB'C$ is a square, $\mathrm{m}\angle B'CO = 45^\circ$, and $\mathrm{m}\angle BCH = 45^\circ$. Hence H lies on the same line as C, O, and P. In terms of r, the length CH is $CO+OP+PH = r\sqrt{2}+2r+r = (3+\sqrt{2})r$. Because $AC = BC = \sqrt{T-3}$, it follows that $CH = \dfrac{\sqrt{T-3}}{\sqrt{2}}$. Thus $r = \dfrac{\sqrt{T-3}}{\sqrt{2}(3+\sqrt{2})} = \dfrac{(3\sqrt{2}-2)\sqrt{T-3}}{14}$. With $T = 101$, $\sqrt{T-3} = \sqrt{98} = 7\sqrt{2}$, and it follows that $r = \mathbf{3-\sqrt{2}}$.

R2-1. Note that $p = 6.6 \times 10^{-27} = 66 \times 10^{-28}$, so $a = \sqrt{66}$ and $b = -14$. Note that $\sqrt{66} > \sqrt{64} = 8$. Because $8.1^2 = 65.61$ and $8.15^2 = 66.4225 > 66$, conclude that $81 < 10\sqrt{66} < 81.5$, hence $10a$ rounded to the nearest integer is 81, and the answer is $81 - 14 = \mathbf{67}$.

R2-2. Suppose there are x children and y adults in the group and each adult's admission ticket costs $\$a$. The given information implies that $5x + ay = 10T$ and $5x + 1.2ay = 11T$. Subtracting the first equation from the second yields $0.2ay = T \to ay = 5T$, so from the first equation, $5x = 5T \to x = T$. With $T = 67$, the answer is **67**.

R2-3. Without loss of generality, let A, U, and N lie on the same side of $\overline{FK}$. Applying the Pythagorean Theorem to triangle AFK, conclude that $FK = \frac{5T}{3}$. Comparing the altitude to $\overline{FK}$ in triangle AFK to $\overline{UF}$, note that the intersection of the two rectangles will be a triangle with area $\frac{2T^2}{3}$ if $\frac{4T}{5} \leq \frac{12}{5}$, or $T \leq 3$. Otherwise, the intersection will be a trapezoid. In this case, using similarity, the triangular regions of $FUNK$ that lie outside of $FAKE$ each have one leg of length $\frac{12}{5}$ and the others of lengths $\frac{16}{5}$ and $\frac{9}{5}$, respectively. Thus their combined areas $\frac{1}{2} \cdot \frac{12}{5}\left(\frac{16}{5} + \frac{9}{5}\right) = 6$, hence the area of the intersection is $\frac{5T}{3} \cdot \frac{12}{5} - 6 = 4T - 6$. With $T = 67$, the answer is therefore **262**.

2019 Tiebreaker Problems

TB-1. Regular tetrahedra $JANE$, $JOHN$, and $JOAN$ have non-overlapping interiors. Compute $\tan \angle HAE$.

TB-2. Each positive integer less than or equal to 2019 is written on a blank sheet of paper, and each of the digits 0 and 5 is erased. Compute the remainder when the product of the remaining digits on the sheet of paper is divided by 1000.

TB-3. Compute the third least positive integer n such that each of n, $n + 1$, and $n + 2$ is a product of exactly two (not necessarily distinct) primes.

2019 Tiebreaker Answers

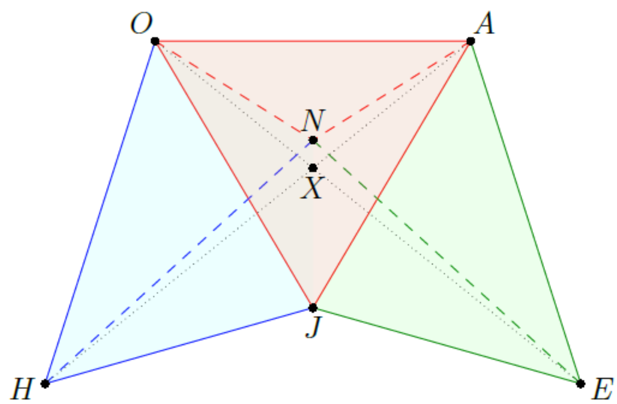
TB-1. $\dfrac{5\sqrt{2}}{2}$

TB-2. 976

TB-3. 93

2019 Tiebreaker Solutions

TB-1. First note that $\overline{JN}$ is a shared edge of all three pyramids, and that the viewpoint for the figure below is from along the line that is the extension of edge $\overline{JN}$.



Let h denote the height of each pyramid. Let X be the center of pyramid $JOAN$, and consider the plane passing through H, A, and E. By symmetry, the altitude in pyramid $JOHN$ through H and the altitude in pyramid $JANE$ through E pass through X. Thus points H, X, and A are collinear, as are points E, X, and O. Hence $AH = OE = 2h$. Using the result that the four medians in a tetrahedron are concurrent and divide each other in a $3:1$ ratio, it follows that $AX = OX = \frac{3h}{4}$ and $XE = OE - OX = \frac{5h}{4}$. Applying the Law of Cosines to triangle AXE yields $\cos\angle XAE = \cos\angle HAE = \frac{2-2h^2}{3h}$. Suppose, without loss of generality, that the common side length of the pyramids is 1. Then $h = \sqrt{\frac{2}{3}}$ and $\cos\angle HAE = \frac{\sqrt{6}}{9}$. Hence $\sin\angle HAE = \frac{\sqrt{75}}{9}$ and therefore $\tan\angle HAE = \boldsymbol{\dfrac{5\sqrt{2}}{2}}$.

TB-2. Count the digits separately by position, noting that 1 is irrelevant to the product. There are a total of 20 instances of the digit 2 in the thousands place. The digit 0 only occurs in the hundreds place if the thousands digit is 2, so look at the numbers 1 through 1999. Each non-zero digit contributes an equal number of times, so there are 200 each of 1, 2, 3, 4, 6, 7, 8, 9. The same applies to the tens digit, except there can be the stray digit of 1 among the numbers 2010 through 2019, but again, these do not affect the product. In the units place, there are 202 of each of the digits. Altogether, there are 602 each of 2, 3, 4, 6, 7, 8, 9, along with 20 extra instances of the digit 2. Note that $9 \cdot 8 \cdot 7 \cdot 6 \cdot 4 \cdot 3 \cdot 2 = 3024 \cdot 24 = 72{,}576$ leaves a remainder of 576 when divided by 1000. Also $2^{20} = 1024^2 \equiv 24^2 \pmod{1000}$, so 2^{20} contributes another factor of 576. The answer is therefore the remainder when 576^{603} is divided by 1000. This computation can be simplified by using the Chinese Remainder Theorem with moduli 8 and 125, whose product is 1000. Note $576^{603} \equiv 0 \pmod{8}$ because 576 is divisible by 8. Also $576 \equiv 76 \pmod{125}$. By Euler's totient theorem, $576^{100} \equiv 1 \pmod{125}$, so $576^{603} \equiv 76^3 \pmod{125}$. This can quickly be computed by noting that $76^3 = (75+1)^3 = 75^3 + 3 \cdot 75^2 + 3 \cdot 75 + 1 \equiv 3 \cdot 75 + 1 \equiv -24 \pmod{125}$. Observing that $-24 \equiv 0 \pmod{8}$, it follows that $576^{603} \equiv -24 \pmod{1000}$, hence the desired remainder is **976**.

TB-3. Define a positive integer n to be a *semiprime* if it is a product of exactly two (not necessarily distinct) primes. Define a *lucky trio* to be a sequence of three consecutive integers, $n, n+1, n+2$, each of which is a semiprime. Note that a lucky trio must contain exactly one multiple of 3. Also note that the middle number in a lucky trio must be even. To see this, note that if the first and last numbers in a lucky trio were both even, then exactly one of these numbers would be a multiple of 4. But neither $2, 3, 4$ nor $4, 5, 6$ is a lucky trio, and if a list of three consecutive integers contains a multiple of 4 that is greater than 4, this number cannot be a semiprime. Using this conclusion and because $3, 4, 5$ is not a lucky trio, it follows that the middle number of a lucky trio cannot be a multiple of 4. Hence it is necessary that a lucky trio has the form $4k+1, 4k+2, 4k+3$, for some positive integer k, with $2k+1$ being a prime. Note that $k \not\equiv 1 \pmod{3}$ because when $k = 1$, the sequence $5, 6, 7$ is not a lucky trio, and when $k > 1$, $4k+2$ would be a multiple of 6 greater than 6, hence it cannot be a semiprime. Trying $k = 2, 3, 5, 6, 8, 9, \ldots$ allows one to eliminate sequences of three consecutive integers that are not lucky trios, and if lucky trios are ordered by their least elements, one finds that the first three lucky trios are $33, 34, 35$; $85, 86, 87$; and $93, 94, 95$. Hence the answer is **93**.

2019 Super Relay Problems

1. Given that a, b, c, and d are integers such that $a + bc = 20$ and $-a + cd = 19$, compute the greatest possible value of c.

2. Let $T = \text{TNYWR}$. Emile randomly chooses a set of T cards from a standard deck of 52 cards. Given that Emile's set contains no clubs, compute the probability that his set contains three aces.

3. Let $T = \text{TNYWR}$. In parallelogram $ABCD$, $\dfrac{AB}{BC} = T$. Given that M is the midpoint of $\overline{AB}$ and P and Q are the trisection points of $\overline{CD}$, compute $\dfrac{[ABCD]}{[MPQ]}$.

4. Let $T = \text{TNYWR}$. Compute the value of x such that $\log_T \sqrt{x-7} + \log_{T^2}(x-2) = 1$.

5. Let $T = \text{TNYWR}$. Let p be an odd prime and let x, y, and z be positive integers less than p. When the trinomial $(px + y + z)^{T-1}$ is expanded and simplified, there are N terms, of which M are <u>always</u> multiples of p. Compute M.

6. Let $T = \text{TNYWR}$. Compute the value of K such that 20, $T-5$, K is an increasing geometric sequence and 19, K, $4T+11$ is an increasing arithmetic sequence.

7. Let $T = \text{TNYWR}$. Cube $\mathcal{C}_1$ has volume T and sphere $\mathcal{S}_1$ is circumscribed about $\mathcal{C}_1$. For $n \geq 1$, the sphere $\mathcal{S}_n$ is circumscribed about the cube $\mathcal{C}_n$ and is inscribed in the cube $\mathcal{C}_{n+1}$. Let k be the least integer such that the volume of $\mathcal{C}_k$ is at least 2019. Compute the edge length of $\mathcal{C}_k$.

8. Let S be the number you will receive from position 7 and let M be the number you will receive from position 9. Sam and Marty each ride a bicycle at a constant speed. Sam's speed is S km/hr and Marty's speed is M km/hr. Given that Sam and Marty are initially 100 km apart and they begin riding towards one another at the same time, along a straight path, compute the number of kilometers that Sam will have traveled when Sam and Marty meet.

9. Let $T = \text{TNYWR}$. At the *Westward House of Supper* ("WHS"), a dinner special consists of an appetizer, an entrée, and dessert. There are 7 different appetizers and K different entrées that a guest could order. There are 2 dessert choices, but ordering dessert is optional. Given that there are T possible different orders that could be placed at the WHS, compute K.

10. Let $T = \text{TNYWR}$. The product of all positive divisors of 2^T can be written in the form 2^K. Compute K.

11. Let $T = \text{TNYWR}$. Given that x, y, and z are real numbers such that $x + y = 5$, $x^2 - y^2 = \frac{1}{T}$, and $x - z = -7$, compute $x + z$.

12. Let $T = \text{TNYWR}$. In $\triangle LEO$, $\sin \angle LEO = \frac{1}{T}$. If $LE = \frac{1}{n}$ for some positive real number n, then $EO = n^3 - 4n^2 + 5n$. As n ranges over the positive reals, compute the least possible value of $[LEO]$.

13. Let $T = \text{TNYWR}$ and let $K = \sqrt{T - 1}$. Compute $\left|(K - 20)(K + 1) + 19K - K^2\right|$.

14. Let $T = \text{TNYWR}$. Let a, b, and c be the three solutions of the equation $x^3 - 20x^2 + 19x + T = 0$. Compute $a^2 + b^2 + c^2$.

15. Square $KENT$ has side length 20. Point M lies in the interior of $KENT$ so that $\triangle MEN$ is equilateral. Given that $KM^2 = a - b\sqrt{3}$, where a and b are integers, compute b.

2019 Super Relay Answers

1. 39
2. 1
3. 6
4. 11
5. 55
6. 125
7. 15
8. 60
9. 10
10. 210
11. 20
12. $\dfrac{1}{40}$
13. 20
14. 362
15. 400

2019 Super Relay Solutions

1. Adding the two given equations yields $bc + cd = c(b + d) = 39$. The greatest possible value of c therefore occurs when $c = \mathbf{39}$ and $b + d = 1$.

2. Knowing that 13 of the cards are *not* in Emile's set, there are $\binom{39}{T}$ ways for him to have chosen a set of T cards. Given that Emile's set contains no clubs, the suits of the three aces are fixed (i.e., diamonds, hearts, and spades). The number of possible sets of cards in which these three aces appear is therefore $\binom{36}{T-3}$. The desired probability is therefore $\binom{36}{T-3}/\binom{39}{T}$. With $T = 39$, this probability is $1/1 = \mathbf{1}$, which is consistent with the fact that Emile's set contains all cards in the deck that are *not* clubs, hence he is guaranteed to have all three of the remaining aces.

3. Let $CD = 3x$ and let h be the length of the altitude between bases $\overline{AB}$ and $\overline{CD}$. Then $[ABCD] = 3xh$ and $[MPQ] = \frac{1}{2}xh$. Hence $\dfrac{[ABCD]}{[MPQ]} = \mathbf{6}$. Both the position of M and the ratio $\dfrac{AB}{BC} = T$ are irrelevant.

4. It can readily be shown that $\log_a b = \log_{a^2} b^2$. Thus it follows that $\log_T \sqrt{x-7} = \log_{T^2}(x-7)$. Hence the left-hand side of the given equation is $\log_{T^2}(x-7)(x-2)$ and the equation is equivalent to $(x-7)(x-2) = T^2$, which is equivalent to $x^2 - 9x + 14 - T^2 = 0$. With $T = 6$, this equation is $x^2 - 9x - 22 = 0 \Longrightarrow (x-11)(x+2) = 0$. Plugging $x = -2$ into the given equation leads to the first term of the left-hand side having a negative radicand and the second term having an argument of 0. However, one can easily check that $x = \mathbf{11}$ indeed satisfies the given equation.

5. A general term in the expansion of $(px + y + z)^{T-1}$ has the form $K(px)^a y^b z^c$, where a, b, and c are nonnegative integers such that $a + b + c = T - 1$. Using the "sticks and stones" approach, the number of nonnegative integral solutions to $a + b + c = T - 1$ is the number of arrangements of $T - 1$ stones and 2 sticks in a row (the sticks act as separators and the "2" arises because it is one less than the number of variables in the equation). Thus there are $\binom{T+1}{2}$ solutions. Each term will be a multiple of p <u>unless</u> $a = 0$. In this case, the number of terms that are *not* multiples of p is the number of nonnegative integral solutions to the equation $b + c = T - 1$, which is T (b can range from 0 to $T - 1$ inclusive, and then c is fixed). Hence $M = \binom{T+1}{2} - T = \frac{T^2-T}{2}$. With $T = 11$, the answer is $\mathbf{55}$.

6. The condition that $20, T-5, K$ is an increasing geometric sequence implies that $\frac{T-5}{20} = \frac{K}{T-5}$, hence $K = \frac{(T-5)^2}{20}$. The condition that $19, K, 4T+11$ is an increasing arithmetic sequence implies that $K - 19 = 4T + 11 - K$, hence $K = 2T + 15$. With $T = 55$, each of these equations implies that $K = \mathbf{125}$. Note that the two equations can be combined and solved without

being passed a value of T. A quadratic equation results, and its roots are $T = 55$ or $T = -5$. However, with $T = -5$, neither of the given sequences is increasing.

7. In general, let cube $\mathcal{C}_n$ have edge length x. Then the diameter of sphere $\mathcal{S}_n$ is the space diagonal of $\mathcal{C}_n$, which has length $x\sqrt{3}$. This in turn is the edge length of cube $\mathcal{C}_{n+1}$. Hence the edge lengths of $\mathcal{C}_1, \mathcal{C}_2, \ldots$ form an increasing geometric sequence with common ratio $\sqrt{3}$ and volumes of $\mathcal{C}_1, \mathcal{C}_2, \ldots$ form an increasing geometric sequence with common ratio $3\sqrt{3}$. With $T = 125$, the edge length of $\mathcal{C}_1$ is 5, so the sequence of edge lengths of the cubes is $5, 5\sqrt{3}, 15, \ldots$, and the respective sequence of the volumes of the cubes is $125, 375\sqrt{3}, 3375, \ldots$. Hence $k = 3$, and the edge length of $\mathcal{C}_3$ is **15**.

8. In km/hr, the combined speed of Sam and Marty is $S + M$. Thus one can determine the total time they traveled and use this to determine the number of kilometers that Sam traveled. However, this is not needed, and there is a simpler approach. Suppose that Marty traveled a distance of d. Then because Sam's speed is $\frac{S}{M}$ of Marty's speed, Sam will have traveled a distance of $\frac{S}{M} \cdot d$. Thus, together, they traveled $d + \frac{S}{M} \cdot d$. Setting this equal to 100 and solving yields $d = \frac{100M}{M+S}$. Thus Sam traveled $\frac{S}{M} \cdot d = \frac{100S}{M+S}$. With $S = 15$ and $M = 10$, this is equal to **60** km.

9. Because dessert is optional, there are effectively $2 + 1 = 3$ dessert choices. Hence, by the Multiplication Principle, it follows that $T = 7 \cdot K \cdot 3$, thus $K = \frac{T}{21}$. With $T = 210$, the answer is **10**.

10. When n is a nonnegative integer, the product of the positive divisors of 2^n is $2^0 \cdot 2^1 \cdot \cdots \cdot 2^{n-1} \cdot 2^n = 2^{0+1+\cdots+(n-1)+n} = 2^{n(n+1)/2}$. Because $T = 20$ is an integer, it follows that $K = \frac{T(T+1)}{2} = \mathbf{210}$.

11. Note that $x^2 - y^2 = (x+y)(x-y) = 5(x-y)$, hence $x - y = \frac{1}{5T}$. Then $x + z = (x+y) + (x-y) + (z-x) = 5 + \frac{1}{5T} + 7 = 12 + \frac{1}{5T}$. With $T = \frac{1}{40}$, the answer is thus $12 + 8 = \mathbf{20}$.

12. Note that $[LEO] = \frac{1}{2}(\sin \angle LEO) \cdot LE \cdot EO = \frac{1}{2} \cdot \frac{1}{T} \cdot \frac{1}{n} \cdot (n^3 - 4n^2 + 5n) = \frac{n^2-4n+5}{2T}$. Because T is a constant, the least possible value of $[LEO]$ is achieved when the function $f(n) = n^2 - 4n + 5$ is minimized. This occurs when $n = -(-4)/(2 \cdot 1) = 2$, and the minimum value is $f(2) = 1$. Hence the desired least possible value of $[LEO]$ is $\frac{1}{2T}$, and with $T = 20$, this is $\mathbf{\frac{1}{40}}$.

13. The expression inside the absolute value bars simplifies to $K^2 - 19K - 20 + 19K - K^2 = -20$. Hence the answer is **20** and the value of K $(= \sqrt{361} = 19)$ is not needed.

14. According to Vieta's formulas, $a + b + c = -(-20) = 20$ and $ab + bc + ca = 19$. Noting that $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca)$, it follows that $a^2 + b^2 + c^2 = 20^2 - 2 \cdot 19 = \mathbf{362}$. The value of T is irrelevant.

15. Let s be the side length of square $KENT$; then $ME = s$. Let J be the foot of the altitude from M to $\overline{KE}$. Then m$\angle JEM = 30°$ and m$\angle EMJ = 60°$. Hence $MJ = \frac{s}{2}$, $JE = \frac{s\sqrt{3}}{2}$, and $KJ = KE - JE = s - \frac{s\sqrt{3}}{2}$. Applying the Pythagorean Theorem to $\triangle KJM$ implies that $KM^2 = \left(s - \frac{s\sqrt{3}}{2}\right)^2 + \left(\frac{s}{2}\right)^2 = 2s^2 - s^2\sqrt{3}$. With $s = 20$, the value of b is therefore $s^2 = \mathbf{400}$.

Part II

ARML Local Contests

ARML Local 2015

Contents

ARML Local 2015 Team Problems

T-1. For a number x, let S_x be the set $\{5, 5, 5, 7, 8, 13, x\}$. Compute the sum of the values of x such that the mean, median, and mode of S_x can be arranged to form an *increasing* arithmetic sequence.

T-2. Compute the value of N for which $\dfrac{1}{\log_2 100} + \dfrac{1}{\log_3 100} + \dfrac{1}{\log_6 100} + \dfrac{1}{\log_9 100} = \dfrac{2}{\log_N 100}$.

T-3. A non-negative integer is called *falling* if its digits are strictly decreasing from left to right. 9876543210 is the largest falling number, 0 is the smallest. If all of the falling numbers were written in a list from smallest to largest, compute the 1000^{th} number on that list.

T-4. Compute the length of the interval of values of x such that $(x-1)^2 \leq 2|x-2|$.

T-5. Pentagon $OPQRS$ has vertices at $O(0,0)$, $P(8,0)$, $Q(8,4)$, $R(2,4)$, and $S(0,8)$. The line $y = mx$ divides $OPQRS$ into two regions of equal area. Compute m.

T-6. Compute the number of ordered triples of integers (x, y, z) such that $xy - |z| = 20$ and $|x+y| - z = 15$.

T-7. Define a positive integer N to be *lucky* if there exists a positive integer a such that a^7 has exactly N positive divisors. Compute the number of positive integers less than 2015 that are lucky.

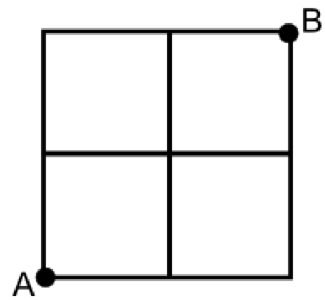
T-8. Compute the sum of all real values of x that satisfy $(x^2 - 16x + 40)^{x^2+3x-18} = 1$.

T-9. Given square $SQUA$ with $SQ = 6$. Equilateral triangles UAR, QUE, SQC, and SAT are constructed in the exterior of $SQUA$. Compute the area of $RECT$.

T-10. For each positive integer k, let S_k denote the increasing arithmetic sequence of 1000 integers, whose first term is 1, and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, \ldots, 2998$. Compute the number of sequences in $S_1, S_2, S_3, \ldots, S_{1000}$ that contain the term 2015.

T-11. If $g(x) = x(x-1)$, then compute the sum of all real solutions to $g(g(x)) = 30$.

T-12. A path along a grid's edges is called *up-right* if it consists solely of movement along segments in the upward or rightward direction. Compute the expected number of up-right paths remaining in the grid below from A to B if exactly one of the 12 segments is randomly removed from the grid.



T-13. For a positive integer n, let $p(n)$ be the product of the digits of n and $s(n)$ be the sum of the digits of n. Compute the sum of the values of n less than 100 such that $n = p(n) + s(n)$.

T-14. A twenty-sided die is rolled twice, getting numbers x and y. Compute the probability that $\log_{10} x + \log_{10} y$ is an integer.

T-15. A deck of 3 red cards, 3 white cards, and 3 blue cards is shuffled and dealt into three rows of three cards. Compute the probability that there are no red cards in the first row, no white cards in the second row, and no blue cards in the third row.

ARML Local 2015 Team Solutions

T-1. The mode of S_x is 5. The median is 5 if $x \leq 5$, x if $5 \leq x \leq 7$, and 7 if $x \geq 7$. The mean of S_x is $M = \frac{43+x}{7}$. If $x \geq 7$, then $M > 7$ so the arithmetic sequence is $5, 7, M$, which means $M = 9$, so $x = 20$. If x is between 5 and 7, then the arithmetic sequence is either $5, x, M$ or $5, M, x$. The former sequence has the solution $M = 7$ and $x = 6$, while the latter sequence is impossible for x between 5 and 7. The sum of the values of x is $\boxed{26}$.

T-2. Changing bases, the equation is equivalent to $\dfrac{\log 2}{\log 100} + \dfrac{\log 3}{\log 100} + \dfrac{\log 6}{\log 100} + \dfrac{\log 9}{\log 100} = \dfrac{2 \log N}{\log 100}$. Multiplying through by $\log 100$, the equation becomes $\log(2 \cdot 3 \cdot 6 \cdot 9) = \log N^2$, so $N = \boxed{18}$.

T-3. There are $2^{10} = 1024$ falling numbers in total. One has 10 digits, 10 have 9 digits, and $C(10, 2) = 45$ have 8 digits. Therefore the answer will be the 33rd largest among those with eight digits. It will omit the 13th smallest pair of digits, which is 5 and 2, so the answer is $\boxed{98764310}$.

T-4. If $x \geq 2$, then $(x-1)^2 \leq 2(x-2) \to x^2 - 2x + 1 \leq 2x - 4 \to x^2 - 4x + 5 \leq 0$, which is never true as $0 \leq (x-2)^2 < (x-2)^2 + 1 = x^2 - 4x + 5$. If $x < 2$, the linear function is $2(2-x) = 4 - 2x$. $x^2 - 2x + 1 = 4 - 2x \to x^2 = 3 \to x = \pm\sqrt{3}$, so the length of the interval is $\boxed{2\sqrt{3}}$.

T-5. Consider $OPQRS$; its area is the area of a rectangle and a triangle, so $[OPQRS] = 8(4) + \frac{1}{2} \cdot 2 \cdot 4 = 36$. The line $y = mx$ leaves a trapezoid to its right. The line $y = mx$ hits side $\overline{QR}$ at a point with y-coordinate 4 and thus x-coordinate $\frac{4}{m}$. Therefore $\frac{1}{2} \cdot 4 \cdot (8 - \frac{4}{m} + 8) = 18$, which can be solved to obtain $m = \boxed{\frac{4}{7}}$.

T-6. For (x, y, z) to be a solution, both x and y must have the same sign and (x, y) and $(-x, -y)$ will have identical corresponding values of z. In addition, to satisfy the first equation, $xy \geq 20$.

If $z \geq 0$, then $xy - |x + y| = 5$. If $x + y \geq 0$, then $xy - (x + y) = 5$, or $(x-1)(y-1) = 6$. Therefore $(x, y) = (7, 2)$, $(2, 7)$, $(4, 3)$, or $(3, 4)$. In each case, $xy < 20$ so the first equation cannot be satisfied.

If $x+y \leq 0$, then $xy+(x+y) = 5$, or $(x+1)(y+1) = 6$. Therefore $(x, y) = (-7, -2)$, $(-2, -7)$, $(-4, -3)$, or $(-3, -4)$. In each case, $xy < 20$ so the first equation cannot be satisfied.

If $z \leq 0$, then $xy + |x + y| = 35$. If $x + y \geq 0$, then $xy + (x + y) = 35$, or $(x+1)(y+1) = 36$. If $x + y \leq 0$, then $xy - (x + y) = 35$, or $(x-1)(y-1) = 36$.

There are five pairs of positive integers (x, y) such that $xy \geq 20$ and $(x+1)(y+1) = 36$. They are $(2, 11)$, $(3, 8)$, $(5, 5)$, $(8, 3)$, and $(11, 2)$. There are also the corresponding pairs of negative integers, so there are a total of $\boxed{10}$ solutions.

T-7. If $a = p_1^{e_1} \cdots p_n^{e_n}$, where the p_i are distinct primes, then the number of positive divisors of a is $\sigma(a) = \sigma(p_1^{e_1}) \cdots \sigma(p_n^{e_n}) = (e_1 + 1) \cdots (e_n + 1)$. Then

$$\sigma(a^7) = \sigma(p_1^{7e_1}) \cdots \sigma(p_n^{7e_n}) = (7e_1 + 1) \cdots (7e_n + 1).$$

All values of $\sigma(a^7)$ are one more than a multiple of seven, and all numbers of the form $7k+1$ for a non-negative integer k are equal to $\sigma(a^7)$ for $a = p^k$ for some prime p, so all numbers of the form $7k+1$ for $k \geq 0$ are lucky. There are $\boxed{288}$ numbers of this form less than 2015 $(287 \cdot 7 + 6 = 2015)$.

T-8. There are three scenarios, one in which a non-zero number is raised to the power of 0, one in which 1 is raised to a power, and one in which -1 is raised to a negative power. The first case requires $x^2+3x-18=0$, and this generates the values $x=3$ or $x=-6$, both of which cause the base to be non-zero. The second case requires $x^2-16x+40=1 \to x^2-16x+39=0$, and this means that $x=13$ or $x=3$. In the third case, $x^2-16x+40=-1$ at $x=8\pm\sqrt{23}$, and the exponent is not an even integer at either of these values. The solutions are $x=13$,$x=3$, and $x=-6$, which sum to $\boxed{10}$.

T-9. Note that $RECT$ is a square and its area is RE^2. Note also that RE is the long side of a triangle whose other sides are 6 and 6 and whose included angle measures $150°$. Now, by the Law of Cosines, $RE^2 = 6^2+6^2-2\cdot 6\cdot 6\cdot\left(-\frac{\sqrt{3}}{2}\right)$, or $RE^2 = 72+36\sqrt{3}$. Therefore $[RECT] = \boxed{72+36\sqrt{3}}$.

T-10. Let $s_{k,n}$ denote the n^{th} element of S_k, with $s_{k,0}=1$ for all k. $s_{k,n}=1+kn$, so in order for an element in S_k to be equal to 2015, $kn=2014$ with $k\leq 1000$ and $n<1000$. $2014 = 2\cdot 19\cdot 53$, and $\sigma(2014)=8$. However, two of the factors (1007 and 2014) are greater than 1000, so four pairs of factors of 2014 (k,n) are invalid, and the answer is $\boxed{4}$.

T-11. Let $w=g(x)$, then $g(w)=30 \to w(w-1)=30 \to w^2-w-30=0 \to w=-5$ or 6. Therefore x satisfies $g(x)=-5$ or $g(x)=6$, or $x^2-x+5=0$ or $x^2-x-6=0$. The first equation has complex solutions, and the second equation has real solutions (-2 and 3) that sum to $\boxed{1}$.

T-12. There are six up-right paths in the full grid. If one of the four segments adjacent to A or B are removed , exactly half of the up-right paths will be removed. If one of the other four boundary segments is removed, 1 up-right path is removed, if one of the four segments adjacent to the center grid point is removed, 2 up-right paths are removed. Accordingly, one-third of the time there will be 3, 4, or 5 up-right paths remaining, so the expected number of paths remaining is $\boxed{4}$.

T-13. Let $n=10t+u$, where t and u are the tens and ones digit of n, respectively. Then $p(n)=tu$ if $t\geq 1$ and u if $t=0$ and $s(n)=t+u$. If $n=p(n)+s(n)$, then $10t+u=tu+t+u \to 9t=tu$, so $u=9$ if $t\geq 1$. The solutions are all two digit numbers with 9 as a units digit, and their sum is $19+29+\cdots+99=\boxed{531}$.

T-14. For $\log_{10} x+\log_{10} y$ to be an integer, xy must be a power of 10. There is one pair of values (x,y) whose product is 1 $(1,1)$, four pairs of values whose product is 10 $((1,10),(2,5),(5,2),(10,1))$, and three pairs whose product is 100 $((5,20),(10,10),(20,5))$. There are 400 possible pairs, so the probability is $\frac{8}{400}=\boxed{\frac{1}{50}}$.

T-15. If k white cards appear in the first row, then $3-k$ white cards and k red cards must appear in the third row, which means that $3-k$ red cards and k blue cards must appear in the second row, so $3 - k$ blue cards must appear in the first row. For each row, there are $C(3, k)$ ways to arrange the cards, so the total number of arrangements of cards that satisfy this criterion is $\sum_{k=0}^{3} C(3, k)^3 = 1^3 + 3^3 + 3^3 + 1^3 = 1 + 27 + 27 + 1 = 56$. The total number of arrangements is $\frac{9!}{3!3!3!} = \frac{362880}{216} = 1680$, so the probability is $\frac{56}{1680} = \boxed{\frac{1}{30}}$.

ARML Local 2015 Individual Problems

I-1. The graph of $y = ax^2 + bx + c$ passes through $(-1, -3)$, $(1, 3)$, and $(2, 12)$. Compute abc.

I-2. Eight rational numbers are written on a blackboard. Their average is 15. One of the numbers is erased, and the average of the numbers remaining on the board is 12. Compute the value of the number that was erased.

I-3. Given real numbers x and y such that $x + y = 4$ and $x^3 + y^3 = 28$. Compute $x^2 + y^2$.

I-4. Triangle NAP has side lengths $NA = 10$, $AP = 17$, and $NP = 21$. R and S are on $\overline{NP}$ so that $\overline{AR}$ is the altitude to segment $\overline{NP}$ and $\overline{AS}$ is the angle bisector from A to $\overline{NP}$. Compute RS.

I-5. Johanna and Edward play a game with a 6-sided die. For each turn, a player rolls a die until they roll a 1, in which case they lose the game, or until they roll a 5 or 6, in which case their turn is over and the other player rolls the die under the same rules. The game continues until some player rolls a 1 and loses. If Johanna plays first, compute the probability that she wins the game.

I-6. The line $x = a$ intersects the graphs of $f(x) = \log_{10}(x)$ and $g(x) = \log_{10}(x + 5)$ at points B and C, respectively. If $BC = 1$, compute a.

I-7. Regular hexagon $HEXAGO$ has vertices $H(4, 1)$ and $E(8, 1)$. The hexagon lies entirely in the first quadrant. If the coordinates of X are (x_1, y_1) and the coordinates of A are (x_2, y_2), compute $x_1 + x_2 + y_1 + y_2$.

I-8. Let $X = 20202020202020202020$, the 20-digit number consisting of 10 copies of the digits 2 and 0. Similarly, let $Y = 15151515151515151515$, the 20-digit number consisting of 10 copies of the digits 1 and 5. Compute the sum of the digits of XY.

I-9. The value of x that solves $(\sqrt{2x+1} + \sqrt{x+3})^2 + \sqrt{2x+1} + \sqrt{x+3} = 12$ is of the form $A - \sqrt{B}$ where A and B are positive integers. Compute $A + B$.

I-10. Square $ARML$ has unit side lengths and center O. D and E are on $\overline{AR}$, with D between A and E. If $\mathrm{m}\angle DOE = 45^\circ$ and $DE = \frac{3}{7}$, compute the largest value of ER.

ARML Local 2015 Individual Solutions

I-1. Substituting $x-$ and $y-$coordinates of the three points into the general equation gives us three equations: $a-b+c=-3$, $a+b+c=3$, and $4a+2b+c=12$. Subtracting the first two equations yields $2b=6 \rightarrow b=3$. Substituting gives two equations: $a+c=0$ and $4a+c=6$. Subtracting these yields $3a=6 \rightarrow a=2 \rightarrow c=-2$. The equation is $y=2x^2+3x-2$, so the answer is $\boxed{-12}$.

I-2. The sum of the original eight numbers is $8 \cdot 15 = 120$. After a number is erased, the sum of the remaining numbers is $7 \cdot 12 = 84$, so the erased number was $\boxed{36}$.

I-3. Note that $x^3+y^3 = 28 = (x+y)^3 - 3xy(x+y) = 64 - 3xy(4) = 64 - 12xy$, so $-36 = -12xy \rightarrow xy = 3$. Now, consider $x^2+y^2 = (x+y)^2 - 2xy = 16 - 2 \cdot 3 = \boxed{10}$. Note that this is true when $x=1$ and $y=3$.

I-4. Either by solving a system of equations or by inspection, we see that $AR=8$, $NR=6$, and $RP=15$. We also know that the angle bisector will divide $\overline{NP}$ into the same ratio as $NA:AP$. Thus $NS:SP = 10:17$ and $NS+SP=21$, so $NS=\frac{210}{27}$, and $RS = NS-NR = \frac{210}{27} - 6 = \boxed{\frac{16}{9}}$.

I-5. Note that on any turn of a game, a player is twice as likely to pass the die to the other player than they are to lose. Accordingly, on any given turn, a player has a $1/3$ probability of losing and a $2/3$ probability of passing the die to the other player. The probability that Johanna wins after one turn by each player is $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$, after two turns is $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}$, and so forth. The sum of the probabilities that Johanna wins in each round is an infinite geometric series with sum $\frac{\frac{2}{9}}{1-\frac{4}{9}} = \boxed{\frac{2}{5}}$.

I-6. The coordinates of the two points of intersection are $(a, \log_{10}(a))$ and $(a, \log_{10}(a+5))$. Because $BC=1$, $\log_{10}(a+5) - \log_{10}(a) = 1 \rightarrow \log_{10}\frac{a+5}{a} = 1 \rightarrow \frac{a+5}{a} = 10 \rightarrow a = \boxed{\frac{5}{9}}$.

I-7. Each side of the regular hexagon measures $\sqrt{(8-4)^2+(1-1)^2} = 4$. Extend side $\overline{HE}$ and drop an altitude from X to that extension, meeting the extension at N. The measure of $\angle XEN$ is $\frac{360}{6} = 60°$, so $EN = 4\sin 30° = 2$ and $XN = 4\sin 60° = 2\sqrt{3}$, so X has coordinates $(8+2, 1+2\sqrt{3})$. Constructing a triangle in a similar way will yield $A(8, 1+2\sqrt{3}+2\sqrt{3})$. The desired sum is $10+8+1+2\sqrt{3}+1+4\sqrt{3} = \boxed{20+6\sqrt{3}}$.

I-8. Note that $X = 20 \cdot Z$, where Z is the 19 digit number consisting of alternating ones and zeroes. Similarly, $Y = 15 \cdot Z$. Therefore $XY = 300 \cdot Z^2$.

$Z^2 = 102030405060708091009080706050403020 1$, so $XY = 300Z^2 = 3Z^2 \cdot 100$. As the 100 factor will not affect the sum of the digits,

$3Z^2 = 3060912151821242730272421181512090603$. Pairs of digits from right to left sum to $3+6+9+3+6+9+3+6+9+3+9+6+3+9+6+3+9+6+3 = \boxed{111}$.

I-9. This problem is easier to solve if $Y = \sqrt{2x+1} + \sqrt{x+3}$, because then the given equation becomes the quadratic $Y^2+Y-12=0 \rightarrow (Y-3)(Y+4)=0$. The second factor in Y has no real solutions for x, so we focus on the factor that implies $Y = 3$. Solving $\sqrt{2x+1}+\sqrt{x+3} = 3 \rightarrow 2x+1 = 9+x+3-6\sqrt{x+3} \rightarrow x-11 = -6\sqrt{x+3}$, which has two solutions: $x = 29 \pm \sqrt{828}$. Only the lesser root checks in the original equation, so $x = 29 - \sqrt{828}$ and the answer is $29 + 828 = \boxed{857}$.

I-10. Let S be the midpoint of $\overline{AR}$, $\alpha = \text{m}\angle DOS$, and $\beta = \text{m}\angle EOS$. Then $OS = \frac{1}{2}$, $SD = \frac{\tan\alpha}{2}$, $SE = \frac{\tan\beta}{2}$, and $\alpha + \beta = 45°$. $DE = SD + SE \rightarrow \frac{3}{7} = \frac{\tan\alpha}{2} + \frac{\tan\beta}{2} \rightarrow \tan\alpha + \tan\beta = \frac{6}{7}$. Because $\tan\beta = \tan(45° - \alpha) = \frac{1-\tan\alpha}{1+\tan\alpha}$, we have that $\tan\alpha + \frac{1-\tan\alpha}{1+\tan\alpha} = \frac{6}{7}$. Let $w = \tan\alpha$, so $w + \frac{1-w}{1+w} = \frac{6}{7} \rightarrow 7w(1+w) + 7(1-w) = 6(1+w) \rightarrow 7w^2 + 7 = 6(1+w) \rightarrow 7w^2 - 6w + 1 = 0$. This equation has solutions $w = \frac{3\pm\sqrt{2}}{7}$. For ER to be as large as possible, $SE = \frac{\tan\beta}{2}$ should be as small as possible, so $\tan\beta = \frac{3-\sqrt{2}}{7} \rightarrow ER = SR - SE = \frac{1}{2} - \frac{3-\sqrt{2}}{14} = \boxed{\frac{4+\sqrt{2}}{14}}$.

ARML Local 2015 Relay Problems

R1-1. If A, B, C, and D are (not necessarily distinct) digits such that $\underline{A}\ \underline{B}\ \underline{C}\ \underline{D} + \underline{A}\ \underline{B}\ \underline{C} + \underline{A}\ \underline{B} + \underline{A} = 2015$. Compute $(A+C)(B+D)$.

R1-2. Let T = TNYWR. Compute the number of positive integers less than or equal to 2015 that are relatively prime to T.

R2-1. The positive integer N can be written in the form $p^a q^b$, where p and q are distinct primes, and a and b are positive integers. If the number $25N$ is a perfect square with exactly 27 positive divisors, compute the smallest possible value of N.

R2-2. Let T = TNYWR. A positive integer is *small-prime-free* if it is not a multiple of 2, 3, or 5. Compute the T^{th} smallest small-prime-free integer.

R2-3. Let T = TNYWR. The product of two consecutive positive odd integers is $3T$. Compute the average of these two integers.

R3-1. Let $\dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{7} + \dfrac{1}{N}$. Compute N.

R3-2. Let T = TNYWR. There are k competitors in a contest. There are T ways to select a winner and a runner-up. Compute k.

R3-3. Let T = TNYWR. A positive integer is a palindrome if its digits read the same forwards as backwards. There are exactly 20 integers between 1 and N inclusive that are palindromes when written in base T. Compute the smallest (base-10) value of N.

R3-4. Let T = TNYWR. Compute the smallest value of n such that the product of any n consecutive integers is a multiple of T.

R3-5. Let T = TNYWR. There are circles of radius T with centers A, R, M, and L. $ARML$ is square of side length $2T$. If x is the area that is inside $ARML$ but outside any of the circles, compute x to the nearest tenth. Pass back your answer in decimal form.

R3-6. Let T = TNYWR. A book that was originally priced K dollars is on sale for T dollars, a discount of D percent. If K and D are both positive integers, compute the minimum value of D.

ARML Local 2015 Relay Solutions

R1-1. The sum in the problem equals $1111A + 111B + 11C + D$. A must be 1, so $111B + 11C + D = 2015 - 1111 = 904$. B must be eight, so $11C + D = 904 - 888 = 16$. By similar arguments, $C = 1$ and $D = 5$, so $(A + C)(B + D) = (1 + 1)(5 + 8) = \boxed{26}$.

R1-2. $T = 26 = 2 \cdot 13$. There are $2015/13 = 155$ multiples of 13 between 1 and 2015, and $\lfloor 2015/2 \rfloor = 1007$ multiples of 2. Of the multiples of 13, 77 of them are multiples of 2 as well, so there are $1007 + 155 - 77 = 1085$ numbers that are not relatively prime to 26, so there are $2015 - 1085 = \boxed{930}$ integers between 1 and 2015 that are relatively prime to 26.

R2-1. Provided p and q are neither 5, $\sigma(25p^aq^b) = \sigma(25)\sigma(p^a)\sigma(q^b) = 3(a + 1)(b + 1)$. N will be minimized when $p = a = b = 2$ and $q = 3$, so $N = 6^2 = \boxed{36}$.

R2-2. A number is small-prime-free if and only if it is relatively prime to 30. In the range of integers between $30k + 1$ and $30k + 30$ for $k \geq 0$, there are $\phi(30) = 30 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$ numbers relatively prime to 30 between $30k + 1$ and $30k + 30$, inclusive. Accordingly, the 36^{th} small-prime-free number is the fourth such number between 121 and 150 inclusive. The numbers less than 30 that are relatively prime to 30 are $\{1, 7, 11, 13, 17, 19, 23, 29\}$, so the answer is $120 + 13 = \boxed{133}$.

R2-3. If the integers are $2n - 1$ and $2n + 1$, $(2n - 1)(2n + 1) = 4n^2 - 1 = 3T = 399 \to 4n^2 = 400 \to n = 10$. The two numbers are 19 and 21 and their average is $\boxed{20}$.

R3-1. The left side difference is $\frac{1}{6}$ and $\frac{1}{6} - \frac{1}{7} = \frac{1}{42}$, so $N = \boxed{42}$.

R3-2. There are $k(k - 1)$ ways to pick a winner and runner up. Because $T = 42$, it follows that $k = \boxed{7}$.

R3-3. Single digits $(\underline{a})_T$ are palindromes, as are numbers of the form $(\underline{a}\ \underline{a})_T$, $(\underline{a}\ \underline{b}\ \underline{a})_T$, and so forth. Because $T = 7$, There are 6 singleton and doubleton palindromes and 7 three-digit palindromes that begin with 1, which makes 19 in total. The first 3-digit palindrome that starts with a 2 is $(202)_7 = \boxed{100}$.

R3-4. $T = 100$. In order for the product of n consecutive integers to be a multiple of 100, at least two of the integers must be multiples of 5. The smallest interval of consecutive integers that is guaranteed to contain two multiples of 5 is contains $\boxed{10}$ integers.

R3-5. Note that a circle of radius T with center at the center of ARML would have the same amount of area outside of it as what is described in the problem. Accordingly, as $T = 10$, the area outside the circles is $400 - 100\pi \approx 400 - 100(3.1416) = \boxed{85.8}$.

R3-6. $\frac{T}{1-\frac{D}{100}} = K \rightarrow T = K - \frac{KD}{100} \rightarrow 100T = 100K - KD \rightarrow 100 - \frac{100T}{K} = D$. To get the minimum positive integral value of D, $\frac{100T}{K}$ must be the largest factor of $100T$ less than 100. For $T = 85.8$, $100T = 8580 = 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 13$. The largest product of factors of 8580 less than 100 is 78, so $D = \boxed{22}$.

ARML Local 2015 Tiebreaker Problem

TB-1. Compute the sum of all integral values of x such that $x^2 + 60x + 1000$ is a perfect square.

ARML Local 2015 Tiebreaker Solution

TB-1. $x^2 + 60x + 1000 = (x + 30)^2 + 100$. If it is equal to a perfect square, say y^2 with $y \geq 0$, then $y^2 - (x + 30)^2 = 100 \rightarrow (y - (x + 30))(y + (x + 30)) = 100$. Note that the two factors will have identical parity, so either they are both equal to 10, in which case $x = -30$ and $y = 10$, or one is equal to 2 and the other is equal to 50. If the first factor is equal to 2 then $x = -6$ and $y = 26$. If the first factor is equal to 50 then $x = -54$ and $y = 26$. The sum of the values of x is $(-30) + (-6) + (-54) = \boxed{-90}$.

Answers to ARML Local 2015

Team Round:

T-1. 26	T-2. 18	T-3. 98764310	T-4. $2\sqrt{3}$	T-5. $\frac{4}{7}$
T-6. 10	T-7. 288	T-8. 10	T-9. $72+36\sqrt{3}$	T-10. 4
T-11. 1	T-12. 4	T-13. 531	T-14. $\frac{1}{50}$	T-15. $\frac{1}{30}$

Individual Round:

I-1. -12	I-2. 36	I-3. 10	I-4. $\frac{16}{9}$	I-5. $\frac{2}{5}$
I-6. $\frac{5}{9}$	I-7. $20+6\sqrt{3}$	I-8. 111	I-9. 857	I-10. $\frac{4+\sqrt{2}}{14}$

Relay Round:

R1-1. 26	R1-2. **930**				
R2-1. 36	R2-2. 133	R2-3. 20			
R3-1. 42	R3-2. 7	R3-3. 100	R3-4. 10	R3-5. 85.8	R3-6. 22

Tiebreaker:

1. -90

ARML Local 2016

Contents

ARML Local 2016 Team Problems

T-1. All the shelves in a library are the same length. When filled, two shelves side-by-side can hold exactly 12 algebra books and 10 geometry books. All algebra books have the same width, as do all the geometry books, but algebra and geometry books have different widths. Three shelves side-by-side can hold exactly 15 algebra books and 21 geometry books. Compute the maximum number of geometry books that can fit on one shelf.

T-2. In the equation $\underline{A}\ \underline{B}\ \underline{C} + \underline{D}\ \underline{E}\ \underline{F} = \underline{G}\ \underline{H}\ \underline{I}$, each letter denotes a distinct non-zero digit. Compute the greatest possible value of $\underline{G}\ \underline{H}\ \underline{I}$.

T-3. If $\frac{\log_b a}{\log_c a} = 2016$, then $\frac{b}{c} = c^k$. Compute k.

T-4. Compute $\cos\frac{\pi}{100} + \cos\frac{2\pi}{100} + \cdots + \cos\frac{98\pi}{100} + \cos\frac{99\pi}{100} + \cos\frac{100\pi}{100}$.

T-5. Compute the smallest positive integer k such that $n^2 - k$ is not a prime number for any positive integer $n \geq \sqrt{k}$.

T-6. A *magic square* is an $n \times n$ grid of numbers where the numbers in each row and column add up to the same sum. Some of the values in the 5×5 magic square below have been erased. Compute x.

	1		8	
21		12		23
	25		17	
7		22		2
	20		x	

T-7. Triangle ABC is inscribed in circle ω. It is given that $BA = BC = 8$ and $AC = 8\sqrt{2}$. The tangent to ω at B intersects the tangents to ω at A and C at points X and Y, respectively. Compute the area of quadrilateral $AXYC$.

T-8. In triangle LMN, it is given that $\mathrm{m}\angle LMN = 90^\circ, LM = 5$, and $MN = 12$. Point P lies on the hypotenuse $\overline{LN}$ so that $\mathrm{m}\angle PMN = 30^\circ$. If $PN = \frac{a-b\sqrt{3}}{c}$, where a, b, and c are positive integers with no common factor greater than one, compute $a+b+c$.

T-9. Compute the sum of all positive integer values of x such that there exists a positive integer y such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$.

T-10. Compute the number of zeroes that 2016! ends with when written in base 2016.

T-11. Compute the number of 2×2 matrices M with the following properties:

- The entries of M are integers between 0 and 29 inclusive.
- The determinant of M is divisible by 30.

T-12. Let $\mathcal{S}$ denote the set of all 4^4 functions from the set $\{1, 2, 3, 4\}$ to itself. Three functions $f, g, h \in \mathcal{S}$ are chosen randomly with replacement, with all functions having an equal probability of being chosen. Compute the probability that the following property holds: for every $y \in \{1, 2, 3, 4\}$, there exists an $x \in \{1, 2, 3, 4\}$ such that $f(g(h(x))) = y$.

T-13. Let $ABCDEF$ be a convex cyclic hexagon. Diagonals $\overline{AC}$ and $\overline{BD}$ meet at X while diagonals $\overline{AE}$ and $\overline{DF}$ meet at Y. Suppose $AF = AB = 3$, $BC = 4$, $DC = DE = 5$, and $EF = 6$. Compute the ratio of the areas of triangles CXF and CYF.

T-14. Initially, three vertices of an equilateral triangle are marked on a blackboard. Every minute, Alice marks the circumcenter of the triangle whose vertices are the three points on the blackboard, then randomly erases one of the four points on the board (each with probability $\frac{1}{4}$). After one hour has elapsed, the probability that the three points on the blackboard form an equilateral triangle is x. Compute the nearest integer to $100x$.

T-15. In the grid below, each of the letters A through P is equal to either 2, 3, 4, or 5. Exactly four letters are equal to each of the four numbers. Some of the sums of numbers in the columns are given, as are some of the products of numbers in the rows. Compute the 4-digit number $\underline{A}\ \underline{F}\ \underline{K}\ \underline{P}$.

$$\begin{array}{ccccccccc}
A & \times & B & \times & C & \times & D & = & 225 \\
+ & & + & & + & & & & \\
E & \times & F & \times & G & \times & H & = & 128 \\
+ & & + & & + & & & & \\
I & \times & J & \times & K & \times & L & = & 120 \\
+ & & + & & + & & & & \\
M & & N & & O & & P & & \\
= & & = & & = & & & & \\
10 & & 12 & & 15 & & & &
\end{array}$$

ARML Local 2016 Team Solutions

T-1. Let a be the width of an algebra book, g be the width of a geometry book, and s be the width of a shelf. Then $12a + 10g = 2s, 15a + 21g = 3s \to 6a + 5g = s = 5a + 7g \to a = 2g$, so $s = 17g$, and $\boxed{17}$ geometry books can fit on one shelf.

T-2. Note that $\underline{G}\ \underline{H}\ \underline{I}$ must be a multiple of nine. The sum of the digits on the left-hand and right-hand side of the equation must be equivalent modulo nine, and because the non-zero digits sum to a multiple of nine, both sides' digits must be equivalent to zero modulo nine. The largest multiple of nine with three distinct non-zero digits is $\boxed{981}$, and there are several sums that work, including $746 + 235 = 981$.

T-3. Note that $2016 = \frac{\log_b a}{\log_c a} = \frac{\left(\frac{\log a}{\log b}\right)}{\left(\frac{\log a}{\log c}\right)} = \frac{\log c}{\log b} = \log_b c$. So $b^{2016} = c \to b = c^{\frac{1}{2016}}$. Therefore $\frac{b}{c} = \frac{c^{\frac{1}{2016}}}{c} = c^{\frac{-2015}{2016}}$, so the answer is $k = \boxed{-\dfrac{2015}{2016}}$.

T-4. Because $\cos(\pi - x) = -\cos(x)$ for all x, removing the last term from the sum, the leftmost and rightmost terms in the sum that remains sum to zero. Repeating this 48 more times leaves $\cos\frac{50\pi}{100}$, which equals 0, so the original sum equals $\cos\frac{100\pi}{100} = \cos(\pi) = \boxed{-1}$.

T-5. The answer is $k = \boxed{16}$. Indeed the number $n^2 - 16 = (n-4)(n+4)$ is always a product of two distinct integers, and when $n - 4 = 1, n = 5$ and the number $5^2 - 16$ is not prime. It remains to rule out all positive integers $k < 16$ as candidates.

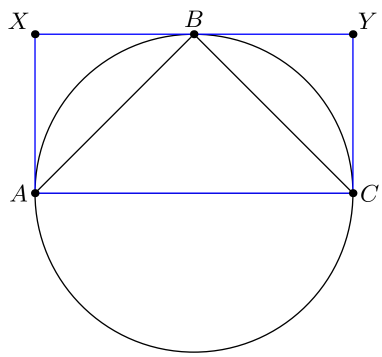
- By taking $n = 2$, we rule out $k = 1, 2$.
- By taking $n = 3$, we rule out $k = 4, 6, 7$.
- By taking $n = 4$, we rule out $k = 3, 5, 9, 11, 13, 14$.
- By taking $n = 5$, we rule out $k = 8, 12$.
- By taking $n = 9$, we rule out $k = 10$.
- By taking $n = 14$, we rule out $k = 15$.

T-6. Consider the erased values in the second and fourth row and column.

	1		8	
21	A	12	B	23
	25		17	
7	C	22	D	2
	20		x	

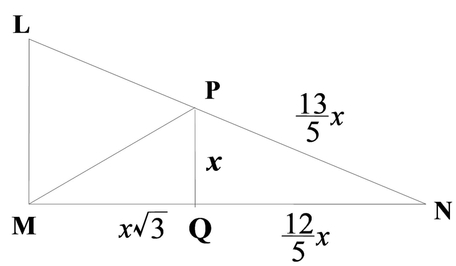
We know that the sum of the values in the second and fourth row is equal to the sum of the values in the second and fourth column. Therefore $21+A+12+B+23+7+C+22+D+2 = 1 + A + 25 + C + 20 + 8 + B + 17 + D + x$. As the terms A, B, C, and D cancel, we are left with $87 = 71 + x \to x = \boxed{16}$.

T-7. By the converse of the Pythagorean Theorem, $\angle ABC$ is a right angle. Hence $\overline{AC}$ is a diameter. Let O be the center of ω. Then $\triangle AOB \cong \triangle COB$ and thus $\angle COB$ is a right angle. Hence $\overline{AX} \parallel \overline{OB} \parallel \overline{CY}$ because angles OAX and OCY are also right angles. Thus quadrilateral $AXYC$ is a rectangle.



Hence the area of $AXYC$ is twice that of the area of ABC, which is $2 \cdot 32 = \boxed{64}$.

T-8. Let Q be the foot of the perpendicular from P to $\overline{MN}$, and let $QP = x$. Then triangles LMN and PQN are similar and $MN = MQ + QN \to 12 = x\sqrt{3} + \frac{12}{5}x \to x = \frac{12}{\sqrt{3}+\frac{12}{5}} = \frac{60}{12+5\sqrt{3}} = \frac{60(12-5\sqrt{3})}{144-75} = \frac{20(12-5\sqrt{3})}{23}$. Because $PN = \frac{13}{5}x = \frac{52(12-5\sqrt{3})}{23}$, it follows that $a + b + c = 52(12+5) + 23 = \boxed{907}$.



T-9. Multiply both sides of the equation by $10xy$ to get $10y + 10x = xy$ or $xy - 10x - 10y = 0$. Add 100 to both sides to get $(x-10)(y-10) = 100$. Accordingly, for every positive factor k of 100, $x = 10 + k$, $y = 10 + \frac{100}{k}$ is a solution. There are 9 factors of 100, $1, 2, 4, 5, 10, 20, 25, 50,$ and 100, and their sum is $(1+2+4) \cdot (1+5+25) = 7 \cdot 31 = 217$. The values of x are 10 more than each of these factors, so the sum of the xs is $217 + 9 \cdot 10 = \boxed{307}$.

T-10. Note that $2016 = 2^5 \cdot 3^2 \cdot 7$, so it is necessary to count the powers of 2, 3, and 7 in 2016! to determine which factor limits the number of powers of 2016 in 2016!. There are $\lfloor \frac{2016}{2^1} \rfloor + \lfloor \frac{2016}{2^2} \rfloor + \cdots + \lfloor \frac{2016}{2^{10}} \rfloor = 2010$ powers of 2, so 402 powers of 2^5. Similarly, there are $\lfloor \frac{2016}{3^1} \rfloor + \lfloor \frac{2016}{3^2} \rfloor + \cdots + \lfloor \frac{2016}{3^6} \rfloor = 1004$ powers of 3, so 502 powers of 3^2. Also, there are $\lfloor \frac{2016}{7^1} \rfloor + \lfloor \frac{2016}{7^2} \rfloor + \lfloor \frac{2016}{7^3} \rfloor = 334$ powers of 7, so 2016! will end with $\boxed{334}$ zeroes.

T-11. First, we count the number of matrices with entries in $\{0, \ldots, p-1\}$ and with determinant divisible by a prime p. This is the number of quadruples (a, b, c, d) such that $ad \equiv bc \pmod{p}$. To count the number of such quadruples we use casework on the common value $x = ad \pmod{p} = bc \pmod{p}$:

- If $x \not\equiv 0 \pmod{p}$, then there are exactly $p-1$ pairs (a, d) and (b, c) such that $x \equiv ad \equiv bc \pmod{p}$, (let a be any non-zero value, let $d = a^{-1}x$), hence there are $(p-1)^2$ cases for each x. Thus over all non-zero x, we have $(p-1)^3$.
- If $x \equiv 0 \pmod{p}$, then there are exactly $2p-1$ pairs (a, d) and (b, c) (at least one of the elements in each pair is zero) such that $0 \equiv ad \pmod{p}$, hence $(2p-1)^2$ cases here.

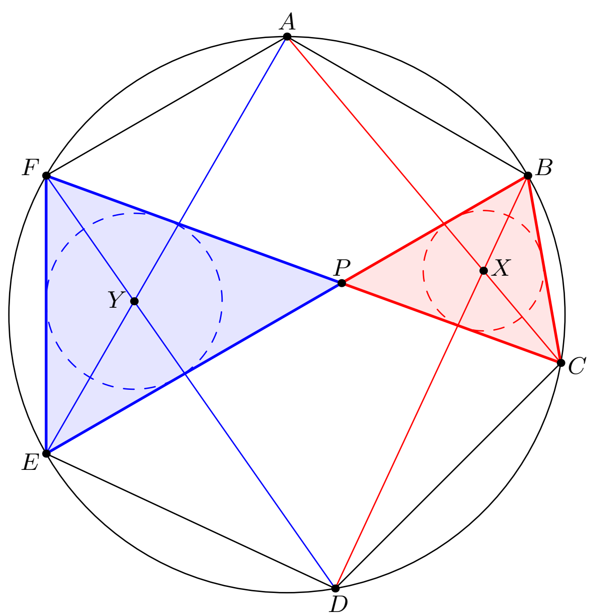
Hence the total number of matrices is $(p-1)^3 + (2p-1)^2$.

Now, by the Chinese Remainder Theorem, it suffices to compute the above value for $p = 2$, $p = 3$ and $p = 5$ and multiple the results. Thus the total number of matrices satisfying the conditions of the problem is $(1^3 + 3^2)(2^3 + 5^2)(4^3 + 9^2) = 10 \cdot 33 \cdot 145 = \boxed{47850}$.

T-12. The condition in the problem is that the composition $f \circ g \circ h$ is surjective. For a function from a finite set to itself, surjectivity implies injectivity, so the condition is equivalent to $f \circ g \circ h$ being a bijection. Note that the composition is a bijection if and only if f, g, h are each themselves a bijection. The number of bijections of $\{1, 2, 3, 4\}$ onto itself is $4! = 24$. The answer is

$$\left(\frac{24}{4^4}\right)^3 = \left(\frac{3}{32}\right)^3 = \boxed{\frac{27}{32768}}.$$

T-13. Let diagonals $\overline{BE}$ and $\overline{CF}$ meet at P. Because $AB = AF$, minor arcs $\widehat{AB}$ and $\widehat{AF}$ have the same length, hence $\angle ACB \cong \angle ACF$ and $\angle AEB \cong \angle AEF$. Similarly, because $DC = DE$, conclude that $\angle DBC \cong \angle DBE$ and $\angle DFC \cong \angle DFE$. Thus X and Y are the incenters of triangles BPC and FPE, respectively.



The ratio of the areas of CXF and CYF is equal to the ratio of the inradius of triangle PBC to the inradius of triangle PFE. But these two triangles are similar, with ratio $BC/FE = 4/6 = \boxed{2/3}$.

T-14. The main observation is that up to similarity, only two types of triangles can arise:

- an equilateral triangle, and
- a 30°-30°-120° triangle.

From the former case there is a $\frac{3}{4}$ chance of transitioning to the latter case, and from the latter case there is a $\frac{1}{2}$ chance of transitioning to the equilateral triangle. Therefore if p_n is the probability of having a equilateral triangle after n minutes, it satisfies the recursion

$$p_0 = 1 \quad \text{and} \quad p_n = \frac{1}{4}p_{n-1} + \frac{1}{2}(1 - p_{n-1}) = \frac{1}{2} - \frac{1}{4}p_{n-1}.$$

From here we can use induction to derive that

$$p_n = \frac{2}{5} + \frac{3}{5}\left(-\frac{1}{4}\right)^n.$$

Taking $n = 60$, $p_n \approx \frac{2}{5}$ so the closest integer to $100p_n$ is $\boxed{40}$.

T-15. The sum of all of the numbers in the grid is 56, so $D + H + L + P = 19$, meaning that last column consists of the numbers 4, 5, 5, and 5 in some order. Similarly, the product of all of the numbers in the grid is 120^4, so $MNOP = 60$, meaning the bottom row consists of the numbers 2, 2, 3, and 5 in some order. The first column must consist of the numbers 2, 2, 2, and 4 or 2, 2, 3, and 3 in some order to sum to 10, but we can discount the former case as the product $ABCD$ is odd so every column must contain an odd number (in fact, we know the first row is the numbers 3, 3, 5, and 5 in some order, so $A = 3$ and $D = 5$). The second row must be the numbers 2, 4, 4, and 4 in some order, so $E = 2$ and $F = G = H = 4$, so $L = P = 5$. The third row must be the numbers 2, 3, 4, 5 in some order. Note that $C + K + O = 11$, and the only sum that works is $5 + 4 + 2$ (because all but one of the 4s are in the second row, and all but one of the 3s are in the first row or column). Accordingly, $C = 5$ and $K = 4$, so $\underline{A}\ \underline{F}\ \underline{K}\ \underline{P} = \boxed{3445}$.

There are, in fact, two solutions to the puzzle, one of which is shown below, the other is derived by swapping the 2s and 3s in the bottom left corner.

$$\begin{array}{cccccccc}
3 & \times & 3 & \times & 5 & \times & 5 & = & 225 \\
+ & & + & & + & & & & \\
2 & \times & 4 & \times & 4 & \times & 4 & = & 128 \\
+ & & + & & + & & & & \\
2 & \times & 3 & \times & 4 & \times & 5 & = & 120 \\
+ & & + & & + & & & & \\
3 & & 2 & & 2 & & 5 & & \\
= & & = & & = & & & & \\
10 & & 12 & & 15 & & & &
\end{array}$$

ARML Local 2016 Individual Problems

I-1. The degree measures of the interior angles of a pentagon are integral, non-equal, and form an arithmetic progression. Compute the number of different values of the smallest degree measure of an interior angle of the pentagon.

I-2. Integers a and b are selected uniformly at random and with replacement from the set $\{-4,-3,-2,-1,0,1,2,3,4\}$. Compute the probability that $a+b^2=a^2+b$.

I-3. Given that $f(x)=\frac{x}{2}+2$, and $f(2f(2f(2z-1)))=\frac{5}{2}$, compute z.

I-4. In square $ABCD$ with side length 1, G lies on $\overline{CD}$, F lies on $\overline{BC}$, and E lies on $\overline{AB}$. If $m\angle GEF=30^\circ$, $\angle GFE$ is right, and $\tan FEB=\frac{2}{3}$, compute BF.

I-5. If $\log_6 48=z$, then $\log_9 6=\frac{b}{cz+d}$ for integers b, c, and d, where $b>0$. Compute the least possible value of the sum $b+c+d$.

I-6. Compute the remainder when 100! is divided by 103.

I-7. Point A is coplanar with circle O and lies outside circle O. Points C and E lie on circle O so that $\overline{AC}$ and $\overline{AE}$ are tangent to circle O. Points B and F lie on $\overline{AC}$ and $\overline{AE}$, respectively, so that $\overline{BF}$ is tangent to circle O. Given that $m\angle CAE=50^\circ$, compute the degree measure of $\angle BOF$.

I-8. For each positive integer $1\le k\le 8$, let $b_k=\binom{8}{0}+\binom{8}{1}+\cdots+\binom{8}{k}$. Compute the value of

$$\frac{\binom{8}{2}}{b_1b_2}+\frac{\binom{8}{3}}{b_2b_3}+\cdots+\frac{\binom{8}{7}}{b_6b_7}.$$

I-9. The integer $Z=104060001$ is the product of three distinct prime numbers. Compute the largest prime factor of Z.

I-10. Let $ABCDEF$ be a hexagon inscribed in circle Γ so that

$$AB=DE=4,\quad BC=EF=6,\quad \text{and}\quad CD=FA=8.$$

Let M, N, L, and K denote the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{EF}$, respectively. Let P be a point in space not coplanar with Γ. If $PM=5$, $PN=7$, $PL=9$, compute PK.

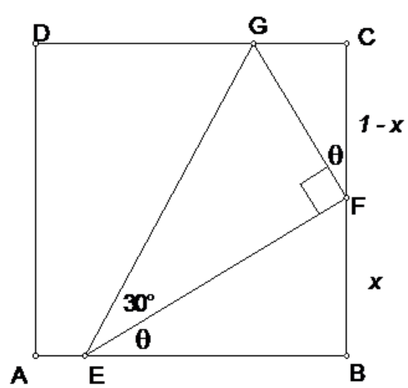
ARML Local 2016 Individual Solutions

I-1. Let the degree measures of the angles form an increasing arithmetic sequence with first term a and common difference d; then $a+(a+d)+(a+2d)+(a+3d)+(a+4d) = 540 \to a+2d = 108$. Both a and d must be positive integers, and a must be even, leading to 53 possible values of a $(2, 4, \ldots, 106)$. However, $a = d = 36$ leads to a degenerate pentagon with the largest angle being $180°$, so there are $\boxed{52}$ pentagons. Note: the degenerate case was not included in the original version of this solution, as a result, we chose to accept 52 or 53 as acceptable answers to this question.

I-2. The given equation can be rewritten as $b^2 - a^2 = b - a$, or $(b-a)(a+b-1) = 0$. The equation is satisfied if and only if $a = b$ or $a + b = 1$. Of the $9^2 = 81$ pairs (a, b), nine of them satisfy the first equation and eight satisfy the second. The answer is $\boxed{\dfrac{17}{81}}$.

I-3. Let $g(x)$ denote the inverse of $f(x)$. Therefore $g(x) = 2x - 4$. So $g(f(2f(2f(2z-1)))) = g(\frac{5}{2}) \to 2f(2f(2z-1)) = 1 \to f(2f(2z-1)) = \frac{1}{2} \to 2f(2z-1) = g(\frac{1}{2}) = -3 \to 2z - 1 = g(-\frac{3}{2}) = -7 \to z = \boxed{-3}$.

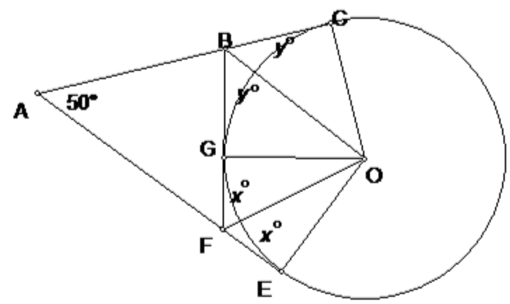
I-4. Let $x = BF$ and $m\angle FEB = \theta$. Then $EF = \frac{x}{\sin\theta}$, $GF = \frac{1-x}{\cos\theta}$, and $\frac{GF}{EF} = \frac{1}{\sqrt{3}}$, so $\frac{1-x}{\cos\theta} \cdot \frac{\sin\theta}{x} = \frac{1}{\sqrt{3}} \to \left(\frac{1}{x} - 1\right)\tan\theta = \frac{1}{\sqrt{3}} \to \left(\frac{1}{x} - 1\right)\frac{2}{3} = \frac{1}{\sqrt{3}} \to \left(\frac{1}{x} - 1\right) = \frac{\sqrt{3}}{2} \to \frac{1}{x} = \frac{2+\sqrt{3}}{2} \to x = \frac{2}{2+\sqrt{3}} = \boxed{4 - 2\sqrt{3}}$.



I-5. $z = \log_6 48 = \log_6 8 + \log_6 6 = 3\log_6 2 + 1 \to \log_6 2 = \frac{z-1}{3}$. $\log_9 6 = \frac{1}{\log_6 9} = \frac{1}{2\log_6 3}$. Because $\log_6 2 + \log_6 3 = 1$, $\frac{1}{2\log_6 3} = \frac{1}{2(1-\log_6 2)} = \frac{1}{2(1-\frac{z-1}{3})} = \frac{3}{8-2z}$, and $b + c + d = 3 - 2 + 8 = \boxed{9}$.

I-6. By Wilson's Theorem, as 103 is prime, $102! \equiv -1 \pmod{103}$, so $101! \equiv 102! \cdot (102)^{-1} \equiv -1 \cdot -1 \equiv 1 \pmod{103}$. Thus $100! \equiv 101! \cdot (101)^{-1} \equiv (101)^{-1} \equiv (-2)^{-1} \equiv (-1)(2)^{-1} \equiv (102)(2)^{-1} \equiv \boxed{51} \pmod{103}$.

I-7. Let G be the point where $\overline{BF}$ is tangent to circle O. Let $m\angle GOF = x°$ and $m\angle BOG = y°$, so $m\angle BOF = (x+y)°$. $m\angle CBG + m\angle GFE = 360° - 130° = 230°$, so $m\angle CBG + m\angle GFE + m\widehat{CG} + m\widehat{GE} = 360° \to m\widehat{CG} + m\widehat{GE} = 130° = (2x+2y)° \to m\angle BOF = \boxed{65°}$.



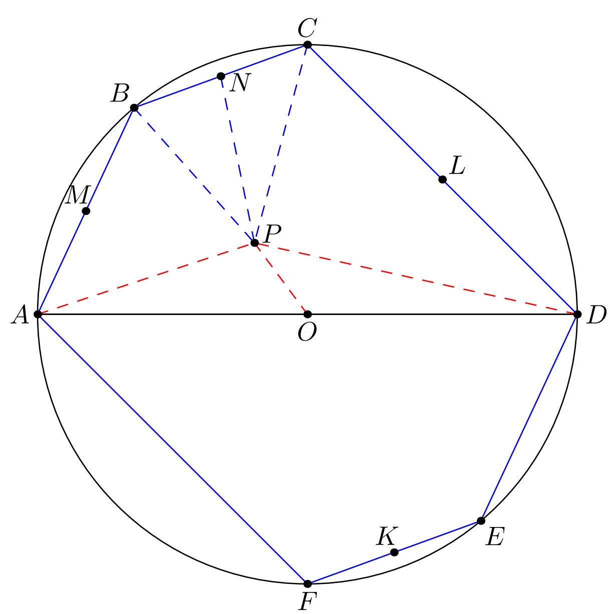
I-8. Note that

$$\frac{\binom{8}{k}}{b_{k-1}b_k} = \frac{1}{b_{k-1}} - \frac{1}{b_k}$$

and so the sum in question equals $\frac{1}{b_1} - \frac{1}{b_7} = \frac{1}{1+8} - \frac{1}{2^8-1} = \boxed{\dfrac{82}{765}}$.

I-9. $104060001 = 104060401 - 400 = 101^4 - 20^2 = (101^2 - 20)(101^2 + 20) = 10181 \cdot 10221$. The latter number is divisible by 3, so the largest prime factor must be $\boxed{10181}$.

I-10. Recall that if XYZ is a triangle and M is the midpoint of $\overline{YZ}$ then $XY^2 + XZ^2 = \frac{1}{2}(YZ^2 + (2XM)^2)$. (This can be seen as, say, a consequence of the parallelogram lemma, or by Stewart's Theorem.) Let O be the center of Γ and $2R$ its diameter. Note that $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are all diameters of Γ.



Applying the above observation to triangles PAD, PBE, PCF gives

$$\frac{1}{2}\left((2R)^2 + (2PO)^2\right) = PA^2 + PD^2 = PB^2 + PE^2 = PC^2 + PF^2.$$

On the other hand

$$PA^2 + PB^2 = \frac{1}{2}(4^2 + 10^2) = 58$$
$$PB^2 + PC^2 = \frac{1}{2}(6^2 + 14^2) = 116$$
$$PC^2 + PD^2 = \frac{1}{2}(8^2 + 18^2) = 194$$

From this we can compute $PA^2 + PB^2 + PC^2 + PD^2 = 252$ and hence $PA^2 + PD^2 = 136$. So $PE^2 + PF^2 = 3 \cdot 136 - 252 = 156$. Hence

$$156 = \frac{1}{2}(6^2 + (2PK)^2) \to (2PK)^2 = 276 \to PK = \boxed{\sqrt{69}}.$$

ARML Local 2016 Relay Problems

R1-1. Compute the number of ordered pairs of integers (x, y) which satisfy $x^2 + 6x + y^2 = 4$.

R1-2. Let T = TNYWR. In triangle ABC, $AB = 2T$ and $BC = 3T$. Let the angle bisectors of triangle ABC meet at point O. If $m\angle AOB = 110^\circ$ and $m\angle BOC = 130^\circ$, compute the area of triangle ABC.

R2-1. An eight-team soccer league is split into two divisions of four teams each. In a season, each team plays two games against each team in its division and one game against each team in the other division. Compute the total number of games played in the league in a season.

R2-2. Let T = TNYWR. Compute the number of distinct prime factors of $1600 - T$.

R2-3. Let T = TNYWR. A fair standard six-sided die is rolled n times. Suppose the probability of rolling exactly $T - 1$ perfect squares is equal to the probability of rolling exactly T perfect squares. Compute the greatest possible value of the positive integer n.

R3-1. Given that $x + 2x + 3x + \cdots + 300x = 3 + 6 + 9 + \cdots + 300$, compute x.

R3-2. Let T = TNYWR. A, R, M, and L are distinct non-zero digits such that $\underline{A}\ \underline{R} \times \underline{M}\ \underline{L} = \underline{L}\ \underline{L}\ \underline{L}$. There are two 4-tuples (A, R, M, L) that satisfy this equation. Pass back the 4-digit number $\underline{A}\ \underline{R}\ \underline{M}\ \underline{L}$ such that $(\underline{M}0\underline{R}) \times T$ is an integer.

R3-3. Let T = TNYWR. For each positive integer x, let $P(x)$ be the product of the digits of x and $S(x)$ be the sum of the digits of x. Let B be the set of all positive integers x such that $P(x) = P(T)$ and $S(x) = S(T)$. Compute the number of elements in B.

R3-4. Let T = TNYWR. A sequence of T positive integers $s_1, s_2, \ldots, s_T$ has the property that the sum of every three consecutive terms in the sequence is T. Compute the least possible value of $s_1 + s_2 + \cdots + s_T$.

R3-5. Let T = TNYWR. Patty's outfits consist of a hat, a blouse, and a skirt. Patty has x different hats, y different blouses, and $x + y$ different skirts. If the total number of different outfits Patty can make is T, compute the number of skirts Patty has.

R3-6. Let T = TNYWR. Compute $\sum_{k=1}^{T} \left\lceil \sqrt{k} \right\rceil = \left\lceil \sqrt{1} \right\rceil + \left\lceil \sqrt{2} \right\rceil + \cdots + \left\lceil \sqrt{T} \right\rceil$.

ARML Local 2016 Relay Solutions

R1-1. $x^2+6x+y^2=4 \to x^2+6x+9=13-y^2 \to (x+3)^2=13-y^2$. When $y=\pm 2$ or ± 3, the right-hand side of the equation is a perfect square, so there are two values of x that satisfy the equation for each value of y, hence there are $\boxed{8}$ different ordered pairs.

R1-2. Let $m\angle A = 2x^\circ$, $m\angle B = 2y^\circ$, and $m\angle C = 2z^\circ$. Then $m\angle AOC = 360^\circ - 110^\circ - 130^\circ = 120^\circ$. Similarly, $x+y+110=180$, $x+z+120=180$, and $y+z+130=180$. This system has the solution $x=40$, $y=30$, and $z=20$. Therefore the area of $ABC = \frac{1}{2}\cdot AB\cdot BC\cdot \sin B = 3T^2\sin(60^\circ) = \frac{3\sqrt{3}T^2}{2}$. As $T=8$, the answer is $\boxed{96\sqrt{3}}$.

R2-1. A team plays six games against teams in its division and four games against teams in the other division, for a total of 10 games per team. There are eight teams for a total of 80 games, but each game is played by two teams, so there are a total of $\boxed{40}$ games played in the league per season.

R2-2. As $T=40$, $1600-40=40(40-1)$. The prime factors of 40 are 2 and 5, and the prime factors of 39 are 3 and 13, for a total of $\boxed{4}$ distinct prime factors.

R2-3. We have $T=4$. Assume $n\geq 3$ for convenience (noting $n=1$ and $n=2$ are trivial solutions). This amounts to solving the equation

$$\binom{n}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{n-3} = \binom{n}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{n-4}.$$

Dividing out the common terms, we obtain that $2=\frac{n-3}{4}$, and solving gives $n=\boxed{11}$.

R3-1. Note that $x=\frac{3+6+9+\cdots+300}{1+2+3+\cdots+300} = 3\cdot\frac{1+2+3+\cdots+100}{1+2+3+\cdots+300} = 3\cdot\frac{\frac{1}{2}\cdot 100\cdot 101}{\frac{1}{2}\cdot 300\cdot 301} = \boxed{\frac{101}{301}}$.

R3-2. Note that $\underline{L}\ \underline{L}\ \underline{L} = 111L = (3\cdot 37)L$. Therefore either one of the factors $\underline{A}\ \underline{R}$ and $\underline{M}\ \underline{L}$ is 37 and the other is $3L$, or one of the factors is 74 and the other is $3L/2$. We can discard this second case, as the only even value of L that results in a two-digit number is $L=8$, and $74\cdot 12 = 888$, but neither factor shares its units digit with 888. If $\underline{M}\ \underline{L}$ is 37, then the other factor is $3L$, so $21\cdot 37 = 777$. If $\underline{A}\ \underline{R}$ is 37, the only value of L such that $37L$ has L as its units digit is 5, so $37\cdot 15 = 555$. Because $T=\frac{101}{301}$, the first case results in $(\underline{M}0\underline{R})\times T$ being an integer, so the answer is $\boxed{2137}$.

R3-3. Consider the following two rules that apply to elements of B:

- Rule 1: If $x\in B$, so is any integer formed by a permutation of the digits of x.

- Rule 2: If $x \in B$ contains one or more digits whose product can be represented by another product of digits with the same sum, then those digits can swap out and the resulting integer (and its permutations) will also be in B. As an example, if x contains 4 as one of its digits, then any integer formed by replacing the 4 with 22 is also in B, and vice versa. Also, if x contains 6 as one of its digits, then any integer formed by replacing the 6 with 123 is also in B, and vice versa, and so forth.

Thankfully, $T = 2137$, and the only application of Rule 2 that applies is that 213 can be replaced by 6 and so $67 \in B$ and $76 \in B$ along with the $4! = 24$ permutations of 2137, resulting in $\boxed{26}$ integers in B.

R3-4. For this condition to hold, the sequence must have the form $a, b, c, a, b, c, \ldots$, with $a+b+c = T$. If $r = \lfloor \frac{T}{3} \rfloor$, then the sum is either rT, $rT+a$, or $rT+a+b$, depending on the value of $T \pmod 3$. Because $T = 26$, the sum is $8 \cdot 26 + a + b$, which is minimized when $a = b = 1$, so the answer is $\boxed{210}$.

R3-5. The total number of outfits is $xy(x+y)$, meaning that T can be broken into three factors whose product is T, one of which is the sum of the other two. As $T = 210 = 3 \cdot 7 \cdot 10$, the answer is $\boxed{10}$.

R3-6. For the integers j such that $(k-1)^2 + 1 \leq j \leq k^2$, $\lceil \sqrt{j} \rceil = k$. There are $2k-1$ numbers in that range, so if N^2 is the largest perfect square less than or equal to T, then $\sum_{k=1}^{T} \left\lceil \sqrt{k} \right\rceil = \sum_{k=1}^{N^2} \left\lceil \sqrt{k} \right\rceil + \sum_{k=N^2+1}^{T} \left\lceil \sqrt{k} \right\rceil = \sum_{j=1}^{N} (2j-1)j + (T - N^2)(N+1)$. As $T = 10$, $N = 3$, and the sum is $(1 \cdot 1) + (3 \cdot 2) + (5 \cdot 3) + (10 - 9) \cdot 4 = \boxed{26}$.

ARML Local 2016 Tiebreaker Problem

TB-1. A sequence a_n is defined for positive integers n as follows: $a_n = 1$ if $n = \frac{N^2+N+2}{2}$ for some non-negative integer N, and $a_n = a_{n-1} + 1$ otherwise. Compute a_{2016}.

ARML Local 2016 Tiebreaker Solution

TB-1. Note that the sequence resets to 1 on the number following every triangular number (because $\frac{N^2+N+2}{2} = \frac{N(N+1)}{2} + 1$), so the sequence is $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, \ldots$. Accordingly, a_n is 1 more than n minus the largest triangular number less than or equal to n. Because 2016 is the 63rd triangular number, $a_{2016} = \boxed{63}$.

Answers to ARML Local 2016

Team Round:

T-1. 17	T-2. 981	T-3. $-\frac{2015}{2016}$	T-4. -1	T-5. 16
T-6. 16	T-7. 64	T-8. 907	T-9. 304	T-10. 334
T-11. 47850	T-12. $\frac{27}{32768}$	T-13. $\frac{2}{3}$	T-14. 40	T-15. 3445

Individual Round:

I-1. 52 or 53	I-2. $\frac{17}{81}$	I-3. -3	I-4. $4-2\sqrt{3}$	I-5. 9
I-6. 51	I-7. 65	I-8. $\frac{82}{765}$	I-9. 10181	I-10. $\sqrt{69}$

Relay Round:

R1-1. 8	R1-2. $96\sqrt{3}$				
R2-1. 40	R2-2. 4	R2-3. 11			
R3-1. $\frac{101}{301}$	R3-2. 2137	R3-3. 26	R3-4. 210	R3-5. 10	R3-6. 26

Tiebreaker:

TB-1. 63

ARML Local 2017

Contents

ARML Local 2017 Team Problems

T-1. A fair six-sided die has faces with values 0, 0, 1, 3, 6, and 10. Compute the smallest positive integer that *cannot* be the sum of four rolls of this die.

T-2. For a positive integer k, let z_k be the number of terminal zeroes of the product $1!2!\cdots k!$. For example, $z_6 = 2$ because $1!2!3!4!5!6! = 24883200$. Compute z_{100}.

T-3. Let $ARML$ be a square of side length 5. A point B on side $\overline{MR}$ and C on side $\overline{ML}$ are selected uniformly at random and independent of one another. Compute the expected area of triangle ABC.

T-4. The edges of regular hexagon $ABCDEF$ are made of mirrors. A laser is fired from A toward the interior of edge $\overline{CD}$, striking it at point G. The laser beam reflects off the interior of exactly one additional edge and returns to A. Compute $\tan(\angle DAG)$.

T-5. Let S be the set of lattice points $\{(x, y) : x, y \in \mathbb{Z}, 0 \leq x \leq 3, 0 \leq y \leq 4\}$. Compute the number of subsets of S of 4 lattice points that form the vertices of a square.

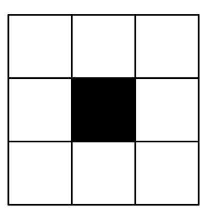
T-6. If A is an acute angle such that $\sin 15^\circ + \cos 15^\circ = \sqrt{2}\sin A$, compute $\cos A$.

T-7. Consider the following algorithm for a given non-negative integer n:
Step 1: Initialize $k = 1$.
Step 2: Replace n with the product of its digits.
Step 3: If $n \leq 9$ return k and exit. If $n > 9$, increment k by 1 and return to Step 2.
For example, if $n = 125$, then the algorithm returns 2 because 125 is replaced by $1 \cdot 2 \cdot 5 = 10$ which is replaced by $1 \cdot 0 = 0$, at which point the algorithm returns 2 and exits. Compute the smallest value of n such that this algorithm returns 3.

T-8. Three co-planar squares, $BAHT$, $CAIN$, and $BCGY$ have areas 16, 16, and 32, respectively. If the squares only intersect pairwise at the vertices A, B, and C, compute the area of the convex hexagon $THINGY$.

T-9. Compute the coefficient of x^8 in the expansion of $(x^2 + x + 1)^8$ after combining like terms.

T-10. Consider the following 3×3 grid with its center square removed, as shown.



Compute the number of distinct ways to fill in the grid with the integers 1 through 8, each appearing exactly once, so that:

- In each of the three rows, the entries are increasing from left to right.
- In each of the three columns, the entries are increasing from top to bottom.

T-11. Kevin and Tarsha are playing a game with four fair standard six-sided dice. The four dice are rolled and if the numbers appearing on top of the four dice are all different, Kevin pays Tarsha \$1. If not, Tarsha pays Kevin \$$k$ dollars. Compute the value of k such that the game is fair, in other words, the expected value of each play of the game to both players is zero.

T-12. Compute the minimum value of ab such that $\log_2(a^4b^{-3}) = 3$ and $\log_2(a^4b^3) = 9$.

T-13. Compute

$$\sum_{n=3}^{10} \frac{20}{(n-2)(n+2)}.$$

.

T-14. The six-digit number 142857 has the property that moving the rightmost digit to the left of the number results in multiplying the number by five ($142857 \cdot 5 = 714285$). Compute the smallest six-digit number with the property that moving the rightmost digit to the left of the number results in multiplying the number by four.

T-15. Compute the number of distinct sequences $(x_1, x_2, \ldots, x_{14})$ with the following properties:

- $x_k \in \{1, 2, \ldots, 14\}$ for each $k = 1, 2, \ldots, 14$.
- The number $20x_{k+1}^2 - 17x_k^2$ is divisible by 14 for each $k = 1, 2, \ldots, 13$.

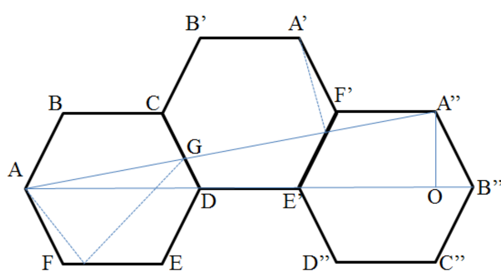
ARML Local 2017 Team Solutions

T-1. The numbers between 1 and 23, inclusive, can be the sum of at most 3 non-zero die rolls ($1, 1+1, 3, 3+1, 3+1+1, 6, 6+1, 6+1+1, 6+3, 10, 10+1+1, 6+6, 10+3, 10+3+1, 6+6+3, 10+6, 10+6+1, 6+6+6, 10+6+3, 10+10, 10+10+1, 10+6+6, 10+10+3$). The numbers 1 through 33 inclusive can be achieved via four rolls by adding an additional roll of 10 to the up to three rolls that sum to 14 to 23. Accordingly, $\boxed{34}$ is the smallest positive integer that cannot be the sum of four rolls of this die.

T-2. If each factorial is expanded out to its individual terms, there would be 96 5s, 91 10s, ..., 6 95s, and 1 100 terms. Each multiple of five term adds one zero to the end of z_{100}, with each multiple of 25 contributing an additional zero to the end of z_{100}. The total number of zeros is $(96+91+\cdots+6+1)+(76+51+26+1)=\boxed{1124}$.

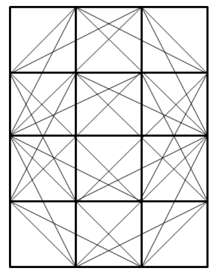
T-3. Put $A=(0,0)$, let $B=(5,y)$ and $C=(x,5)$. Then the area of ABC is equal to $\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 5 & y \\ 1 & x & 5 \end{vmatrix} = \frac{1}{2}(25-xy)$. Because x and y are independent, $E[xy]=E[x]E[y]=(5/2)^2=25/4$. The expected area of ABC is $\frac{1}{2}\left(25-\frac{25}{4}\right)=\boxed{\dfrac{75}{8}}$.

T-4. Reflect the hexagon $ABCDEF$ across the segment $\overline{CD}$ to get the hexagon $A'B'CDE'F'$. In order to strike exactly one additional side and return to A, the laser should strike $\overline{EF}$ next before returning to A. Reflecting the second hexagon about $\overline{E'F'}$ to get the hexagon $A''B''C''D''E'F'$, the segment $\overline{AA''}$ traces the path of the laser beam when reflected back upon the initial hexagon as shown below.



Dropping the perpendicular from A'' to $\overline{AB''}$ at O, $AO=\frac{9}{2}AB$ and $A''O=\frac{\sqrt{3}}{2}AB$, and $\tan(\angle DAG)=\tan(\angle OAA'')=\frac{\sqrt{3}/2}{9/2}=\boxed{\dfrac{\sqrt{3}}{9}}$.

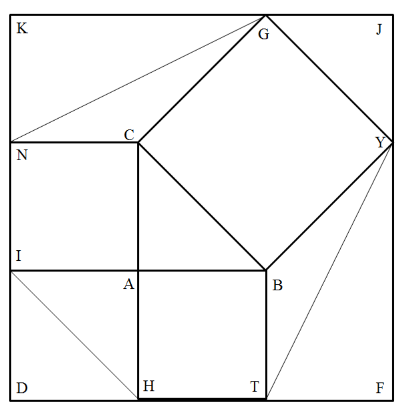
T-5. In the 3×4 grid, there are 12 squares with side length 1, 6 of side length 2, and 2 of side length 3. There are also squares with edges not aligned with the grid; 6 of side length $\sqrt{2}$ and 4 of side length $\sqrt{3}$, for a total of $\boxed{30}$.



T-6. Multiplying both sides of the equation by $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}\sin 15^\circ + \frac{\sqrt{2}}{2}\cos 15^\circ = \sin A \rightarrow \sin A = \cos 45^\circ \sin 15^\circ + \sin 45^\circ \cos 15^\circ = \sin(45^\circ + 15^\circ) = \sin(60^\circ)$. Because A is acute, $A = 60^\circ$, so $\cos A = \boxed{\dfrac{1}{2}}$.

T-7. For this process, all integers up to 24 return 1. The integers 25 to 29 return 2. The smallest integer for which the product of its digits is between 25 and 29, inclusive, is $\boxed{39}$.

T-8. As the three squares have side lengths of 4, 4, and $4\sqrt{2}$, it is clear that ABC is an isosceles right triangle. Let $DFJK$ be the square of side length 12 for which $\overline{HT}$ is a subsegment of $\overline{DF}$ and $\overline{IN}$ is a subsegment of $\overline{DK}$. Then $[THINGY] = [DFJK] - [HID] - [FTY] - [JGY] - [GNK] = 144 - 8 - 16 - 8 - 16 = \boxed{96}$.



T-9. There are five ways to represent x^8 as the product of exactly eight terms of the form x^2, x, and 1: $(x^2)^4(x)^0(1)^4$, $(x^2)^3(x)^2(1)^3$, $(x^2)^2(x)^4(1)^2$, $(x^2)^1(x)^6(1)^1$, and $(x^2)^0(x)^8(1)^0$. The coefficient of x^8 will be $\binom{8}{4,0,4} + \binom{8}{3,2,3} + \binom{8}{2,4,2} + \binom{8}{1,6,1} + \binom{8}{0,8,0} = \frac{8!}{4!0!4!} + \frac{8!}{3!2!3!} + \frac{8!}{2!4!2!} + \frac{8!}{1!6!1!} + \frac{8!}{0!8!0!} = 70 + 560 + 420 + 56 + 1 = \boxed{1107}$.

T-10. First place 1 and 8 in the upper-left and bottom-right. WLOG 2 is the entry to the right of 1 (by symmetry the final answer is twice the number of configurations with 2 to the right of 1). Then, the grid is filled in as follows:

1	2	a
x	■	b
y	z	8

Observe that $x \le y \le z$, $a \le b$, and $x \le b$. There are $\binom{5}{2}$ ways to pick a and b, and all of them result in valid configurations except $(a, b) = (3, 4)$. Hence the final answer is $2\left(\binom{5}{2} - 1\right) = \boxed{18}$.

T-11. Rolling the dice in sequence, it is clear to see that the probability that all four die rolls are different is $\frac{6\cdot5\cdot4\cdot3}{6\cdot6\cdot6\cdot6} = \frac{5}{18}$. The game is fair provided $\frac{5}{18}(1) + \frac{13}{18}(-k) = 0 \to k = \boxed{\dfrac{5}{13}}$.

T-12. Note that while b must be positive, a may be negative. The two equations in the problem are equivalent to $2\log_2(a^2) - 3\log_2(b) = 3$ and $2\log_2(a^2) + 3\log_2(b) = 9$. Adding the two equations together gives $4\log_2(a^2) = 12 \to \log_2(a^2) = 3 \to \log_2(b) = 1$. Thus $b = 2$ and $a = \pm 2\sqrt{2}$, so the minimum value of ab is $\boxed{-4\sqrt{2}}$.

T-13. $\frac{20}{(n-2)(n+2)} = \frac{5}{n-2} - \frac{5}{n+2}$, so

$$\begin{aligned}
\sum_{n=3}^{10} \frac{20}{(n-2)(n+2)} &= 5\left(\sum_{n=3}^{10}\frac{1}{n-2} - \sum_{n=3}^{10}\frac{1}{n+2}\right) \\
&= 5\left(\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{8}\right) - \left(\frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{12}\right)\right) \\
&= 5\left(\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12}\right)\right) \\
&= 5\left(\frac{1980 + 990 + 660 + 495}{1980} - \frac{220 + 198 + 180 + 165}{1980}\right) \\
&= 5\left(\frac{4125}{1980} - \frac{763}{1980}\right) = \frac{3362}{396} = \boxed{\frac{1681}{198}}.
\end{aligned}$$

T-14. Let N be the six-digit number with the property described in the problem. Let a_0 be the rightmost digit of N such that $N = 10k + a_0$, where $10000 \le k \le 99999$. Therefore $4N = 10^5 a_0 + k$, and combining the two equations gives $4(10k + a_0) = 10^5 a_0 + k \to 39k = 99996a_0 \to k = 2564a_0$. k is greater than or equal to 10000 for the first time when $a_0 = 4$, and the resulting six-digit number is $\boxed{102564}$.

T-15. By the Chinese Remainder Theorem, it suffices to look modulo 2 and modulo 7. Modulo 2, there are clearly only two solutions because x_k must be even for $k = 1, 2, \dots 13$.

Modulo 7, there are two possible situations:

- The values of $x_k^2 \pmod 7$ cycle between 1, 4, 2 in that order. There are 3 ways to choose the value x_1 modulo 7, and afterwards for each k there are 2 ways to choose x_k based on x_{k-1}. Hence $3 \cdot 2^{14}$ sequences in this case.
- The value $x_k^2 \pmod 7$ is always zero. There is only 1 sequence in this case, the constant zero sequence.

So the final answer is $2\,(3 \cdot 2^{14} + 1) = \boxed{98306}$.

ARML Local 2017 Individual Problems

I-1. A positive integer m is *stable* if $m = 2^n - n^2$ for some positive integer n. Compute the number of stable positive integers less than 2017.

I-2. Compute the area of the quadrilateral with vertices at $(1, 1)$, $(4, 7)$, $(5, 3)$, and $(2, 0)$.

I-3. The sum of the squares of five consecutive positive integers, the largest integer being n, is equal to the sum of the squares of the next four consecutive integers. Compute n.

I-4. The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the sum of its coefficients is equal to the sum of its roots and is also equal to the product of its roots. If $P(0) = 4$, compute b.

I-5. $ABCDEFG$ is a pyramid with a hexagonal base. Compute the number of distinct ways all seven vertices of $ABCDEFG$ can be colored one of either red, blue, or green so that no vertices that share an edge are identically colored.

I-6. In triangle ABC, if $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, compute the smallest possible value of $\cos C$.

I-7. Compute $\log_2(3136) - \log_2(1764) + \log_2(900) - \log_2(400) + \log_2(144) - \log_2(36)$.

I-8. The roots of $10x^2 - 14x + k$ are $\sin\alpha$ and $\cos\alpha$ for some real value of α. Compute k.

I-9. Compute the rightmost non-zero digit in the base-8 expansion of 17!.

I-10. Let S be a set of 100 points inside a square of side length 1. An ordered pair of not necessarily distinct points (P, Q) is *bad* if $P \in S$, $Q \in S$ and $|PQ| < \frac{\sqrt{3}}{2}$. Compute the minimum possible number of bad ordered pairs in S.

ARML Local 2017 Individual Solutions

I-1. Let $f(n) = 2^n - n^2$. The initial values of $f(n)$ are $f(1) = 1$, $f(2) = 0$, $f(3) = -1$, and $f(4) = 0$. For $n \geq 5$ the function f is increasing, and $f(11) = 1927$ and $f(12) = 3952$. So $f(1)$, $f(5)$, $\ldots$, $f(11)$ correspond to stable positive integers less than 2017, of which there are $\boxed{8}$.

I-2. Break the quadrilateral into two triangles; then the area is equal to

$$\frac{1}{2}\left(\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 5 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 3 \\ 1 & 4 & 7 \end{vmatrix}\right) = \frac{1}{2}(6+18) = \boxed{12}.$$

I-3. The property stated in the problem translates into $(n-4)^2+(n-3)^2+(n-2)^2+(n-1)^2+n^2 = (n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2$. Pairing the $(n-k)^2$ and the $(n+k)^2$ terms on both sides of the equation, the n^2 and constant terms cancel, leaving $n^2 - 20n = 20n \to n^2 - 40n = 0 \to n = \boxed{40}$. Note that the case $n = 0$ is omitted as the five consecutive integers must be positive.

I-4. The sum of the coefficients of P is $1 + a + b + c$, the sum of its roots is $-a$ and the product of its roots is $-c$. As $P(0) = 4$, it follows that $a = c = 4$ and $1 + 4 + b + 4 = -4 \to b = \boxed{-13}$.

I-5. Without loss of generality, let $ABCDEF$ be the base of the pyramid. Once a color is assigned to G, the colors of $ABCDEF$ must alternate between the two remaining colors. As there are three ways to pick the color for G and then two ways to assign colors to the remaining vertices, the answer is $\boxed{6}$.

I-6. Noting that $\cos C = \cos(\pi - (A+B)) = \cos\pi\cos(A+B) + \sin\pi\sin(A+B) = -\cos(A+B)$, $-\cos(A+B) = -(\cos(A)\cos(B) - \sin(A)\sin(B))$. To minimize $\cos C$, $\cos(A)\cos(B)$ should be positive, so $\cos C = -\left(\frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13}\right) = \boxed{-\dfrac{33}{65}}$.

I-7. It is hopefully observed that the terms in the alternating sum are of the form $(n(n+1))^2$ for $2 \leq n \leq 7$. The sum equals

$$\begin{aligned} &2\left(\log_2(56) - \log_2(42) + \log_2(30) - \log_2(20) + \log_2(12) - \log_2(6)\right) \\ &= 2\left((\log_2(8) + \log_2(7)) - (\log_2(7) + \log_2(6)) + (\log_2(6) + \log_2(5))\right. \\ &\qquad \left. - (\log_2(5) + \log_2(4)) + (\log_2(4) + \log_2(3)) - (\log_2(3) + \log_2(2))\right) \\ &= 2\left(\log_2(8) - \log_2(2)\right) = \boxed{4}. \end{aligned}$$

I-8. Noting that $\sin\alpha + \cos\alpha = \frac{7}{5}$, $\frac{49}{25} = (\sin\alpha + \cos\alpha)^2 = (\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha) = 1 + 2\left(\frac{k}{10}\right) \to k = \boxed{\dfrac{24}{5}}$.

I-9. The prime factorization of 17! is

$$(2)(3)(2^2)(5)(2\cdot3)(7)(2^3)(3^2)(2\cdot5)(11)(2^2\cdot3)(13)(2\cdot7)(3\cdot5)(2^4)(17) = 2^{15}\cdot3^6\cdot5^3\cdot7^2\cdot11\cdot13\cdot17.$$

Because of the 2^{15} term, $17!$ will have 5 terminal zeroes when written in base 8. Accordingly, it is necessary to determine the product of the remaining terms in the prime factorization modulo 8 to determine the rightmost non-zero digit. Each terms' remainder modulo 8 is, in order, 1, 5, 1, 3, 5, and 1, for a product of 75, which has a remainder of $\boxed{3}$ modulo 8. Note: $17! = 355687428096000_{10} = 12067735663300000_8$.

I-10. First, note that among any five points there is a bad pair. To see this, divide the unit square into four identical sub-squares, and observe that any two points in the same sub-square are separated by a distance of at most $1/\sqrt{2}$.

Consider a graph with the vertices representing the 100 points of S and an edge between two vertices if they are *not* a bad pair. Because of the above observation, the graph contains no complete subgraph on five vertices (in other words, no set of five vertices has every vertex joined to every other vertex in the set). Consequently, by Turán's theorem, the graph that contains no complete subgraph of five vertices that maximizes the number of edges in the graph (in other words, minimizes the number of bad pairs) has the vertices partitioned into 4 groups of 25 and connects every vertex to every other vertex provided they are in different groups. This corresponds to a construction of S where each corner of the square contains 25 points of S, for example. For this arrangement, there are $4 \cdot 25^2 = \boxed{2500}$ bad ordered pairs, and this is the fewest possible.

ARML Local 2017 Relay Problems

R1-1. If x and y are non-negative integers such that $x < y$ and $x!y! = 10!$, compute the maximum possible value of x.

R1-2. Let T = TNYWR. $HAIRNETS$ is an equilateral concave octagon of side length T. The interior angle measurements of H, I, N, and T are 60 degrees, and the interior angle measurements of A, R, E, and S are 210 degrees. Compute the area of $HAIRNETS$.

R2-1. The side lengths of a triangle are $\sqrt{20}$, $\sqrt{17}$, and x. Compute the greatest possible integer value of x.

R2-2. Let T = TNYWR. The finite set S has exactly T distinct subsets each containing an even number of elements. Compute the number of elements in S.

R2-3. Let T = TNYWR. Let S be the set of non-negative integers less than T^3. Compute the (base-10) sum of the digits of all of the elements of S written in base T.

R3-1. Compute the number of distinct ways to erase two of the decimal digits of 9876543210 and obtain an eight-digit number that is divisible by 9.

R3-2. Let T = TNYWR. ABC is an isosceles right triangle. If the longest median has length T, compute the area of ABC.

R3-3. Let T = TNYWR. The polynomial $x^3 + 3x^2 + px + T$ is evenly divisible by $x + 2$. Compute p.

R3-4. Let T = TNYWR. If p and q are the roots of $x^2 + x + 3$, then $x^2 + bx + c$ has roots $p + T$ and $q + T$. Compute c.

R3-5. Let T = TNYWR. Phil has a stack of money consisting of 5-dollar and 20-dollar bills worth 780 dollars in total. If there are T bills in total, compute the number of 5-dollar bills.

R3-6. Let T = TNYWR. Compute the remainder when the eight-digit number $\underline{T}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{2}\ \underline{3}\ \underline{4}\ \underline{T}$ is divided by 13.

ARML Local 2017 Relay Solutions

R1-1. Observing that $10 \cdot 9 \cdot 8 = 720 = 6!$, $10! = 6!7!$, so the answer is $\boxed{6}$.

R1-2. The octagon $HAIRNETS$ consists of a square ($ARES$) and four equilateral triangles (SHA, AIR, RNE, and ETS). The total area is $T^2 + 4\frac{\sqrt{3}T^2}{4}$. As $T = 6$, the answer is $\boxed{36 + 36\sqrt{3}}$.

R2-1. By the Triangle Inequality, the sum of two of the side lengths cannot exceed the length of the third side. Accordingly, if x is the length of the longest side, it is bounded above by $\sqrt{20} + \sqrt{17}$. As both values are between 4 ($\sqrt{16}$) and 4.5 ($\sqrt{20.25}$), their sum is greater than 8 but less than 9, so the greatest possible integer value of x is $\boxed{8}$.

R2-2. The set S has $2^{|S|}$ subsets, of which $2^{|S|-1}$ contain an even number of elements. As $T = 8$, $|S| - 1 = 3 \rightarrow |S| = \boxed{4}$.

R2-3. Including trailing zeroes, the elements of S when written in base T are $\{000_T, 001_T, \ldots, \underline{(T-1)}\ \underline{(T-1)}\ \underline{(T-1)}_T\}$. By symmetry, every digit between 0 and $T - 1$ inclusive appears in each position T^2 times, so the sum of all of the digits is $3 \cdot T^2 \cdot \frac{(T-1)(T)}{2}$. As $T = 4$, the answer is $3 \cdot 16 \cdot 6 = \boxed{288}$.

R3-1. The given number has sum of digits 45, so two digits whose sum is divisible by 9 should be erased. There are $\boxed{5}$ ways to do this, namely $0+9$, $1+8$, $2+7$, $3+6$, and $4+5$.

R3-2. Let A be the right angle of the triangle. The longest median goes from one of the vertices of the hypotenuse (say C) to the opposite side (say a point D on $\overline{AB}$). For simplicity, let $AB = 2x$, then the area of ABC is $2x^2$. Additionally, $CD^2 = CA^2 + AD^2 = 5x^2$, so the area of the triangle is $\frac{2}{5}$ the square of the length of the longest median. As $T = 5$, the area is $\boxed{10}$.

R3-3. As -2 is a root of the polynomial, $(-2)^3 + 3(-2)^2 + p(-2) + T = 0 \rightarrow 2p = T + 4$. As $T = 10$, $p = \boxed{7}$.

R3-4. Noting that $x^2 + bx + c = (x - (p + T))(x - (q + T)) = x^2 - (p + q + 2T)x + (p + T)(q + T)$, then $c = (p+T)(q+T) = pq + (p+q)T + T^2$. The sum and product of the roots of $x^2 + x + 3$ are -1 and 3, respectively, so $c = T^2 - T + 3$. As $T = 7$, $c = 49 - 7 + 3 = \boxed{45}$.

R3-5. Let F be the number of 5-dollar bills. Then there are $T - F$ 20-dollar bills and $5F + 20(T - F) = 780$, so $F = \frac{20T-780}{15}$. As $T = 45$, $F = \frac{900-780}{15} = \boxed{8}$.

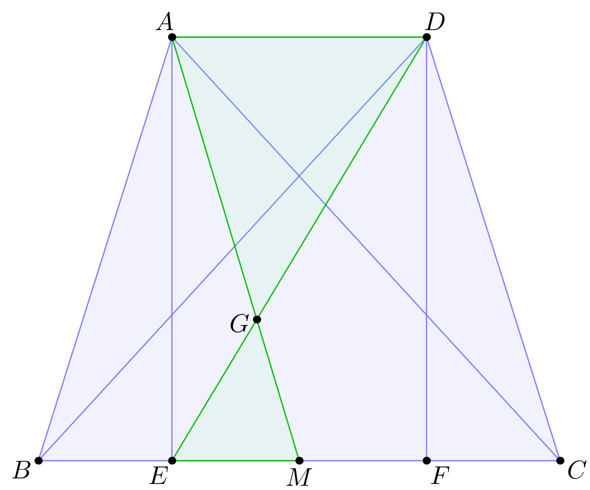
R3-6. Let N be the number above, then $N = 10000001T + 2342340 = 10000001T + 234 \cdot 1001 \cdot 10$. As $1001 = 7 \cdot 11 \cdot 13$, $N \equiv 10000001T \pmod{13}$. As $10000001 \equiv 11 \pmod{13}$, $N \equiv 11T \pmod{13}$. As $T = 8$, $N \equiv 88 \equiv \boxed{10} \pmod{13}$.

ARML Local 2017 Tiebreaker Problem

TB-1. $ABCD$ is an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$ and $AD < BC$. E lies on $\overline{BC}$ so that $\overline{AE} \perp \overline{BC}$ and let M be the midpoint of $\overline{BC}$. Lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{AM}$ meet at G. Given that triangle GEM has area 20 and $AB = AM = 17$, compute the area of triangle ABC.

ARML Local 2017 Tiebreaker Solution

TB-1. First note that $AM = 3GM$ (in particular, G is the centroid of ABC). To see this, let F be the foot of the altitude from D. Then $AEFD$ is a rectangle and $EM = \frac{1}{2}AD$. So $\triangle GEM \sim \triangle GDA$ with ratio 2, hence $AG : GM = 2$ as claimed. Then $[GEM] = 20$ implies $[AEM] = 60$. because $AB = AM$, $[ABM] = 120$, and finally $[ABC] = 2[ABM] = \boxed{240}$.



Answers to ARML Local 2017

Team Round:

T-1. 34 T-2. 1124 T-3. $\dfrac{75}{8}$ T-4. $\dfrac{\sqrt{3}}{9}$ T-5. 30

T-6. $\dfrac{1}{2}$ T-7. 39 T-8. 96 T-9. 1107 T-10. 18

T-11. $\dfrac{5}{13}$ T-12. $-4\sqrt{2}$ T-13. $\dfrac{1681}{198}$ T-14. 102564 T-15. 98306

Individual Round:

I-1. 8 I-2. 12 I-3. 40 I-4. -13 I-5. 6

I-6. $-\dfrac{33}{65}$ I-7. 4 I-8. $\dfrac{24}{5}$ I-9. 3 I-10. 2500

Relay Round:

R1-1. 6 R1-2. $36 + 36\sqrt{3}$

R2-1. 8 R2-2. 4 R2-3. 288

R3-1. 5 R3-2. 10 R3-3. 7 R3-4. 45 R3-5. 8 R3-6. 10

Tiebreaker:

TB-1. 240

ARML Local 2018

Contents

ARML Local 2018 Team Problems

T-1. A sphere with surface area 2118π is circumscribed around a cube and a smaller sphere is inscribed in the cube. Compute the surface area of the smaller sphere.

T-2. $REBS$ is a square of side length 3. If $SU = UO = OR = RT = TX = XE = ED = DI = IB = BM = MA = AS = 1$, compute the sum of the areas of the distinct rectangles whose vertices consist of four of the points $A, M, B, I, D, E, X, T, R, O, U, S$.

T-3. Compute the value of x for which $\log_8(\log_4 x) = \log_{64}(\log_2 x)$.

T-4. For non-negative integers A and B, with $A \geq B$, let $A \star B$ denote the number formed by the concatenation of $(A + B)$, $(A - B)$, and $(A \cdot B)$. For example, $7 \star 5 = 12235$. Compute the number of ordered pairs (A, B), $1 \leq B \leq A \leq 9$, such that $A \star B$ contains at most three distinct digits (to clarify, the example contains four distinct digits).

T-5. Compute the greatest integer n for which $(6!)^n$ is a factor of $60!$.

T-6. The ARML Local staff have ranked each of the 10 individual problems in order of difficulty from 1 to 10. They wish to order the problems so that the problem with difficulty i is before the problem with difficulty $i + 2$ for all $1 \leq i \leq 8$. Compute the number of orderings of the problems that satisfy this condition.

T-7. Let ABC be a triangle with $AB = 5$, $BC = 12$, and an obtuse angle at B. It is given that there exists a point P in plane ABC for which $\angle PBC = \angle PCB = \angle PAB = \gamma$, where γ is an acute angle satisfying $\tan\gamma = \frac{3}{4}$. Compute $\sin\angle ABP$.

T-8. Suppose a, b, and c are real numbers such that a, b, c and $a, b + 1, c + 2$ are both geometric sequences in their respective orders. Compute the smallest possible value of $(a + b + c)^2$.

T-9. Circles ω_1 and ω_2 have radii 5 and 12 respectively and have centers distance 13 apart. Let A and B denote the intersection points of ω_1 and ω_2. A line ℓ passing through A intersects ω_1 again at X and ω_2 again at Y so that $\angle ABY = 2\angle ABX$. Compute XY.

T-10. Compute the sum of all possible values of $a + b$, where a and b are integers such that $a > b$ and $a^2 - b^2 = 2016$.

T-11. If u and v are complex numbers such that $u + v = 9$ and $u^3 + v^3 = 81$, compute $u^2 + v^2$.

T-12. On the planet ARMLia, there are three species of sentient creatures: Trickles, who count in base 3, Quadbos, who count in base 4, and Quinters, who count in base 5. One thing all three species agree on is the magic of the positive integer Zelf, which ends in the digits "11" when represented in all three bases. Compute the least possible value (in base 10) of Zelf.

T-13. Compute the number of ordered pairs of integers (a, b, c) with $a \geq b \geq c$ such that a, b, and c are the side lengths of a non-degenerate triangle with perimeter 218.

T-14. Compute the sum of the real roots of $f(x) = x^6 + 3x^4 - 2018x^3 + 3x^2 + 1$.

T-15. Place in the 10×10 grid of cells below one 4×4 square, two 3×3 squares, three 2×2 squares, and four 1×1 squares so that:

- All square sides are parallel to boundaries of the grid.
- No squares overlap nor share a common boundary, not even a corner point.
- The total number of 1×1 cells covered by squares in each row and column is equal to the number to the right or below the row or column, respectively. The numbered or lettered cells may not be covered by tiles.

Enter on your answer sheet the letter coordinates (given in the first row and column) of the four cells containing the 1×1 squares, for example, if the four corners of the grid contained the 1×1 tiles, you would enter AK, JK, AT, and JT.

	A	B	C	D	E	F	G	H	I	J	
K											3
L											2
M											7
N											4
O											7
P											7
Q											3
R											5
S											7
T											5
	3	6	4	8	7	3	2	3	7	7	

ARML Local 2018 Team Solutions

T-1. If the cube has side length D, the sphere inscribed inside the cube has diameter D, and the circumscribed sphere has diameter $\sqrt{3}D$. The surface area of a cube of side length D is πD^2, so the inscribed sphere has surface area $\frac{1}{3}$ of the circumscribed one, so the answer is $\boxed{706\pi}$.

T-2. The other eight points given are the trisection points for the edges of $REBS$. There are rectangles of five different dimensions: 3×1 (there are six), 3×2 (there are four), 3×3 (there is one), $\sqrt{2} \times \sqrt{8}$ (there are two), and $\sqrt{5} \times \sqrt{5}$ (there are two). The sum of the areas is $6 \cdot 3 + 4 \cdot 6 + 1 \cdot 9 + 2 \cdot 4 + 2 \cdot 5 = \boxed{69}$.

T-3. Let $x = 2^u$, then $\log_2 x = u$ and $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{u}{2}$. Then $\log_8(u/2) = \log_{64}(u) = \frac{\log_8(u)}{\log_8 64} = \frac{\log_8(u)}{2} \to \log_8(u) - \log_8(2) = \frac{\log_8(u)}{2} \to \frac{\log_8(u)}{2} = \frac{1}{3} \to \log_8(u) = \frac{2}{3} \to u = 4 \to x = \boxed{16}$.

T-4. The resulting number will be at most five digits for single-digit values of A and B. The cases that satisfy the conditions of the problem are:

- Three digits long: The sum, difference, and product all being single digits: this happens for $B = 1$, $1 \leq A \leq 8$, $B = 2$, $2 \leq A \leq 4$, and $B = 3$, $A = 3$.
- Four digits: The sum must be a single digit and will be distinct from the difference. Therefore the product must share a digit with either the sum or the difference. This only occurs when $A = 4$ and $B = 3$ (7112).
- Five digits: The tens digit of the sum is always one. If the difference is also one, then either the sum is 11 ($6 \star 5 = 11130$), or the sum shares a digit with the product ($8 \star 7 = 15156$ and $9 \star 8 = 17172$). If the difference is not one, then either the sum is 11 and the product contains a one ($9 \star 2 = 11718$) or the difference and product contain a common digit ($8 \star 2 = 10616$), or the sum and product contain two common digits ($9 \star 9 = 18081$), or the sum, difference, and product contain a common digit ($9 \star 5 = 14445$) .

Counting the cases, the answer is $\boxed{20}$.

T-5. $(6!)^n = 2^{4n}3^{2n}5^n$. Therefore we determine the greatest power of 2, 3, and 5 that divides 60!. In general, the greatest power of p that divides $k!$ is $\sum_{j=1}^{\infty} \lfloor \frac{k}{p^j} \rfloor$. When $p = 2$, this sum is $30 + 15 + 7 + 3 + 1 = 56$, when $p = 3$, this sum is $20 + 6 + 2 = 28$, and when $p = 5$, the sum is $12 + 2 = 14$. Accordingly, $\boxed{14}$ is greatest power of 6! that divides 60! evenly.

T-6. Any acceptable ordering of the problems must have the odd numbered problems and even numbered problems in order of increasing difficulty. Therefore it is merely a question of how many ways two sequences of 5 numbers can be interleaved. There are $C(10, 5) = \boxed{252}$ ways to pick the positions of the questions of odd difficulty, and then a unique ordering of the odd difficulty questions in those positions and the even difficulty questions in the positions left over.

T-7. Note that by Law of Sines on $\triangle ABP$ and $\triangle CBP$,

$$\frac{AB}{\sin \angle APB} = \frac{PB}{\sin \gamma} = \frac{BC}{\sin \angle BPC},$$

from which
$$\sin\angle APB = \frac{AB}{BC}\cdot\sin\angle BPC = \frac{5}{12}\cdot\sin 2\gamma = \frac{5}{12}\cdot\frac{24}{25} = \frac{2}{5}.$$
It follows that
$$\begin{aligned}\sin\angle ABP &= \sin(\angle BAP + \angle APB)\\ &= \sin\angle BAP\cos\angle APB + \sin\angle APB\cos\angle APB\\ &= \frac{2}{5}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{\sqrt{21}}{5} = \boxed{\frac{8+3\sqrt{21}}{25}}.\end{aligned}$$

T-8. Note that the conditions imply $b^2 = ac$ and $(b+1)^2 = a(c+2)$. Subtracting yields $2b+1 = 2a$, so $b+\frac{1}{2} = a$. Now recall that from the first equation $c = \frac{b^2}{a}$, so
$$a+b+c = a+\left(\frac{1}{2}+a\right)+\frac{(\frac{1}{2}+a)^2}{a} = 3a+\frac{1}{4a}-\frac{3}{2}.$$
If a is positive, then this expression is at least $2\sqrt{\frac{4}{3}}-\frac{3}{2} = \sqrt{3}-\frac{3}{2}$ by the AM-GM inequality; in a similar vein, a is negative this expression is at most $-\sqrt{3}-\frac{3}{2}$. From both of these cases we conclude that the smallest possible value of $(a+b+c)^2$ is $(\sqrt{3}-\frac{3}{2})^2 = \boxed{\frac{21}{4}-3\sqrt{3}}$.

T-9. Denote by O_1 and O_2 the centers of ω_1 and ω_2 respectively. Then $O_1A^2 + O_2A^2 = O_1O_2^2$ from the lengths given in the problem statement, so $O_1A \perp O_2A$. Now set $\angle XAO_1 = \alpha$ and $\angle YAO_2 = \beta$; then $\alpha+\beta = 90^\circ$. Furthermore, angle chasing reveals
$$\angle ABX = \tfrac{1}{2}\angle AO_1X = 90^\circ-\alpha \qquad \text{and} \qquad \angle ABY = \tfrac{1}{2}\angle AO_2Y = 90^\circ-\beta,$$
and so $90^\circ-\beta = 2(90^\circ-\alpha) \Rightarrow 2\alpha-\beta = 90^\circ$. Solving this system of equations yields $\alpha = 60^\circ$ and $\beta = 30^\circ$, so
$$XY = XA+AY = 2\cdot 5\cos 60^\circ + 2\cdot 12\cos 30^\circ = \boxed{5+12\sqrt{3}}.$$

T-10. The condition in the problem to be satisfied is equivalent to $(a-b)(a+b) = 2016$. Note that $a-b>0$, so $a+b>0$ also. For $d = a+b$, note that $d \mid 2016$, and furthermore d and $2016/d$ must have the same parity. It follows that $d = 2^a3^b7^c$ for $a\in\{1,2,3,4\}$, $b\in\{0,1,2\}$, and $c\in\{0,1\}$; the requested sum is
$$(2+2^2+2^3+2^4)(1+3+3^2)(1+7) = 30\cdot 13\cdot 8 = \boxed{3120}.$$

T-11. $81 = (u+v)(u^2-uv+v^2) \to (u^2-uv+v^2) = 9$. $u+v = 9 \to u^2+2uv+v^2 = 81$. Combining the two equations to eliminate the u^2 and v^2 terms gives $3uv = 72 \to uv = 24$, so $u^2+v^2 = 9+uv = \boxed{33}$.

T-12. Let z be the least possible value of Zelf. This becomes a Chinese Remainder Theorem problem, where $z \equiv (b+1) \bmod b^2$ for each base b. Looking first at Trickles and Quadbos, $z\equiv 4 \bmod 9$ and $z \equiv 5 \bmod 16$. Using the Euclidean algorithm, we establish that $4\cdot 16 - 7\cdot 9 = 1$, so $z \equiv 4\cdot 4\cdot 16 - 5\cdot 7\cdot 9 \equiv -59 \bmod 144 \equiv 85 \bmod 144$. Additionally, $z\equiv 6 \bmod 25$, and $4\cdot 144 - 23\cdot 25 = 1$, so $z \equiv 4\cdot 144\cdot 6 - 23\cdot 25\cdot 85 \equiv 3456 - 48875 = -45419 \equiv \boxed{1381} \bmod 3600$.

T-13. To satisfy the triangle inequality, $a > b + c$. The longest side a can take values between 108 $(108{-}108{-}2, 108{-}107{-}3, \ldots, 108{-}55{-}55)$ and 73 $(73{-}73{-}72)$. For a fixed value of a, b can vary between $\lceil \frac{218-a}{2} \rceil$ and a, inclusive, while c is fixed given the choice of a and b $(c = 218 - a - b)$.

$$\begin{aligned}\sum_{a=73}^{108} a - \left\lceil \frac{218-a}{2} \right\rceil + 1 &= \sum_{a=73}^{108} a - \sum_{a=73}^{108} \left\lceil \frac{218-a}{2} \right\rceil + \sum_{a=73}^{108} 1 \\ &= 181 \cdot 18 - (73 + 72 + 72 + \cdots + 56 + 56 + 55) + 36 \\ &= 3258 - 2304 + 36 = \boxed{990}.\end{aligned}$$

T-14. If $f(x) = 0$, then $2018x^3 = x^6 + 3x^4 + 3x^2 + 1 = (x^2+1)^3$. Dividing both sides by x^3 gives $2018 = (\frac{x^2+1}{x})^3 = (x + \frac{1}{x})^3$. By symmetry, this means that if x is a root, so is $\frac{1}{x}$, so the sum of the real roots is $(x + \frac{1}{x}) = \boxed{\sqrt[3]{2018}}$.

T-15. The only consecutive rows and columns that contain 4 or more filled squares are M to P and B to E, respectively, so the 4×4 must lie there. Additionally, no other filled squares are in A-F $\times$ L-Q, and no other filled squares in column C or row N. Because column D contains eight filled squares, DK must be filled in with a 1×1 and D-F $\times$ R-T must be filled in with one of the 3×3 squares. This eliminates EK and FK from being filled in, as the conditions on columns E and F are now satisfied, as well as G $\times$ Q-T, as those squares are adjacent to the 3×3. The only way that rows O and P can contain 7 filled in squares is if the other 3×3 square is at H-J $\times$ O-Q, which leaves the only way to satisfy the condition on row R to place a 2×2 square at A-B $\times$ R-S, and then 2×2 squares at I-J $\times$ S-T and I-J $\times$ L-M. This leaves the positions of the other three 1×1 squares at AK, GK, and GM. The answer is $\boxed{\text{AK, DK, GK, and GM}}$. The fully filled in solution is below.

	A	B	C	D	E	F	G	H	I	J	
K	X			X			X				3
L									X	X	2
M		X	X	X	X		X		X	X	7
N		X	X	X	X						4
O		X	X	X	X			X	X	X	7
P		X	X	X	X			X	X	X	7
Q								X	X	X	3
R	X	X		X	X	X					5
S	X	X		X	X	X			X	X	7
T				X	X	X			X	X	5
	3	6	4	8	7	3	2	3	7	7	

ARML Local 2018 Individual Problems

I-1. The number $N = 2^{12} - 1$ is the product of 5 (not necessarily distinct) prime numbers p_1, p_2, p_3, p_4, and p_5. Compute $p_1 + p_2 + p_3 + p_4 + p_5$.

I-2. If $a_0, a_1, a_2, \ldots$ is an arithmetic sequence and $18a_{20} + 20 = 20a_{18} + 18$, compute a_0.

I-3. If x and y are real numbers such that $2x + y = 4$ and $5x + 3y = 9$, compute the value of $16x + 9y$.

I-4. For each real number x, let $f(x) = \sin(\frac{\pi x}{3}) + \cos(\frac{\pi x}{2})$. Compute the least $p > 0$ such that $f(x + p) = f(x)$ for all real x.

I-5. Compute the number of triples of consecutive positive integers less than 50 whose product is both a multiple of 20 and 18.

I-6. Suppose that x is a complex number that satisfies $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$. Compute the product $x^{20} \cdot x^{18} \cdot x^2 \cdot x^0 \cdot x^1 \cdot x^8$.

I-7. Let $S = \{1, 2, 3, 4, 5, 6\}$. Compute the number of invertible 2×2 matrices with entries that are distinct elements of S.

I-8. Let $ABCD$ be a rectangle with $AB = 15$ and $BC = 19$. Points E and F are located inside $ABCD$ so that quadrilaterals $AECF$ and $BEDF$ are both parallelograms with areas 107 and 88, respectively. Compute the maximum value of EF.

I-9. Compute the number of positive integers N between 1 and 100 inclusive have the property that there exist distinct divisors a and b of N such that $a + b$ is also a divisor of N.

I-10. It is given that there exist real numbers $a_0, \ldots, a_{18}$ such that

$$\frac{\sin^2(10x)}{\sin^2(x)} = a_0 + a_1 \cos(x) + a_2 \cos(2x) + \cdots + a_{18} \cos(18x)$$

for all x with $\sin x \neq 0$. Compute $a_0^2 + a_1^2 + \cdots + a_{18}^2$.

ARML Local 2018 Individual Solutions

I-1. $2^{12} - 1 = (2^6 - 1)(2^6 + 1) = (2^3 - 1)(2^3 + 1)(2^6 + 1) = 7 \cdot 9 \cdot 65 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. The sum of the primes is $\boxed{31}$.

I-2. Let d be the common difference of the sequence, then $a_{20} = a_{18} + 2d \rightarrow 18(a_{18} + 2d) + 20 = 20a_{18} + 18 \rightarrow 2a_{18} = 36d + 2$ or $a_{18} = 18d + 1$. Because $a_{18} = a_0 + 18d$, the answer is $\boxed{1}$. Alternatively, because the sequence isn't specified, one could assume it is a constant sequence, in which case $18a_0 + 20 = 20a_0 + 18$, and the answer follows.

I-3. Summing three times the third equation and two times the second, you get that $3(2x + y) + 2(5x + 3y) = 3 \cdot 4 + 2 \cdot 9 \rightarrow 16x + 9y = 12 + 18 = \boxed{30}$.

I-4. The first summand $\sin(\frac{\pi x}{3})$ has period 6, because $\sin(x)$ has period 2π. Similarly, the second summand has period 4. The period of the sum has period equal to the least common multiple of the periods of the two summands, which is $\boxed{12}$.

I-5. The least common multiple of 20 and 18 is 180. One of the values in the triple must be a multiple of 5 (say m), and then if the greatest common divisor of m and 180 is d, the product of the other two terms must be a multiple of $\frac{180}{d}$. This leads to the following triples that satisfy the condition:

- $m = 10$, $d = 18$, and the sequence 8,9,10
- $m = 20$, $d = 9$, and the sequence 18,19,20
- $m = 35$, $d = 36$, and the sequences 34,35,36 and 35,36,37
- $m = 45$, $d = 4$, and the sequences 43,44,45 and 44,45,46

Therefore the answer is $\boxed{6}$.

I-6. The left hand side of the equation is equal to $\frac{x^7-1}{x-1}$, so the values that satisfy the equation are the seventh roots of unity not equal to 1. $x^{20} \cdot x^{18} \cdot x^2 \cdot x^0 \cdot x^1 \cdot x^8 = x^{49}$, so for any seventh root of unity, its 49^{th} power will be $\boxed{1}$.

I-7. There are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ 2×2 matrices with entries that are distinct elements of S. If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, M is not invertible if and only if $ad = bc$. There are two sets of pairs of distinct elements of S with equal products, $1 \cdot 6 = 2 \cdot 3$ and $2 \cdot 6 = 3 \cdot 4$. For each set of pairs, any of the four values can be equal to a, which then sets the corresponding value in d, and then the other pair can fill the remaining positions in two ways in M. Therefore sixteen of the matrices are non-invertible, and the answer is $360 - 16 = \boxed{344}$.

I-8. Note that because $AECF$ is a parallelogram, EF passes through the midpoint of $\overline{AC}$, which just so happens to be the center of the rectangle $ABCD$. Embed the figure into a coordinate system with the origin the center of $ABCD$, and WLOG set $A = (-\frac{19}{2}, -\frac{15}{2})$, $B = (-\frac{19}{2}, \frac{15}{2})$, $C = (\frac{19}{2}, \frac{15}{2})$, and $D = (\frac{19}{2}, -\frac{15}{2})$. Let $E = (x, y)$ (WLOG assume that E is clockwise from A

in $AECF$), so that $F = (-x, -y)$. Using the shoelace formula on parallelogram $AECF$ gives:

$$\begin{aligned} 107 = [AECF] &= \frac{\left|\frac{-19}{2}y + \frac{15}{2}x + \frac{-19}{2}y + \frac{15}{2}x - (\frac{-15}{2}x + \frac{19}{2}y + \frac{-15}{2}x + \frac{19}{2}y)\right|}{2} \\ &= |15x - 19y|. \end{aligned}$$

Applying similar reasoning to $BEDF$ yields the equation $|15x + 19y| = 88$.

Assuming WLOG that $y > 0$, there are two possible cases: either $15x - 19y = -107$ and $15x + 19y = -88$ (which has the solution $x = \frac{-13}{2}$, $y = \frac{1}{2}$) or $15x - 19y = -107$ and $15x + 19y = 88$ (which has the solution $x = \frac{-19}{30}$, $y = \frac{195}{38}$). $EF = 2\sqrt{x^2 + y^2}$, and the greater value occurs in the first case, where $EF = 2\sqrt{(\frac{-13}{2})^2 + (\frac{1}{2})^2} = \boxed{\sqrt{170}}$.

I-9. First note that by scaling downward we may assume $\gcd(a, b) = 1$. This implies $\gcd(a, a+b) = 1$ and $\gcd(b, a+b) = 1$, so N in fact is divisible by $ab(a+b)$. We now compute numbers of that form which are less than or equal to 100 and remove all the ones which are multiples of any others. This leaves 6, 20, 56, and 70 as the only four possibilities (most of the other numbers are multiples of six). It thus suffices to compute the number of integers between 1 and 100 which are multiples of at least one of these four numbers. There are $\lfloor \frac{100}{6} \rfloor = 16$ multiples of six between 1 and 100. Then there are four more multiples of 20 (20, 40, 80, 100). Including 56 and 70 gives $\boxed{22}$ total integers.

I-10. Recall that $\sin t = \dfrac{e^{it} - e^{-it}}{2}$ for all t. Thus

$$\frac{\sin(10x)}{\sin(x)} = \frac{e^{10ix} - e^{-10ix}}{e^{ix} - e^{-ix}} = e^{9ix} + e^{7ix} + \cdots + e^{-7ix} + e^{9ix}$$

for all x. This means that

$$\begin{aligned} \frac{\sin^2(10x)}{\sin^2(x)} &= \left(e^{9ix} + e^{7ix} + \cdots + e^{-7ix} + e^{-9ix}\right)^2 \\ &= \left(e^{18ix} + e^{-18ix}\right) + 2\left(e^{16ix} + e^{-16ix}\right) + \cdots + 9\left(e^{2ix} + e^{-2ix}\right) + 10 \\ &= 2\cos(18x) + 4\cos(16x) + \cdots + 18\cos(2x) + 10. \end{aligned}$$

The requested answer is $10^2 + (2^2 + 4^2 + \cdots + 18^2) = \boxed{1240}$.

ARML Local 2018 Relay Problems

R1-1. Compute the sum of the values of x such that $x^{(x-4)^2} = x^{16}$.

R1-2. Let T = TNYWR. Compute the remainder when $\sum_{i=1}^{3}(x-4i)^{(4-i)}$ is divided by $x - T$.

R2-1. Leah rolls a fair standard six-sided die three times. Compute the probability that the product of the three rolls is prime.

R2-2. Let T = TNYWR. Compute the greatest integer N such that $8^{NT} < 4$.

R2-3. Let T = TNYWR. Compute the sum of the perimeters of all distinct rectangles with integer side lengths and area T.

R3-1. Compute the sum of all prime numbers p such that $p^{2018} + p^{2019}$ is a perfect square.

R3-2. Let T = TNYWR. Compute the number of integers in the domain of the function $f(x) = \sqrt{\log((2-x)(x+T))}$.

R3-3. Let T = TNYWR. Compute the number of ordered triples (a, b, c) such that $1 \leq a, b, c \leq T$ and $a + b + c$ is odd.

R3-4. Let T = TNYWR. The lengths of the two perpendicular legs of a right triangle sum to T. A circle of radius r is tangent to all three sides of the triangle, and a circle of radius R passes through all three vertices of the triangle. Compute $r + R$.

R3-5. Let T = TNYWR. Compute the number of integers between 100 and 999 inclusive whose digits sum to T that are *not* multiples of 5.

R3-6. Let T = TNYWR. The number of subsets of size N of the letters in the word TEAMWORK that do not contain all of the vowels (A, E, and O) is T. Compute N.

ARML Local 2018 Relay Solutions

R1-1. There are three cases where the equality holds: either the exponents are equal ($(x-4)^2 = 16 \to x = 0$ or 8), or the base $x = 0$ or 1. The other possibility is $x = -1$ if the exponent is even, but $(-5)^2$ is odd. The answer is $\boxed{9}$.

R1-2. The sum expands to $(x-4)^3 + (x-8)^2 + (x-12)$ which can be rewritten as $((x-T)+(T-4))^3 + ((x-T)+(T-8))^2 + ((x-T)+(T-12))$ which when divided by $x-T$ leaves $(T-4)^3 + (T-8)^2 + (T-12) = 5^3 + 1^2 - 3 = \boxed{123}$.

R2-1. The product of the three rolls is prime if and only if two of the rolls are 1 and one of the rolls is 2, 3, or 5. There are nine sequences of three rolls that satisfy these conditions out of 6^3, so the probability is $\frac{9}{216} = \boxed{\frac{1}{24}}$.

R2-2. Taking the logarithm (base-2) of both sides becomes $3NT < 2 \to N < \frac{2}{3T}$. As $T = \frac{1}{24}$, $N < 16$, which means the answer is $\boxed{15}$.

R2-3. Let the side lengths of the rectangle be x and y, and WLOG, $x \le y$. The area of the rectangle is xy and the perimeter is $2x + 2y$. because $T = 15$, there are two pairs of factors (x, y) such that $x \le y$ and $xy = 15$: $(1, 15)$ and $(3, 5)$. The sum of the perimeters is $32 + 16 = \boxed{48}$.

R3-1. Factoring out p^{2018}, we see that the value is a perfect square if and only if $1 + p = k^2$ for some integer k. $p = k^2 - 1 = (k-1)(k+1)$, so p is prime only when $k = 2$, or $p = \boxed{3}$.

R3-2. In order for $f(x)$ to be defined, $\log((2-x)(x+T) \ge 0 \to (2-x)(x+T) \ge 1 \to (x-2)(x+T) \le 1 \to x^2 + (T-2)x - (2T-1) \le 0$. As $T = 3$, $x^2 + x - 5 \le 0 \to \frac{-1-\sqrt{21}}{2} \le x \le \frac{-1+\sqrt{21}}{2}$. The set of integers that satisfy these inequalities are $-2 \le x \le 1$, so the answer is $\boxed{4}$.

R3-3. There are T^3 triples in total. If T is even, there are an equal number of triples that have even and odd sums, so the answer is $\frac{4^3}{2} = \boxed{32}$.

R3-4. The inradius of a right triangle with legs a and b is $\frac{a+b-\sqrt{a^2+b^2}}{2}$. The radius the circumcircle is $\frac{\sqrt{a^2+b^2}}{2}$, their sum is $\frac{a+b}{2} = \frac{T}{2} = \boxed{16}$.

R3-5. If $k = \underline{X}\,\underline{Y}\,\underline{Z}$. Let $A(n)$ be the number of pairs of digits $\underline{X}\,\underline{Y}$ that sum to n. Note that $A(n) = n$ if $T \le 9$ and $19 - n$ for $10 \le T \le 18$, and 0 otherwise. Ignoring the multiples of 5, the total number of three-digit numbers is $A(T-1) + A(T-2) + A(T-3) + A(T-4) + A(T-6) + A(T-7) + A(T-8) + A(T-9)$. As $T = 16$, the sum is $4+5+6+7+9+9+8+7 = \boxed{55}$.

R3-6. The total number of subsets of size N is $\binom{8}{N}$. Of them, $\binom{N}{N-3}$ contain all of the vowels, so $T = \binom{8}{N} - \binom{N}{N-3}$. Because $T = 55$, $N = \boxed{3}$.

ARML Local 2018 Tiebreaker Problem

TB-1. For all nonnegative integers c, let $f(c)$ denote the unique real number x satisfying the equation

$$\frac{x^5 + x^4 - 48}{x^9 - 1} = c.$$

Compute $f(0)f(1)f(2)\cdots f(47)$.

ARML Local 2018 Tiebreaker Solution

TB-1. Rewrite the original condition as $P_c(x) := cx^9 - x^5 - x^4 + (48 - c) = 0$. Now remark that

$$P_{48-c}(x) = (48 - c)x^9 - x^5 - x^4 + c = x^9 P_c(\tfrac{1}{x});$$

this means that the roots of $P_c(x)$ and $P_{48-c}(x)$ (which are nonzero because $c \neq 48$) are reciprocals of each other, so $f(c)f(48 - c) = 1$ for any c. Now just compute $f(0) = 2$ and $f(24) = -1$; this gives the final answer as $\boxed{-2}$.

Answers to ARML Local 2018

Team Round:

T-1. 706π	T-2. 69	T-3. 16	T-4. 20	T-5. 14
T-6. 252	T-7. $\dfrac{8+3\sqrt{21}}{25}$	T-8. $\dfrac{21-12\sqrt{3}}{4}$	T-9. $5+12\sqrt{3}$	T-10. 3120
T-11. 33	T-12. 1381	T-13. 990	T-14. $\sqrt[3]{2018}$	T-15. AK, DK, GK, GM

Individual Round:

I-1. 31	I-2. 1	I-3. 30	I-4. 12	I-5. 6
I-6. 1	I-7. 344	I-8. $\sqrt{170}$	I-9. 22	I-10. 1240

Relay Round:

R1-1. 9	R1-2. 123				
R2-1. $\dfrac{1}{24}$	R2-2. 15	R2-3. 48			
R3-1. 3	R3-2. 4	R3-3. 32	R3-4. 16	R3-5. 55	R3-6. 3

Tiebreaker:

TB-1. -2

ARML Local 2019

Contents

ARML Local 2019 Team Problems

T-1. Compute the number of ordered pairs of positive integers (x, y) that satisfy the equality $x^y = 2^{9!}$.

T-2. Let $PQRS$ be a rectangle and let A, B, C, and D be points such that A is on $\overline{PQ}$, D is on $\overline{RS}$, and B and C are inside the rectangle so that $\overline{AB} \parallel \overline{PS}$, $\overline{AB} \perp \overline{BC}$, and $\overline{BC} \perp \overline{CD}$. Point E lies on $\overline{RS}$ so that $\overline{AE}$ intersects $\overline{BC}$ and $[PABCDS] = [PAES]$. Given that $AB = 30$, $BC = 24$, and $CD = 10$, compute DE.

T-3. Given that the numbers $\log_2(\log_2 x)$, $\log_4(\log_4 x)$, and $\log_{16}(\log_{16} x)$ form an arithmetic progression in that order, compute x.

T-4. Compute the maximum value of the function $f(x) = \sin(x) + \sin(x + \cos^{-1} \frac{4}{7})$ as x varies over all real numbers.

T-5. Lulu the Magical Pony has developed a wagering strategy for seven days at the casino. He plays the same game five times each day, but wagers different amounts each day. For $1 \leq N \leq 7$, on day N, for each play of the game he either wins $\$2 \cdot 5^{N-1}$ or loses $\$5^{N-1}$. At the end of the week, Lulu has a profit of \$20,119. Compute the total number of games that Lulu won over the week.

T-6. Let $\mathcal{T}$ be an isosceles trapezoid such that there exists a point P in the plane of $\mathcal{T}$ whose distances to the four vertices of $\mathcal{T}$ are 5, 6, 8, and 11. Compute the greatest possible ratio of the length of the longer base of $\mathcal{T}$ to the length of its shorter base.

T-7. Let $f(x) = x^2 - 2\sqrt{2}x + 3$. Define the functions $g_n\big(f(x)\big)$ such that $g_1\big(f(x)\big) = f(x)$ and $g_{n+1}\big(f(x)\big) = f(x)^{g_n(f(x))}$ for all $n \geq 1$. Compute the sum of all distinct real values x such that $g_{2019}(f(x)) = 2019$.

T-8. Integers a, b, c, d, e, and f are chosen uniformly at random and with replacement from the set $\{1, 2, \ldots, 12\}$. Compute the probability that $a^b c^d e^f - 1$ is divisible by 3.

T-9. Positive numbers x, y, and z satisfy $x^3y^2z = 36$. Compute the least possible value of $2x+y+3z$.

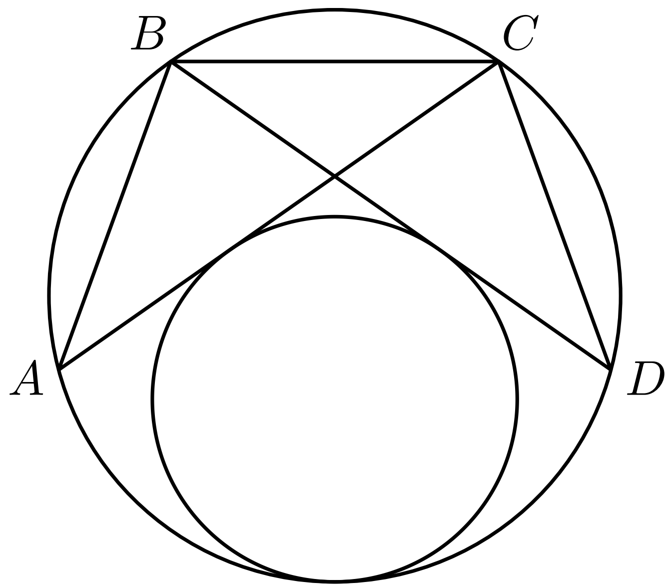
T-10. Compute the least positive value of θ in radians for which $\sin\theta + \sin(2\theta) + \cdots + \sin(2019\theta) = 0$.

T-11. In $\triangle ABC$, D is on $\overline{BC}$ so that $\overline{AD} \perp \overline{BC}$. Let P and Q be the incenters of $\triangle ABD$ and $\triangle ADC$, respectively. Given that $BC = 14$, $AD = 12$, $CD = 5$, and O is the circumcenter of $\triangle ABC$, compute $[OPQ]$.

T-12. For integers $n \geq 2$, let $h(n)$ be the number of positive integers $x \leq n$ such that $x^2 \equiv 1 \pmod{n}$. Compute the greatest integer $n \geq 2$ such that $h(n) = \phi(n)$, where $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

T-13. Compute the number of functions $f : \{1, 2, \ldots, 20\} \to \{1, 2, 3, 4\}$ such that $f(m)$ divides $f(n)$ whenever m divides n, and m and $f(m)$ have the same parity for all $m \in \{1, 2, \ldots, 20\}$.

T-14. Let Ω be a circle with radius 5, and suppose that A, B, C, and D are points on Ω in that order such that $AB = BC = CD = 4$. Compute the radius of the circle shown in the figure below that is tangent to $\overline{AC}$, $\overline{BD}$, and minor arc $\widehat{AD}$.



T-15. Given that $\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{5^k} = \sqrt{5}$, compute the value of the sum

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{5^k}.$$

ARML Local 2019 Team Solutions

T-1. The given equation implies x is a power of 2, so let $x = 2^z$ for some nonnegative integer z. Then $2^{zy} = 2^{9!}$, so $yz = 9!$. The number of pairs satisfying the equality is equal to the number of positive divisors of $9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$, and $\sigma(9!) = (7+1)(4+1)(1+1)(1+1) = \boxed{160}$.

T-2. Let F be the point of intersection of $\overline{AE}$ and $\overline{BC}$. Let $y = DE$ and G be on FC so that $\overline{EG} \perp \overline{BC}$. Because the two polygon areas are the same, the area of $\triangle AFB$ and trapezoid $FEDC$ are the same. Note $EG = CD = 10$ is the height of the trapezoid. Furthermore, $\triangle AFB \sim EFG$ so $\frac{AB}{EG} = \frac{BF}{GF}$, implying $BF = 3GF$. Letting $x = GF$, it follows that $BF = 3x$ so $BC = 4x + GC$. Because $GC = DE$, it follows that $4x + y = 24$. Then the area of $\triangle AFB$ is $\frac{30 \cdot 3x}{2} = 45x$ and the area of $FEDC$ is $\frac{10(y+x+y)}{2} = 5(x+2y)$. Equating these two areas yields $9x = x + 2y$ or $y = 4x$. Therefore $24 = y + 4x = 2y$ so $y = \boxed{12}$.

T-3. The given condition translates to the equation $\log_2(\log_2 x) + \log_{16}(\log_{16} x) = 2\log_4(\log_4 x)$. To solve this equation, let $y = \log_2 x$. Then, by the change-of-base formula, $\log_{16} x = \frac{\log_2 x}{\log_2 16} = \frac{y}{4}$ and $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{y}{2}$, so it follows that $\log_2 y + \log_{16} \frac{y}{4} = 2\log_4 \frac{y}{2}$, or $\log_2 y + \log_{16} y - \log_{16} 4 = 2\log_4 y - 2\log_4 2$, which simplifies to $\log_2 y + \log_{16} y + \frac{1}{2} = 2\log_4 y$. Let $z = \log_2 y$. Then, by similar logic, $\log_{16} y = \frac{z}{4}$ and $\log_4 = \frac{z}{2}$, so it follows that $z + \frac{z}{4} + \frac{1}{2} = 2 \cdot \frac{z}{2}$. Solving for z gives $z = -2$. Then $y = 2^z = 2^{-2} = \frac{1}{4}$, so $x = 2^y = \boxed{\sqrt[4]{2}}$.

T-4. Note that

$$\begin{aligned} f(x) &= \sin(x)[1 + \cos(\cos^{-1} \tfrac{4}{7})] + \cos(x)[\sin \cos^{-1} \tfrac{4}{7}] \\ &\le \sqrt{(1 + \cos(\cos^{-1} \tfrac{4}{7}))^2 + (\sin(\cos^{-1} \tfrac{4}{7}))^2} \\ &= \sqrt{2 + 2\cos(\cos^{-1} \tfrac{4}{7})} = \sqrt{2 + \tfrac{8}{7}} = \sqrt{\tfrac{22}{7}} = \boxed{\tfrac{\sqrt{154}}{7}}. \end{aligned}$$

T-5. Let a_i be the number of times Lulu wins on day i. Then, his profit, in dollars, on day i is $2a_i 5^{i-1} - (5 - a_i)\, 5^{i-1} = 3a_i \cdot 5^{i-1} - 5^i$. Summing this expression from $i = 1$ to $i = 7$ yields $3\displaystyle\sum_{i=1}^{7} a_i 5^{i-1} - \frac{5^8 - 5}{5 - 1} = 3\sum_{i=1}^{7} a_i 5^{i-1} - 97655 \to 3\sum_{i=1}^{7} a_i 5^{i-1} = 97655 + 20119 = 117774$. Therefore $\displaystyle\sum_{i=1}^{7} a_i 5^{i-1} = 39258$ and the goal is to compute $\displaystyle\sum_{i=1}^{7} a_i$. Each a_i can be determined by representing 39258 in base 5. Because $39258 = 2224013_5$, Lulu won $\displaystyle\sum_{i=1}^{7} a_i = 2+2+2+4+0+1+3 = \boxed{14}$ games.*

T-6. Suppose $ABCD$ is the trapezoid with $BC \parallel AD$ and $BC < AD$. Let $PA = a$, $PB = b$, $PC = c$, and $PD = d$; the problem stipulates that $\{a, b, c, d\} = \{5, 6, 8, 11\}$.

*Because $0 \le a_i \le 5$ for all a_i, there is another base-5 representation of 39258 where fives are allowed as a digit: 2223513_5. As a result, $\boxed{18}$ was also accepted.

Let Q be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and set $QB = QC = s$, $BA = CD = t$, and $PQ = r$. Stewart's Theorem on $\triangle APQ$ and $\triangle BPQ$ yields the system of equations

$$r^2t + a^2s = b^2(s+t) + st(s+t) \qquad \text{and} \qquad r^2t + d^2s = c^2(s+t) + st(s+t).$$

Subtracting these two equations and utilizing $\triangle QBC \sim \triangle QAD$ yields

$$\frac{AD}{BC} = \frac{s+t}{s} = \frac{d^2-a^2}{c^2-b^2}.$$

This ratio has a maximum of $\boxed{\frac{57}{11}}$, achieved at $(a,b,c,d) = (8,5,6,11)$.

T-7. If $g_{2019}(f(x)) = 2019$, it follows that $f(x) > 1$. Let $a = f(x)$ so $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}} = 2019$ (where the exponent is taken 2019 times), and it also follows that $a > 1$. By definition, x is a solution to $g_{2019}(f(x)) = 2019$ if and only if x is a solution to $x^2 - 2\sqrt{2}x + (3-a) = 0$. The discriminant of the left hand side shows that there are two distinct real solutions, and therefore, by Vieta, the sum of all such x is $\boxed{2\sqrt{2}}$.

T-8. Consider the expression $a^bc^de^f - 1$ modulo 3. The integers a, c, and e are equally likely to be any of the three residues modulo 3. If any of them is $0 \pmod 3$, then the entire expression becomes $0 - 1 \equiv 2 \pmod 3$, so none of a, c, and e can be $0 \pmod 3$; they must each be either $1 \pmod 3$ or $2 \pmod 3$. With probability $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$, all three of a, c, and e are $1 \pmod 3$, which makes

$$a^bc^de^f - 1 \equiv 1^b1^d1^f - 1 = 0 \pmod 3,$$

as desired. The probability that each variable is either $1 \pmod 3$ or $2 \pmod 3$ is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$; the probability that that occurs *but* not all the variables are $1 \pmod 3$ is $\frac{8}{27} - \frac{1}{27} = \frac{7}{27}$. In this case, the expression reduces to

$$a^bc^de^f - 1 \equiv 2^x - 1 \pmod 3,$$

where x is the sum of the exponents on the variables which are $2 \pmod 3$. (For example, if $a \equiv 1 \pmod 3$ and $c \equiv e \equiv 2 \pmod 3$, then $a^bc^de^f - 1 \equiv 2^{d+f} - 1 \pmod 3$, so $x = d + f$.) Because b, d, and f are equally likely to be odd or even, the variable x is also equally likely to be odd or even. (In general, if $a_1, \ldots, a_k$ are equally likely to be odd or even, then their sum $a_1 + \cdots + a_k$ is *also* equally likely to be odd or even, because regardless of the values of $a_1, \ldots, a_{k-1}$, the parity of a_k is equally likely to be the same as the parity of $a_1 + \cdots + a_{k-1}$, making the sum even, or different, making the sum odd.) Because $2^2 \equiv 1 \pmod 3$, it follows that $2^x - 1$ is equally likely to be $2^1 - 1 \equiv 1 \pmod 3$ or $2^2 - 1 \equiv 0 \pmod 3$. So, the probability that $a^bc^de^f - 1$ is divisible by 3 in this case is $\frac{7}{27} \cdot \frac{1}{2} = \frac{7}{54}$. In total, the desired probability is $\frac{1}{27} + \frac{7}{54} = \boxed{\frac{1}{6}}$.

T-9. Note that $2x + y + 3z = \left(\frac{2x}{3} + \frac{2x}{3} + \frac{2x}{3}\right) + \left(\frac{y}{2} + \frac{y}{2}\right) + 3z$. Therefore, by the AM-GM inequality, $\frac{2x+y+3z}{6} \geq \sqrt[6]{\frac{8}{27} \cdot \frac{1}{4} \cdot 3 \cdot x^3y^2z} = \sqrt[6]{8} = \sqrt{2}$. Equality occurs when $\frac{2x}{3} = \frac{y}{2} = 3z$ so $(x,y,z) = \left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}, \frac{\sqrt{2}}{3}\right)$ and $2x + y + 3z = \boxed{6\sqrt{2}}$.

T-10. In general:

$$\begin{aligned}\sum_{k=1}^{n}\sin k\theta &= \frac{\sum_{k=1}^{n} 2\sin\frac{\theta}{2}\sin k\theta}{2\sin\frac{\theta}{2}}\\ &= \frac{\sum_{k=1}^{n}\left(\cos\left(k-\frac{1}{2}\right)\theta-\cos(k+\frac{1}{2})\theta)\right)}{2\sin\frac{\theta}{2}}\\ \textit{(telescoping sum)} &= \frac{\cos\frac{\theta}{2}-\cos\left(n\theta+\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}.\end{aligned}$$

Therefore, for the above expression to be equal to zero, for some integer k either $n\theta = 2\pi k$ or $n\theta + \frac{\theta}{2} = 2\pi k - \frac{\theta}{2} \to (n+1)\theta = 2\pi k$. When $n = 2019$, the least positive value of θ that satisfies one of these equations occurs when $k = 1$ and $\theta = \boxed{\frac{\pi}{1010}}$.

T-11. Using coordinates: let $B = (0,0)$, $C = (14,0)$, $A = (9,12)$, and $D = (9,0)$. Suppose $O = (7,y)$, then $7^2 + y^2 = OC^2 = OA^2 = 2^2 + (y-12)^2 \to y = \frac{33}{8}$ so $O = (7, \frac{33}{8})$. Then, the inradius of $\triangle ABD$ is $\frac{9+12-15}{2} = 3$ and the inradius of $\triangle ADC$ is $\frac{5+12-13}{2} = 2$ because both are right triangles. Therefore $P = (6,3)$ and $Q = (11,2)$. By the shoelace formula, the area of OPQ is $\frac{1}{2}\left|6(2) + 11(\frac{33}{8}) + 7(3) - 3(11) - 2(7) - (\frac{33}{8})(6)\right| = \boxed{\frac{53}{16}}$.

T-12. The function h is multiplicative so it suffices to determine when $h\left(p^\alpha\right) = \phi\left(p^\alpha\right)$ for prime p. If p is odd, then p^α divides $x^2 - 1 = (x-1)(x+1)$ implies either p^α divides $x - 1$ or p^α divides $x + 1$ (otherwise, p divides $(x+1) - (x-1) = 2$, contradiction). Therefore $x^2 \equiv 1 \bmod p^\alpha$ if and only if $x \equiv \pm \bmod p^\alpha$, so $h\left(p^\alpha\right) = 2$. Because $\phi\left(p^\alpha\right) = p^{\alpha-1}(p-1)$, it follows that $\phi\left(p^\alpha\right) = 2$ implies $p = 3$ and $\alpha = 1$. Consider the case where $h\left(2^\alpha\right) = \phi\left(2^\alpha\right)$. This is true for $0 \le \alpha \le 3$. When $\alpha > 3$, 2^α divides $(x-1)(x+1)$. Because only one of these factors are divisible by 4, it follows that $x \equiv \pm 1 \bmod 2^{\alpha-1}$. Therefore $h\left(2^\alpha\right) = 4 < 2^{\alpha-1} = \phi\left(2^\alpha\right)$ so there are no such powers of two. Therefore $h(n) = \phi(n)$ if and only if $n = 2^a \cdot 3^b$ where $a \in \{0,1,2,3\}$ and $b \in \{0,1\}$. Therefore the greatest such integer is $n = \boxed{24}$.

T-13. Note that $f(n) = 1$ for all odd $n \le 10$ (if $f(n) = 3$, then $f(2n)$ is even and divisible by 3, which is a contradiction because no value in the range is a multiple of 6). The function values for odd numbers greater than 10 can either be 1 or 3, so there are 2^5 possible assignments of values to the odd integers in the domain. Consider the greatest positive integer α such that $f\left(2^\alpha\right) = 2$.

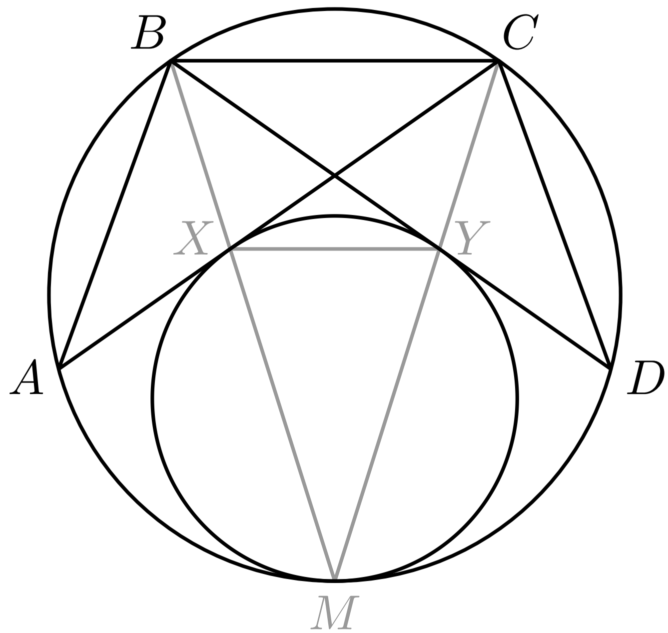
$f(2) = 4$: In this case $f(n) = 4$ for all even values of n.

$2^\alpha = 2$: In this case, $f(4) = f(8) = f(12) = f(16) = f(20) = 4$ while $f(10)$ and $f(14)$ can be either 2 or 4, but $f(6) \mid f(18)$. This gives 12 possibilities.

$2^\alpha \ge 4$: In these cases, $f(14)$ can be 2 or 4, $f(6) \mid f(12)$, $f(6) \mid f(18)$, and $f(10) \mid f(20)$. This results in 30 possible cases for each.

Therefore there is a total of $2^5(1 + 12 + 3 \cdot 30) = \boxed{3296}$ such functions.

T-14. Let X, Y, and M denote the tangency points of the circle to $\overline{AC}$, $\overline{BD}$, and $\widehat{AD}$ respectively. Note that by Archimedes' Lemma, B, X, and M are collinear; C, Y, and M are collinear; and M is the midpoint of $\widehat{AD}$.



The crucial claim is that $AX = XY = YD$. Indeed, note that from $\angle ABM = \angle DBM$ and $\angle BAC = \angle DAC$, X is the incenter of $\triangle ABD$; and therefore M is the circumcenter of $\triangle AXD$[†] which gives $AM = MX = MD$. By the same argument MY is also equal to all these lengths. But now $\angle AMB = \angle BMC = \angle CMD$ yields $\triangle AMX \cong \triangle XMY \cong \triangle YMD$, which gives $AX = XY = YD$. To finish, note that $\triangle AMX$ being isosceles implies that $\triangle BCX$ is isosceles, so $BX = CX = 4$. The Law of Sines yields $\sin\angle BCA = \frac{2}{5}$, and it follows that $AC = \frac{8}{5}\sqrt{21}$. Thus $XY = AX = 4(\frac{2}{5}\sqrt{21} - 1)$, so letting r denote the radius of the desired circle, using homothety gives

$$\tfrac{r}{5} = \tfrac{XY}{4} = \tfrac{2}{5}\sqrt{21} - 1 \quad \Longrightarrow \quad r = \boxed{2\sqrt{21} - 5}.$$

T-15. Note that from the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, it follows that

$$\binom{2k+1}{k} = \frac{k+1}{2k+2}\binom{2k+2}{k+1} = \frac{1}{2}\binom{2k+2}{k+1}.$$

Thus

$$\sum_{k=0}^{\infty}\binom{2k+1}{k}\frac{1}{5^k} = \frac{1}{2}\sum_{k=0}^{\infty}\binom{2k+2}{k+1}\frac{1}{5^k} = \frac{5}{2}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{5^n} = \boxed{\frac{5\sqrt{5}-5}{2}}.$$

[†]See `http://web.evanchen.cc/handouts/Fact5/Fact5.pdf`.

ARML Local 2019 Individual Problems

I-1. Compute the least positive integer n such that $2n$ and $3n$ both contain the digit 7 when written in base ten.

I-2. Circle $\mathcal{C}$ lies entirely inside rectangle $\mathcal{R}$, and $\mathcal{C}$ is tangent to three of the sides of $\mathcal{R}$. Given that the ratio of the area of $\mathcal{C}$ to the area of $\mathcal{R}$ is $\frac{\pi}{5}$, compute the ratio of the circumference of $\mathcal{C}$ to the perimeter of $\mathcal{R}$.

I-3. Compute the sum of all integers k with $1 \leq k \leq 2019$ such that $(2k-1)^2 = 1 + k \cdot 10^{k/13}$.

I-4. Exactly five distinct vertices of a regular 12-gon are randomly colored red, with all vertices equally likely to be colored. Compute the probability that no edge of the 12-gon has both its vertices colored red.

I-5. Triangle SAM lies in the coordinate plane with $S = (-15, 0)$, $A = (0, 36)$, and $M = (15, 0)$. There is a unique point $B = (x, y)$ in the interior of $\triangle SAM$ such that the perimeters of $\triangle BSA$, $\triangle BAM$, and $\triangle BMS$ are equal. Compute $x + y$.

I-6. The function $f(x) = x^4 + ax^3 + bx^2 + 2000x + d$ has four distinct roots. Two of the roots sum to 5; the other two roots also sum to 5. Compute b.

I-7. Compute the sum of all possible values for d that satisfy the following property: there exists a positive integer n with exactly 15 divisors such that $2n$ has exactly d divisors.

I-8. Let $\theta = \frac{1}{2}\sin^{-1}\frac{2}{3}$. Compute $\sin^4\theta + \cos^4\theta$.

I-9. Let P be a set of five points in the plane, no three of which are colinear. Let S be the set of all line segments between two points in P. Compute the number of subsets of S such that exactly four triangles have all their edges in S.

I-10. Let m and n be positive integers. When the point $(15, 17)$ is reflected across the line $y = mx$ and then the reflected point is reflected across the line $y = nx$, the resulting point is $(17, 15)$. Compute $m + n$.

ARML Local 2019 Individual Solutions

I-1. Because $2n$ is always even, if $2n$ contains the digit 7, then $2n \geq 70$, or $n \geq 35$. Trying values of n starting with 35, $n = \boxed{39}$ is the first one that works, with $2n = 78$ and $3n = 117$.

I-2. Let r be the radius of $\mathcal{C}$. Then one of the side lengths of $\mathcal{R}$ is $2r$. Let the other side length be s. Then the area of $\mathcal{C}$ is πr^2 and the area of $\mathcal{R}$ is $2rs$, so $\frac{\pi r^2}{2rs} = \frac{\pi}{5}$. This gives $s = \frac{5r}{2}$. Now, the circumference of $\mathcal{C}$ is $2\pi r$ and the perimeter of $\mathcal{R}$ is $2(2r + s) = 2(2r + \frac{5r}{2}) = 9r$, so the desired ratio is $\frac{2\pi r}{9r} = \boxed{\frac{2\pi}{9}}$.

I-3. Expanding the left hand side gives $4k^2 - 4k = k \cdot 10^{k/13}$ so $4k - 4 = 10^{k/13}$. Note that $10^{k/13}$ must be an integer so k is divisible by 13. Therefore when $k \geq 39$, $10^{k/13} > 4k$ so it suffices to check $k = 13$ and $k = 26$. Only $k = \boxed{26}$ is a solution.

I-4. Label the vertices in order $v_1, v_2, \ldots, v_{12}$. Let R be the set of red vertices. Assume without loss of generality that $v_1 \in R$, then there are $\binom{11}{4} = 330$ ways to select the remaining four vertices in R. Call a set R *edge-free* if it does not contain both vertices of an edge of the 12-gon. Consider the five possible cases of the vertex (or vertices) in an edge-free R that is/are furthest from v_1: either v_6, v_7, v_8, v_6 and v_8, or v_5 and v_9. For any other cases, it is impossible to have the remaining elements in R not share an edge with another vertex in R. Note that neither v_2 nor v_{12} can be in an edge-free R that contains v_1.

- v_6: R cannot contain v_5, v_7, or v_8, which forces one of v_3 or v_4, plus v_9 and v_{11} to be in R for it to be edge-free. By symmetry, there are only two edge-free sets for the case of v_8 being furthest as well.
- v_7: R cannot contain v_6 or v_8. There will be two remaining vertices of R clockwise from v_7 to v_1 and one counter-clockwise, or vice versa. The side that has two is forced to be v_9 and v_{11} or v_3 and v_5, the vertex on the other side can be any of the three not adjacent to v_1 or v_7. This makes six edge-free sets in total.
- v_6 and v_8: R cannot contain v_5, v_7, or v_9. The two remaining vertices must be one of v_{10} or v_{11} and v_3 or v_4. This results in four edge-free sets.
- v_5 and v_9: R can only contain v_3 and v_{11} and be edge-free.

In total, there are 15 edge-free sets of size five, so the probability is $\frac{15}{330} = \boxed{\frac{1}{22}}$.

I-5. Note that $\triangle SAM$ is isosceles with $SM = 15 - (-15) = 30$ and

$$AS = AM = \sqrt{(0 - 15)^2 + 36^2} = \sqrt{15^2 + 36^2} = \sqrt{3^2(5^2 + 12^2)} = 3 \cdot 13 = 39.$$

Note that point B must lie on the altitude $\overline{AH}$ from A to $\overline{SM}$, for if this were not the case, then by symmetry, the reflection of B across $\overline{AH}$—call it B'—would also satisfy the given constraints, implying that B was not unique. Hence $BS = BM$, and it therefore suffices to equate the perimeters of $\triangle BAM$ and $\triangle BSM$. It also follows that H is the midpoint of $\overline{SM}$, hence $x = \frac{-15+15}{2} = 0$. Thus

$$\begin{aligned} & BA + BM + AM = BS + BM + SM \\ \Longrightarrow\ & BA + AM = BS + SM \\ \Longrightarrow\ & (36 - y) + 39 = \sqrt{(0 - (-15))^2 + (y - 0)^2} + 30 \\ \Longrightarrow\ & 45 - y = \sqrt{15^2 + y^2} \\ \Longrightarrow\ & y^2 - 90y + 45^2 = y^2 + 15^2 \\ \Longrightarrow\ & 90y = 45^2 - 15^2. \end{aligned}$$

Factoring the right-hand side of the last equation gives $90y = (45-15)(45+15) = 30 \cdot 60$ and dividing each side by 90 yields $y = 20$, hence $x + y = \boxed{20}$.

Note: The point B is known as the *Kimberling point* of triangle SAM.

I-6. Let r_1, r_2, s_1, and s_2 be the four roots, with $r_1 + r_2 = s_1 + s_2 = 5$. By Vieta's formulas:

$$-2000 = r_1r_2s_1 + r_1r_2s_2 + r_1s_1s_2 + r_2s_1s_2.$$

This equation factors as follows:

$$-2000 = r_1r_2(s_1 + s_2) + (r_1 + r_2)s_1s_2.$$

The terms in parentheses both equal 5, so $-400 = r_1r_2 + s_1s_2$. Now, by Vieta again,

$$\begin{aligned} b &= r_1r_2 + r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2 + s_1s_2 \\ &= (r_1r_2 + s_1s_2) + (r_1 + r_2)(s_1 + s_2) \\ &= -400 + (5)(5) = \boxed{-375}. \end{aligned}$$

I-7. Because n has $15 = 3^15^1$ divisors, either $n = p^{14}$ for some prime p, or $n = p^4q^2$ for some distinct primes p and q. If $n = p^{14}$, then either $p = 2$, in which case $2n = 2^{15}$ has 16 divisors, or $p \neq 2$, in which case $2n = 2^1p^{14}$ has $2 \cdot 15 = 30$ divisors. If $n = p^4q^2$ and $p = 2$, then $2n = 2^5q^2$ has $6 \cdot 3 = 18$ divisors. If $n = p^4q^2$ and $q = 2$, then $2n = p^42^3$ has $5 \cdot 4 = 20$ divisors. If $p \neq 2$ and $q \neq 2$, then $2n = 2^1p^4q^2$ has $2 \cdot 5 \cdot 3 = 30$ divisors, a value already found. Therefore the answer is $16 + 30 + 18 + 20 = \boxed{84}$.

I-8. Notice that

$$\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta = 1 - 2\sin^2\theta\cos^2\theta.$$

Using the double-angle formula, this can be rewritten as

$$1 - 2\sin^2\theta\cos^2\theta = 1 - \tfrac{1}{2}\left(2\sin\theta\cos\theta\right)^2 = 1 - \tfrac{1}{2}\sin^2(2\theta).$$

Because $2\theta = \sin^{-1}\frac{2}{3}$, it follows that $\sin(2\theta) = \frac{2}{3}$, so $1 - \frac{1}{2}\sin^2(2\theta) = 1 - \frac{1}{2}\left(\frac{2}{3}\right)^2 = \boxed{\frac{7}{9}}$.

I-9. The segments in S form ten triangles. The removal of any single segment from S eliminates three triangles. Accordingly, removing any pair of segments that share no points in common will eliminate six triangles, leaving four. Removing two edges that share a point (say p) in common removes five triangles in total, leaving five. If another edge containing p as a vertex is removed, another triangle is removed, leaving four. Finally, if the last edge containing p is removed, no additional triangles are removed. There are 15 pairs of disjoint segments in S, and there are 5 choices of p and 5 choices of a single (or no) edge to retain containing p, so there are $15 + 25 = \boxed{40}$ subsets of S that contain exactly four triangles.

I-10. Let θ be the angle from the ray $y = mx$ to the ray $y = nx$ in the first quadrant. Then a reflection of $(15, 17)$ over these two lines in succession constitutes a rotation through an angle of 2θ. Therefore θ must be in the clockwise direction, so $m > n$, and 2θ must equal the angle between ℓ_1 and ℓ_2, where ℓ_1 is the line $y = \frac{17}{15}x$ and ℓ_2 is the line $y = \frac{15}{17}x$. Letting ℓ denote the line $y = x$, it follows that ℓ_1 and ℓ_2 are reflections of each other over ℓ, so θ must equal the angle between ℓ and ℓ_2. The angle between the positive x-axis and the line $y = kx$ is $\operatorname{Arctan} k$, so it follows that

$$\operatorname{Arctan} m - \operatorname{Arctan} n = \operatorname{Arctan} 1 - \operatorname{Arctan} \frac{15}{17}.$$

Taking the tangent of both sides and using the difference identity for tangents gives

$$\frac{m-n}{1+mn} = \frac{1-\frac{15}{17}}{1+\frac{15}{17}} = \frac{1}{16}.$$

Cross-multiplying gives $16m - 16n = 1 + mn$, or $mn - 16m + 16n + 1 = 0$. Subtracting 257 from both sides, this factors as

$$(m+16)(n-16) = -257.$$

Because 257 is a prime number and $m, n > 0$, the only possibility is that $m + 16 = 257$ and $n - 16 = -1$, which gives $(m, n) = (241, 15)$, so $m + n = \boxed{256}$.

ARML Local 2019 Relay Problems

R1-1. Three 1×1 squares in a 3×3 grid are chosen randomly and colored orange. Compute the probability that no row or column of the grid contains more than one orange square.

R1-2. Let $T = \text{TNYWR}$. Compute the sum of the prime factors of $\dfrac{1-T^2}{T^3}$.

R2-1. Let $a_1, a_2, a_3, \ldots$ be a nonconstant arithmetic sequence. Suppose that a_1, a_{11}, a_{111} is a geometric sequence. Compute a_{11}/a_1.

R2-2. Let $T = \text{TNYWR}$. Frank has a $3 \times T$ grid of squares. Compute the number of ways that Frank can shade some (possibly none) of the squares in the grid so that each row and column contains at most one shaded square.

R2-3. Let $T = \text{TNYWR}$. Let N be the positive integer created by joining T copies of T. (For example, if $T = 15$, then N is the 30-digit integer $151515\ldots15$.) Compute the remainder when N is divided by 99.

R3-1. Given that $(20^{19} + 100)^2 - (20^{19} - 100)^2 = 20^N$, compute N.

R3-2. Let $T = \text{TNYWR}$. Given that a rectangle has area T and perimeter $T - 1$, compute the length of the shorter side of the rectangle.

R3-3. Let $T = \text{TNYWR}$. Let N be the sum of all positive integers which divide $2^7(T^8 - 1)$. Compute the remainder when N is divided by 100.

R3-4. Let $T = \text{TNYWR}$. Fran flips four fair coins and rolls a standard, fair six-sided die. Let p be the probability that the number rolled on the die is equal to the number of heads flipped. Compute pT.

R3-5. Let $T = \text{TNYWR}$. In $\triangle ABC$, point M is the midpoint of $\overline{BC}$ and D is the foot of the altitude from A to $\overline{BC}$, with D between B and M. Given that $AC = T$, $AM = 4$, and $BD = \dfrac{3}{4}$, compute AB.

R3-6. Let $T = \text{TNYWR}$. For a positive integer n, let S_n be the sum of the first n positive integers. Compute the remainder of S_{T^4} when divided by 9.

ARML Local 2019 Relay Solutions

R1-1. There are $\binom{9}{3} = 84$ possible colorings of 3 out of 9 squares in the grid. Exactly six of them have exactly one orange square in each row and column. Such a coloring would have to have one orange square in each column, let r_i be the row of the orange square in column i, the r_i must be distinct, and there are $3! = 6$ possible assignments of the r_i. Therefore the desired probability is $\frac{6}{84} = \boxed{\frac{1}{14}}$.

R1-2. Let $N = \dfrac{1-T^2}{T^3}$, then $N = \dfrac{1}{T^3} - \dfrac{1}{T} = \dfrac{1}{T}\left(\dfrac{1}{T^2} - 1\right) = \dfrac{1}{T}\left(\dfrac{1}{T} - 1\right)\left(\dfrac{1}{T} + 1\right)$. As $T = \frac{1}{14}$, $N = 14 \cdot 13 \cdot 15$, which has prime factors 2, 3, 5, 7, and 13. The sum of these prime factors is $\boxed{30}$.

R2-1. If r is the common ratio, then $a_{11} = ra_1$ and $a_{111} = r^2 a_1$, so $a_{11} - a_1 = (r-1)a_1$ and $a_{111} - a_{11} = (r^2 - r)a_1$. Therefore

$$\frac{a_{111} - a_{11}}{a_{11} - a_1} = \frac{(r^2 - r)a_1}{(r-1)a_1} = r.$$

If d is the common difference of the arithmetic sequence, then $a_{111} - a_{11} = 100d$ and $a_{11} - a_1 = 10d$, so

$$\frac{a_{111} - a_{11}}{a_{11} - a_1} = \frac{100d}{10d} = 10.$$

Thus $r = \boxed{10}$.

R2-2. Frank can shade 0, 1, 2, or 3 squares. There is 1 way for Frank to shade 0 squares, and there are $3T$ ways for Frank to shade 1 square. To shade 2 squares, Frank can pick the first square in $3T$ ways and the second square in $2(T-1)$ ways; but the order of the squares does not matter, so the total number of ways to shade 2 squares is $3T \cdot 2(T-1) \cdot \frac{1}{2} = 3T(T-1)$. Similarly, to shade 3 squares, Frank can pick the squares in $3T \cdot 2(T-1) \cdot 1(T-2)$ ways, but because the order of the squares does not matter, the total number of ways is $3T \cdot 2(T-1) \cdot 1(T-2) \cdot \frac{1}{3!} = T(T-1)(T-2)$. So the answer is

$$1 + 3T + 3T(T-1) + T(T-1)(T-2) = T^3 + 2T + 1.$$

With $T = 10$, the answer is $10^3 + 2 \cdot 10 + 1 = \boxed{1021}$.

R2-3. It suffices to determine N modulo 9 and modulo 11. To find N modulo 9, note that if the sum of the digits of T is s, then the sum of the digits of N is $T \cdot s$. But by the divisibility rule for 9, $T \equiv s \pmod 9$, so $N \equiv T \cdot s \equiv s^2 \pmod 9$. To find N modulo 11, use the divisibility rule for 11. First, if T has an odd number of digits and T is even, then in the alternating sum of digits of N, all the terms will cancel out, making $N \equiv 0 \pmod{11}$. If T has an odd number of digits and T is odd, then all the terms will cancel except for the rightmost appearance of T, which means that $N \equiv T \pmod{11}$. On the other hand, if T has an even number of digits, then the alternating sum of digits of N equals T times the alternating sum of digits of T. If s' is the alternating sum of digits of T, then $N \equiv T \cdot s' \equiv (s')^2 \pmod{11}$. With $T = 1021$, the digit sum is $s = 4$, so $N \equiv s^2 \equiv 7 \pmod 9$. Also, T has an even number of digits and $s' = 1 - 2 + 0 - 1 = -2$, so $N \equiv (s')^2 \equiv 4 \pmod{11}$. These two congruences imply that $N \equiv \boxed{70} \pmod{99}$.

Alternate Solution: Because $100 \equiv 1 \pmod{99}$, if $N = \overline{a_{2k}a_{2k-1}\dots a_2a_1}$, then $N \equiv \overline{a_{2k}a_{2k-1}} + \cdots + \overline{a_2a_1} \pmod{99}$. If T has an even number of digits, and S is the pairwise-digit sum of T, then $N \equiv S^2 \pmod{99}$. If T has an odd number of digits (say k), then let R be the $2k$ digit number consisting of two copies of T, and $N \equiv T + R \cdot (T-1)/2 \pmod{99}$. Because $T = 1021$, $S = 31$ and $N \equiv 961 \pmod{99} \equiv 9 + 61 \equiv \boxed{70} \pmod{99}$.

R3-1. Let $x = 20^{19}$. Then the left-hand side is

$$(x+100)^2 - (x-100)^2 = (x^2 + 200x + 10000) - (x^2 - 200x + 10000) = 400x = 20^2x,$$

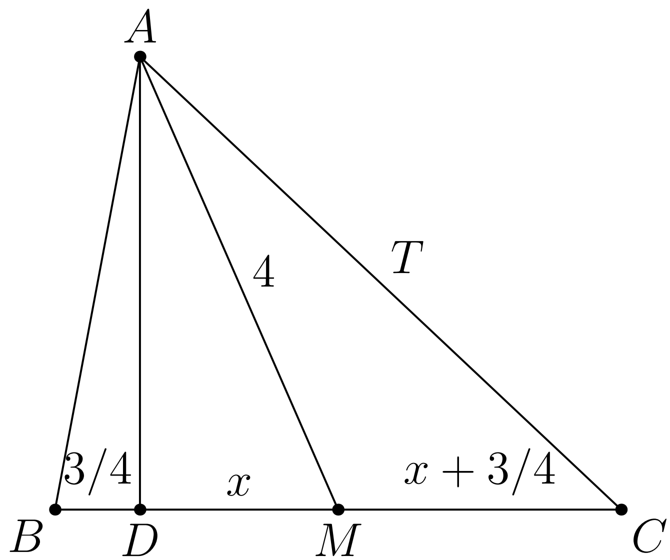
so $N = \boxed{21}$.

R3-2. Let x and y be the lengths of the sides of the rectangle, then $2x + 2y = T - 1$ and $xy = T$, combining the two equations gives $x = \dfrac{2y+1}{y-2}$. Provided $y \neq 2$, plugging this expression into the rectangle area formula gives $2y^2 + y = T(y-2)$. As $T = 21$, this simplifies to $2y^2 + y = 21y - 42 \to 2y^2 - 20y + 42 = 0$, which factors to $2(y-3)(y-7) = 0$. Therefore the shorter edge of the rectangle has length $\boxed{3}$.

R3-3. Let $S = 2^7(T^8 - 1)$. By difference of squares, $S = 2^7(T^2-1)(T^2+1)(T^4+1) = 2^{12} \cdot 5 \cdot 41$ for $T = 3$. Note that $N = \sigma(S) = \sigma(2^{12})\sigma(5)\sigma(41) = (2^{13}-1)(6)(42)$. Therefore $N \equiv 91 \cdot 52 \equiv \boxed{32} \bmod 100$.

R3-4. If Fran flips 0 heads, then it is impossible for the result of the die roll to equal the number of heads flipped. On the other hand, if Fran flips a nonzero number of heads ($1, 2, 3$, or 4 heads), then there is always a $\frac{1}{6}$ probability that the die roll matches the number of heads flipped. The probability that Fran flips 0 heads is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, so the probability that Fran flips a nonzero number of heads is $1 - \frac{1}{16} = \frac{15}{16}$. Therefore $p = \frac{15}{16} \cdot \frac{1}{6} = \frac{5}{32}$, and $pT = \boxed{5}$.

R3-5. Let $DM = x$. Because M is the midpoint of $\overline{BC}$, it follows that $CM = BM = x + \frac{3}{4}$.



By the Pythagorean Theorem on $\triangle ADM$ and $\triangle ADC$, it follows that $AD^2 + x^2 = 4^2$ and $AD^2 + (2x + \frac{3}{4})^2 = T^2$. Subtracting these two equations, $(2x + \frac{3}{4})^2 - x^2 = T^2 - 4^2$, which for $T = 5$ simplifies to $3x^2 + 3x - \frac{135}{16} = 0$, or $x^2 + x - \frac{45}{16} = 0$. This factors as $(x + \frac{9}{4})(x - \frac{5}{4}) = 0$, so $x = \frac{5}{4}$. Then $AD = \sqrt{4^2 - (\frac{5}{4})^2} = \sqrt{\frac{231}{16}}$, so $AB = \sqrt{AD^2 + (\frac{3}{4})^2} = \sqrt{\frac{240}{16}} = \boxed{\sqrt{15}}$.

R3-6. The numbers described are the triangular numbers, whose residues modulo 9 form a 9-cycle: $1, 3, 6, 1, 6, 3, 1, 0, 0, 1, 3, 6, \ldots$. $T = \sqrt{15}$, so $T^4 = 15^2 = 225$, a multiple of 9, so the remainder is $\boxed{0}$.

ARML Local 2019 Tiebreaker Problem

TB-1. Compute the number of orderings of the numbers $1, 2, \ldots, 9$ with the following property: if m comes before n in the ordering, then $m < n^2$.

ARML Local 2019 Tiebreaker Solution

TB-1. The given condition amounts to the following: 1 must come first in the permutation; 2 must come before $4, 5, 6, \ldots, 9$; and 3 must come before 9. After placing 1, it follows that 2 must either be in the second or third spot. If 2 is in the second spot, then of the 7! possible permutations of $3, 4, \ldots, 9$, exactly $\frac{1}{2}$ of them have 3 before 9, so there are $\frac{7!}{2}$ possible permutations. If 2 is in the third spot, then 3 must be in the second spot, and all 6! possible permutations of the remaining numbers $4, 5, \ldots, 9$ work. So the total number of permutations is $\frac{7!}{2} + 6! = \frac{5040}{2} + 720 = 2520 + 720 = \boxed{3240}$.

Answers to ARML Local 2019

Team Round:

T-1. 160	T-2. 12	T-3. $\sqrt[4]{2}$	T-4. $\dfrac{\sqrt{154}}{7}$	T-5. 14 or 18
T-6. $\dfrac{57}{11}$	T-7. $2\sqrt{2}$	T-8. $\dfrac{1}{6}$	T-9. $6\sqrt{2}$	T-10. $\dfrac{\pi}{1010}$
T-11. $\dfrac{53}{16}$	T-12. 24	T-13. 3296	T-14. $2\sqrt{21}-5$	T-15. $\dfrac{5\sqrt{5}-5}{2}$

Individual Round:

I-1. 39	I-2. $\dfrac{2\pi}{9}$	I-3. 26	I-4. $\dfrac{1}{22}$	I-5. 20
I-6. -375	I-7. 84	I-8. $\dfrac{7}{9}$	I-9. 40	I-10. 256

Relay Round:

R1-1. $\dfrac{1}{14}$ R1-2. 30

R2-1. 10 R2-2. 1021 R2-3. 70

R3-1. 21 R3-2. 3 R3-3. 32 R3-4. 5 R3-5. $\sqrt{15}$ R3-6. 0

Tiebreaker:

TB-1. 3240

ARML Local 2020

Contents

ARML Local 2020 Team Problems

T-1. The *primorial* of a natural number is the product of all primes less than or equal to that natural number. For example, the primorial of 4 is $2 \cdot 3 = 6$. Compute the number of natural numbers whose primorial is 210.

T-2. Compute the number of whole numbers less than 1000 whose digit sum is 4.

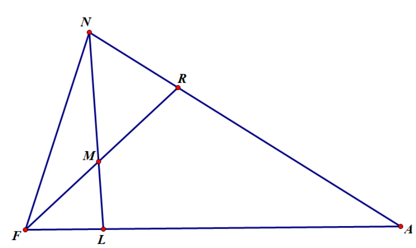
T-3. The *ARML Averages* is a baseball team for which every batter has a batting average of 0.270. That is, the probability that a batter will make a hit when they bat is 0.270; otherwise, the batter makes an out. Each time at bat is considered independent of all other times at bat. Given that the *ARML Averages* make 27 outs in a baseball game, and that the game ends when the 27th out is made, compute the integer closest to the number of times at bat they should expect to have in a game.

T-4. Twenty fair 20-sided dice, each of whose sides are numbered $1, 2, \ldots, 20$, are rolled, and all rolls are independent. Any dice that rolled prime numbers are discarded, and the remaining dice are rolled again. Compute the expected value of the sum of the results of the rerolled dice.

T-5. Compute the least positive integer n for which $10n + 1$ and $10n + 9$ are prime but $10n + 3$ and $10n + 7$ are composite.

T-6. Let n be a positive integer, and consider the list $1, 2, 2, 3, 3, 3, \ldots, n, n, \ldots, n$ where the integer k appears k times in the list for $1 \leq k \leq n$. The integer n will be called "*ARMLy*" if the median of the list is <u>not</u> an integer. The least ARMLy integer is 3. Compute the least ARMLy integer greater than 3.

T-7. In $\triangle FAN$, point R lies on $\overline{AN}$ so that $RA = 2 \cdot RN$ and point L lies on $\overline{AF}$ so that $LA = 3 \cdot LF$. Point M is the intersection of $\overline{FR}$ and $\overline{NL}$. Given that $[ARML] = 1974$, compute $[FAN]$.



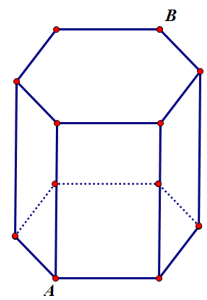
T-8. Triangle YOU and square $FOUR$ lie in the same plane, with R on $\overline{UY}$. Suppose that the intersection of $\overline{OY}$ and $\overline{FR}$ is point A and the intersection of $\overline{OY}$ and diagonal $\overline{UF}$ is point K. Given that $OK = 15$ and $KA = 9$, compute AY.

T-9. Let a positive integer be called *diverse* if all its decimal digits are distinct. For example, 2019 is diverse, but 2020 is not diverse. Compute the arithmetic mean of all diverse integers between 1000 and 2020 inclusive.

T-10. The polynomial $f(x) = x^3 - 6x^2 + 1$ has distinct real roots p, q, r. Compute the arithmetic mean of the values of $a^2b + b^2c + c^2a$ over all permutations (a, b, c) of (p, q, r).

T-11. The diagonals of a convex cyclic quadrilateral $ABCD$ are perpendicular and meet at point P. Suppose $\{PA, PB, PC, PD\}$ is a collection of four integers such that no element of the collection divides any other element of the collection. Compute the least possible area of $ABCD$.

T-12. An ant is at vertex A of the regular right hexagonal prism shown below. The side length of the hexagon is 1 and the height of the prism is 2.



Compute the shortest distance the ant can walk on the surface of the prism, passing through the interiors of at most three faces and no more than one hexagonal face, to get from point A to point B.

T-13. Compute the greatest value of n that satisfies $n + 20\lfloor\sqrt{n}\rfloor - 20\lfloor\sqrt[3]{n}\rfloor = 1000$.

T-14. The *tribonacci sequence* $T_0, T_1, T_2, \ldots$ is defined by $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 3$. Compute the number of non-empty subsets of $\{T_0, T_1, \ldots, T_{20}\}$ whose elements sum to a number in the tribonacci sequence.

T-15. Two externally tangent circles are drawn inside triangle ABC so that one is tangent to side $\overline{AB}$, the other is tangent to side $\overline{AC}$, and both are tangent to side $\overline{BC}$. Suppose that the product of the two circles' radii is 9. Given that $AB + BC = 35$ and $AC = 13$, compute the sum of the two circles' radii.

ARML Local 2020 Team Solutions

T-1. The set of primes is $\{2, 3, 5, 7, 11, \ldots\}$. Notice that $210 = 2 \cdot 3 \cdot 5 \cdot 7$, and the least prime number greater than 7 is 11. Therefore 7, 8, 9, and 10 have primorials of 210. Notice that natural numbers less than 7 have primorials no greater than $2 \cdot 3 \cdot 5 = 30$, and natural numbers greater than 10 have primorials that are at least $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. Thus the answer is $\boxed{4}$.

T-2. Suppose that leading 0's are allowed. For example, the whole number 020 has a leading 0 and represents the number 20.

If the digit sum of a whole number less than 1000 is 4, then the digits are either 4, 0, and 0, in some order; 3, 1, and 0, in some order; 2, 2, and 0, in some order; or 2, 1, and 1, in some order.

There are 3 whole numbers less than 1000 for which the digits are 4, 0, and 0, in some order: 400, 40, and 4.

There are 6 whole numbers less than 1000 for which the digits are 3, 1, and 0, in some order: 310, 301, 130, 103, 31, and 13.

There are 3 whole numbers less than 1000 for which the digits are 2, 2, and 0, in some order: 220, 202, and 22.

There are 3 whole numbers less than 1000 for which the digits are 2, 1, and 1, in some order: 211, 121, and 112.

Thus the answer is $3 + 6 + 3 + 3 = \boxed{15}$.

Alternate Solution: Use the "sticks-and-stones" approach. Allow leading 0's; then the problem is equivalent to finding solutions to $a + b + c = 4$ where a, b, and c are integers with $a, b, c \geq 0$. There are $\binom{4+2}{2} = 15$ such solutions.

T-3. If a batter's probability of getting a hit is 0.270, then the probability of making an out is 0.730. Because the 27th out ends the game, solve $\dfrac{27}{A} = \dfrac{730}{1000}$ to obtain the number of at-bats, which is $A = \dfrac{2700}{73} = 36\dfrac{72}{73} \approx \boxed{37}$.

T-4. The probability that a die survives to the next roll is $\frac{12}{20}$, as there are eight prime numbers less than or equal to 20, namely 2, 3, 5, 7, 11, 13, 17, and 19. Therefore the expected number of dice rerolled is 12, and the expected value of each die's roll is 10.5, so the expected value of the sum is $12 \cdot 10.5 = \boxed{126}$.

T-5. If $n = 3k$ for some integer k, then $10n + 9 = 3(10k + 3)$ is composite. Similarly, if $n = 3k + 2$ for some integer k, then $10n + 1 = 3(10k + 7)$ is composite. Thus $n = 3k + 1$ for some integer k. Note that $10n + 3 = 30k + 13$ is prime for $k = 0, 1, 2, 3, 5, 6, 7$ by inspection, with respective values of $10n + 3$ equal to 13, 43, 73, 103, 163, 193, and 223. Note also that $10n + 7 = 30k + 17$ is prime for $k = 4$ with $10n + 7 = 137$. Note also that $10n + 9 = 30k + 19$ is composite for $k = 8, 9, 10$ (with $10n + 9$ equaling $259 = 7 \cdot 37$, $289 = 17^2$, and $319 = 11 \cdot 29$, respectively) and $10n + 1 = 30k + 11$ is composite for $k = 11$ and 12 (with $10n + 1$ equaling $341 = 11 \cdot 31$ and $371 = 7 \cdot 53$, respectively).

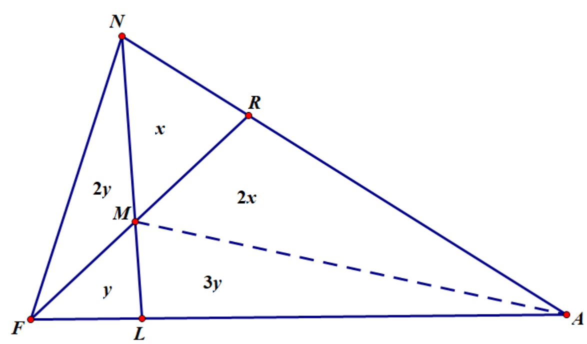
Finally, if $k = 13$, then $n = 40$, and $10n + 1 = 401$ and $10n + 9 = 409$ are prime while $10n + 3 = 403 = 13 \cdot 31$ and $10n + 7 = 407 = 11 \cdot 37$ are composite, so the answer is $n = \boxed{40}$.

T-6. The condition is satisfied if there exists a positive integer m such that $1, 2, 2, 3, 3, 3, \ldots, n, n, \ldots, n$ can be split into two sequences of equal length: $1, 2, 2, 3, 3, 3, \ldots, m, m, \ldots, m$ and $m+1, m+1, \ldots, n, n, \ldots, n$. Therefore $\frac{n(n+1)}{4} = \frac{m(m+1)}{2}$ or $n^2 + n = 2m^2 + 2m$. Multiplying both sides by 2 yields $2n^2 + 2n = 4m^2 + 4m$ or $n^2 + (n+1)^2 = (2m+1)^2$. The smallest Pythagorean triples satisfying the last equation are $(3, 4, 5)$, $(20, 21, 29)$, and $(119, 120, 169)$. Thus the three least integers satisfying the condition are 3, $\boxed{20}$, and 119. Note that these values can also be found by solving $2\binom{m}{2} = \binom{n}{2}$ using Pascal's Triangle and looking down the second column:

$$1, \underline{3}, \underline{6}, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, \underline{\mathit{105}}, 120, 136, 153, 171, 190, \underline{\mathit{210}}, \ldots$$

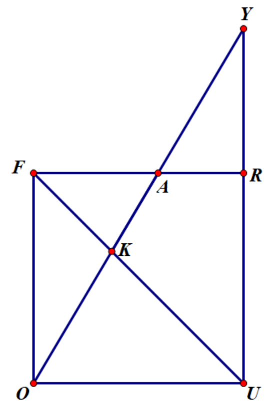
One can rewrite $2n^2 + 2n = 4m^2 + 4m$ as $(2n+1)^2 - 1 = 2((2m+1)^2 - 1)$. If $a = 2n + 1$ and $b = 2m + 1$, this is equivalent to Pell's equation $a^2 - 2b^2 = -1$, with solutions $(1, 1), (7, 5), (41, 29), \ldots$, from which it follows that $n = 3, 20, 119, \ldots$. One can also look at $n(n+1) = 2m(m+1)$ and factor quickly to reach $20 \cdot 21 = 2 \cdot 10 \cdot 21 = 2 \cdot 2 \cdot 5 \cdot 3 \cdot 7 = 2 \cdot 14 \cdot 15$.

T-7. Draw $\overline{AM}$. Because $RA = 2 \cdot RN$, it follows that $[ARM] = 2 \cdot [NRM]$. Similarly, $[ALM] = 3 \cdot [FLM]$. Also notice that $[ARF] = 2 \cdot [NRF]$ so $[AMF] = 2 \cdot [NMF]$ as shown in the figure.



Because $[ALN] = 3 \cdot [FLN]$, it follows that $3x + 3y = 3(3y)$, so $x = 2y$. Now, $[ARML] = 2x + 3y = 7y = 1974$, so $y = 282$ and $x = 564$. Thus $[FAN] = 3x + 6y = 3(564) + 6(282) = \boxed{3384}$.

T-8. Consider the diagram below.



Notice that $\triangle FAK \sim \triangle UOK$, so $\frac{FA}{UO} = \frac{AK}{OK} = \frac{3}{5}$. This implies $AR = \frac{2}{5}OU$. Also notice that $\triangle AFO \sim \triangle ARY$, which implies $\frac{AR}{AF} = \frac{AY}{AO}$, so $\frac{2}{3} = \frac{AY}{24}$. The answer is $AY = \boxed{16}$.

T-9. First consider diverse integers of the form $1\,\underline{A}\,\underline{B}\,\underline{C}$. There are $9 \cdot 8 \cdot 7 = 504$ diverse integers of this type. Note that A can be any digit in the set $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$ and for each choice of A, there are $8 \cdot 7 = 56$ diverse integers where A is the hundreds digit. Note also that B can be any digit in the set $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$ and for each choice of B, there are $8 \cdot 7 = 56$ diverse integers where B is the tens digit. Note further that C can be any digit in the set $\{0, 2, 3, 4, 5, 6, 7, 8, 9\}$ and for each choice of C, there are $8 \cdot 7 = 56$ diverse integers where C is the ones digit. Thus the sum of all diverse integers of the form $1\,\underline{A}\,\underline{B}\,\underline{C}$ is $504 \cdot 1000 + 56 \cdot (0 + 2 + 3 + \cdots + 9) \cdot 100 + 56 \cdot (0 + 2 + 3 + \cdots + 9) \cdot 10 + 56 \cdot (0 + 2 + 3 + \cdots + 9) \cdot 1 = 504000 + 56 \cdot 44 \cdot 111 = 777504$.

There are an additional 7 diverse integers 2013, 2014, ..., 2019 exceeding 1999 but not exceeding 2020, whose sum is $2016 \cdot 7 = 14112$. Therefore the answer is

$$\frac{777504 + 14112}{504 + 7} = \frac{791616}{511} = \boxed{\frac{113088}{73}}.$$

T-10. Let $S(a, b, c) = a^2b + b^2c + c^2a$. It suffices to compute the sum of $S(a, b, c)$ as (a, b, c) ranges through all permutations of the roots of $f(x)$. First note that $S(a, b, c) = S(c, a, b) = S(b, c, a)$ because the sum is cyclic. Similarly, $S(a, c, b) = S(b, a, c) = S(c, b, a)$, so it suffices to compute $S(a, b, c) + S(a, c, b)$. Note that $S(a, b, c) + S(a, c, b) = a^2(6 - a) + b^2(6 - b) + c^2(6 - c)$ by Vieta's formulas. Because $r^3 = 6r^2 - 1$ for $r = a, b, c$, it follows that $S(a, b, c) + S(a, c, b) = 3$. Therefore the average value of $a^2b + b^2c + c^2a$ over all permutations (a, b, c) of (p, q, r) is $\frac{3 \cdot 3}{6} = \boxed{\frac{3}{2}}$.

T-11. Consider the following lemma: If positive integers a, b, c, and d satisfy $ac = bd$, then there exist positive integers x, y, z, and t such that $a = xy$, $c = zt$, $b = xt$, and $d = yz$. The proof of the lemma proceeds as follows. Let $x = \gcd(a, b)$ and $z = \gcd(c, d)$. Then $a = xy_1$,

$b = xt_1$, $c = zt_2$, and $d = zy_2$ where y_1, y_2, t_1, and t_2 are positive integers. Note that $\gcd(y_1, t_1) = \gcd(y_2, t_2) = 1$. Because $ac = bd$, it follows that $xy_1zt_2 = xt_1zy_2 \to y_1t_2 = t_1y_2$. Because y_1t_2 is a multiple of t_1 and because y_1 is relatively prime to t_1, it follows that t_2 is a multiple of t_1. Similarly, t_1 is a multiple of t_2. This implies that $t_2 = t_1 = t$ and $y_2 = y_1 = y$. This completes the proof of the lemma.

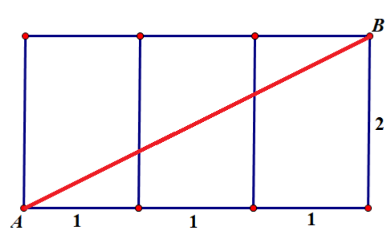
By the Power of a Point Theorem, it follows that $PA \cdot PC = PB \cdot PD$, so by the lemma, there exist positive integers x, y, z, t such that $PA = xy$, $PC = zt$, $PB = xt$, $PD = yz$. Note that $PA \nmid PB \iff y \nmid t$, and similarly, it follows that $t \nmid y$, $x \nmid z$, $z \nmid x$. In particular, none of x, y, z, t can be 1.

It must be that $\max(x, y, z, t) \geq 5$. Otherwise, two of x, y, z, t must be equal, say $x = y$, so $\{PA, PB, PC, PD\} = \{x^2, xt, zt, xz\}$. Then none of x, z, t divides one another, so it is impossible for them to be contained in $\{2, 3, 4\}$. (Note that $x = y$ is the only case that needs to be considered. If $x = z$, then $x \mid z$, which is a contradiction. If $x = t$, the results are similar to the case where $x = y$.) Thus assume, without loss of generality, that $t \geq 5$. Because x and z are distinct, assume $x \geq 2$ and $z \geq 3$. Also, assume $y \geq 2$. Note that $[ABCD] = \frac{AC \cdot BD \cdot \sin(\angle APB)}{2} = \frac{AC \cdot BD}{2} = \frac{(PA+PC)(PB+PD)}{2}$. Then

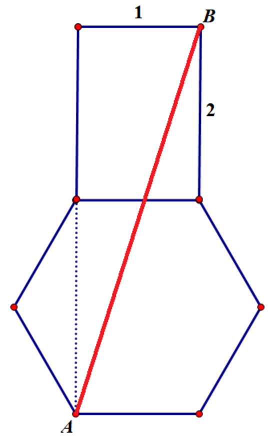
$$[ABCD] = \frac{1}{2}(xy + zt)(xt + yz) \geq \frac{1}{2}(2 \cdot 2 + 3 \cdot 5)(2 \cdot 5 + 2 \cdot 3) = \boxed{152}.$$

This is indeed achievable: one ordered quadruple satisfying these conditions is $(x, y, z, t) = (2, 2, 3, 5)$, and this implies $PA = 4$, $PB = 10$, $PC = 15$, $PD = 6$.

T-12. There are many different ways that the ant could travel along the surface of the prism from point A to point B. However, it appears that only three do so in ways that result in short straight-line distances from A to B across the unfolded net of the prism: a path across 3 rectangles, a path across one hexagon and two rectangles, and a path across one hexagon and one rectangle. Calculate each distance and determine which is the shortest. First consider the shortest path that crosses three rectangles. The ant would travel a path of length $\sqrt{3^2 + 2^2} = \sqrt{13}$, as shown below.

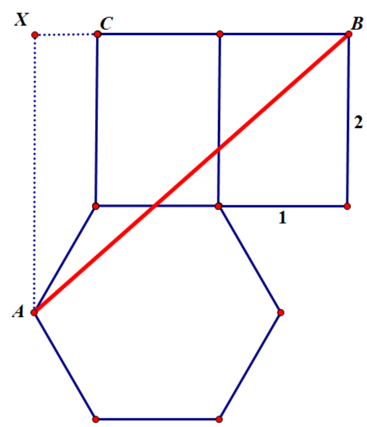


Next consider paths that cross one hexagon and one rectangle. One such path is shown below. Other paths pass through a rectangle and then a hexagon before ending at B. By symmetry, there is no need to consider them separately. Because the hexagon has side length 1, by the Law of Cosines, the dotted segment has length $\sqrt{1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos(120^\circ)} = \sqrt{3}$. Thus the ant would travel a path of length $\sqrt{1^2 + (2 + \sqrt{3})^2} = \sqrt{8 + 4\sqrt{3}}$.



Finally consider paths that cross through one hexagon and two rectangles. One such path is shown below, and there is no need to consider other such paths because of symmetry. Again, because the hexagon has side length 1, the ant would travel a path of length

$$\sqrt{\left(2+\frac{\sqrt{3}}{2}\right)^2+\left(2+\frac{1}{2}\right)^2}=\sqrt{11+2\sqrt{3}}.$$



Compare the radicands to establish which distance is the least. Note that $11+2\sqrt{3} > 11+2(1.5)=13$, so $\sqrt{13}<\sqrt{11+2\sqrt{3}}$. Note also that $8+4\sqrt{3}>8+4(1.5)=14>13$, so $\sqrt{13}<\sqrt{8+4\sqrt{3}}$. It can be seen that the path across three rectangles is the shortest of the three. Thus the minimum distance is $\boxed{\sqrt{13}}$.

T-13. First, suppose that $\lfloor\sqrt{n}\rfloor = \lfloor\sqrt[3]{n}\rfloor = k$, where $k \geq 3$. The equation $\lfloor\sqrt{n}\rfloor = k$ implies the inequality

$$(k+1)^2 > n \geq k^2 \tag{1}$$

and the equation $\lfloor\sqrt[3]{n}\rfloor = k$ implies the inequality

$$(k+1)^3 > n \geq k^3. \tag{2}$$

But by induction, it follows that

$$k^3 > (k+1)^2 \text{ for } k \geq 3. \tag{3}$$

Hence, by combining (1), (2), and (3), it follows that

$$n \geq k^3 > (k+1)^2 > n,$$

which is impossible. Hence $\lfloor \sqrt{n} \rfloor - \lfloor \sqrt[3]{n} \rfloor \geq 1$ for $n \geq 3^2 = 9$. Note further that it is impossible to have $\lfloor \sqrt[3]{n} \rfloor \geq 10$, because otherwise,

$$n + 20\lfloor \sqrt{n} \rfloor - 20\lfloor \sqrt[3]{n} \rfloor \geq 10^3 + 20(1) > 1000.$$

Suppose that $\lfloor \sqrt[3]{n} \rfloor \geq 9$. Then by (2), it follows that $10^3 - 1 \geq n \geq 9^3$, which is equivalent to $729 \leq n \leq 999$. By (1), it follows that $\sqrt{n} \geq \sqrt{729} = 27$. Hence

$$n + 20\lfloor \sqrt{n} \rfloor - 20\lfloor \sqrt[3]{n} \rfloor \geq 729 + 20(27 - 9) = 1089 > 1000.$$

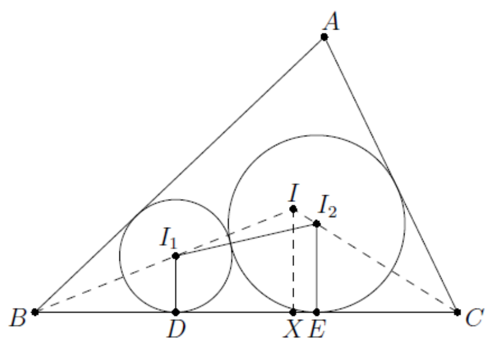
Thus $\lfloor \sqrt[3]{n} \rfloor \leq 8$. Next, suppose that $\lfloor \sqrt[3]{n} \rfloor = 8$. The previously established inequalities imply that

$$728 \geq n \geq 512 \quad \text{and thus} \quad 26 \geq \lfloor \sqrt{n} \rfloor \geq 22.$$

If $\lfloor \sqrt{n} \rfloor = 26$, then $n = 20(50 - \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor) = 20(50-26+8) = 640$, but this leads to a contradiction because $\lfloor \sqrt{640} \rfloor = 25$. However, if $\lfloor \sqrt{n} \rfloor = 25$, then $n = 20(50 - \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor) = 20(50 - 25 + 8) = 660$, and indeed, $\lfloor \sqrt{660} \rfloor = 25$. Because decreasing values of n have been considered up until this point, it follows that $n = \boxed{660}$ is the greatest solution to the given equation. Moreover, by continuing the preceding analysis, it can be established that $n = 660$ is the *unique* solution to the given equation.

T-14. Note that every $\{T_i\}$ for $0 \leq i \leq 20$ is a valid subset and if S is a valid subset with $0 \notin S$, then $S \cup \{0\}$ is also a valid subset. Therefore it suffices to examine the valid subsets of $X = \{T_1, T_2, \ldots, T_{20}\}$ with more than one element. Note that it can be shown by induction that $\sum\limits_{j=1}^{k} T_j < T_{k+2}$ for all natural numbers k. Let B_k be the set of desired subsets, with more than one element, where its largest element is T_k. Furthermore, let $n_k = |B_k|$ and suppose $S \in B_k$. Then the sum of elements in S is strictly between T_k and T_{k+2} by the main inequality, so it must be equal to T_{k+1}. Setting up for a recursive method of computing n_k, consider a set $Y \in B_k$ where $k \geq 3$. Furthermore, let $S_0 = T_{k+1}$ be the sum of elements in Y and let $Y_0 = \{T_1, T_2 \ldots, T_k\}$. Therefore it suffices to find the subsets of $Y_1 = Y_0 \setminus \{T_k\}$ with sum $S_1 = T_{k+1} - T_k = T_{k-1} + T_{k-2}$. If $T_{k-1} \notin Y$, then $S_1 \leq T_{k-2} + \left(\sum\limits_{j=1}^{k-3} T_j\right) < T_{k-2} + T_{k-1}$, which is a contradiction. If $T_{k-2} \in Y$, then $Y = \{T_k, T_{k-1}, T_{k-2}\}$ is a valid subset, and so there cannot be any more elements in Y. Otherwise, it suffices to find the subsets of $Y_2 = Y_1 \setminus \{T_{k-1}, T_{k-2}\}$ with sum $S_2 = S_1 - T_{k-1} = T_{k-2}$. By definition, there are n_{k-3} such sets so this yields $n_k = 1 + n_{k-3}$. Computing the first few values, note that $n_1 = 0$, $n_2 = 1$, and $n_3 = 1$, so $n_k = \left\lfloor \frac{k+1}{3} \right\rfloor$ for all $k \in \mathbb{N}$. Therefore the number of valid subsets of X with more than one element is $\sum_{j=1}^{20} n_j = 3(1+2+\cdots+6)+7 = 70$. Because 0 can be appended to each of these subsets, this creates 140 valid subsets. However, $\{T_i\}$ for $0 \le i \le 20$ and $\{0, T_i\}$ for $1 \le i \le 20$ are also valid subsets. Therefore there are $140+21+20 = \boxed{181}$ non-empty subsets of $\{T_0, T_1, \ldots, T_{20}\}$ whose elements sum to a number in the tribonacci sequence.

T-15. Let the circles be Γ_1 and Γ_2 with radii r_1 and r_2, respectively. Let Γ_1, Γ_2, and the incircle of triangle ABC have centers I_1, I_2, and I, respectively, and touch $\overline{BC}$ at D, E, and X, respectively. By definition, $DI_1 = r_1$, $EI_2 = r_2$, and let $XI = r$. Then $\triangle BDI_1$ and $\triangle BXI$ are similar right triangles, so $\frac{BD}{r_1} = \frac{BX}{r} = \frac{s-b}{r}$, where s is the semiperimeter of $\triangle ABC$. It follows that $BD = \frac{r_1(s-b)}{r}$ and similarly, $CE = \frac{r_2(s-c)}{r}$. Note that $s-a$, $s-b$, and $s-c$ are all positive.



From right trapezoid DI_1I_2E, it follows that $DE^2 + (r_2-r_1)^2 = (r_2+r_1)^2$ by the Pythagorean Theorem. Simplifying this gives $DE = 2\sqrt{r_1r_2}$, so

$$a = BD + DE + EC = \frac{r_1(s-b)}{r} + \frac{r_2(s-c)}{r} + 2\sqrt{r_1r_2}.$$

Applying the AM-GM Inequality on $r_1(s-b)$ and $r_2(s-c)$ gives $\frac{1}{2}\left(r_1(s-b)+r_2(s-c)\right) \ge \sqrt{r_1r_2(s-b)(s-c)}$, so

$$a \ge \frac{2\sqrt{r_1r_2(s-b)(s-c)}}{r} + 2\sqrt{r_1r_2} = 2\sqrt{r_1r_2}\left(\sqrt{\frac{s}{s-a}}+1\right)$$

because $r = \frac{[ABC]}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$ by Heron's Formula. Applying the AM-GM Inequality to $\frac{4(s-a)}{s}$, $\frac{1}{2}\sqrt{\frac{s}{s-a}}$, and $\frac{1}{2}\sqrt{\frac{s}{s-a}}$, implies $\frac{1}{3}\left(\frac{4(s-a)}{s} + \sqrt{\frac{s}{s-a}}\right) \ge 1$, so

$$a \ge 2\sqrt{r_1r_2}\left(4 - \frac{4(s-a)}{s}\right) = a \cdot \frac{8\sqrt{r_1r_2}}{s}.$$

Because $r_1r_2 = 9$ and $s = 24$, the right-hand side is just a, so equality must hold in each inequality. Thus $r_1(s-b) = r_2(s-c)$ and $\frac{4(s-a)}{s} = \frac{1}{2}\sqrt{\frac{s}{s-a}}$. The latter implies $\frac{s-a}{s} = \frac{1}{4}$, so

$a = \frac{3s}{4} = 18$. With $b = 13$, it follows that $c = 17$. Then $11r_1 = 7r_2$, but $r_1r_2 = 9$, so $r_1 = 3\sqrt{\frac{7}{11}}$ and $r_2 = 3\sqrt{\frac{11}{7}}$. Thus the answer is

$$r_1 + r_2 = 3\left(\sqrt{\frac{7}{11}} + \sqrt{\frac{11}{7}}\right) = \frac{3}{77}\left(7\sqrt{77} + 11\sqrt{77}\right) = \boxed{\frac{54\sqrt{77}}{77}}.$$

ARML Local 2020 Individual Problems

I-1. There are four distinct prime factors of 999039. Compute the sum of these four prime factors.

I-2. Suppose that m and n are real numbers such that $\frac{7}{2020} < \frac{m}{m+n} < \frac{9}{2020}$. Compute the number of different possible integer values of $\frac{n}{m}$.

I-3. In rectangle $ARML$, diagonal $\overline{RL}$ is drawn along with altitudes $\overline{AX}$ of $\triangle LAR$ and $\overline{MY}$ of $\triangle RML$. Given that $AR = 3$ and $RM = 4$, compute XY.

I-4. A real number x is selected uniformly at random between 0 and π. Compute the probability that $\sin(x)\cos(x) < \frac{1}{4}$.

I-5. In a Taekwondo class of 21 students, the students compete in "sparring teams" of three students. Suppose that there are 56 different sparring teams that have ever been formed by the students in the class (here, two teams are the same if they consist of the same three students and different otherwise). Given that each student has been, on average, a part of N different sparring teams, compute N.

I-6. Define points P_k in the plane so that $P_n = (n^2, n^2)$ if n is even and $P_n = (n^2, n^2 - 1)$ if n is odd. Given that points X_k lie on the x-axis, compute the least possible length of the path $P_1X_1P_2X_2 \cdots P_{98}X_{98}P_{99}$.

I-7. Let $ABCD$ be a unit square. Points E and F are chosen randomly, uniformly, and independently on $\overline{AB}$ and $\overline{CD}$, respectively, with all points on each segment equally likely to be chosen. Compute the probability that

$$\max\{[AEFD], [BEFC]\} \geq \frac{3}{4}.$$

I-8. Amy adds some positive numbers together and gets 17. Bella multiplies those same positive numbers together and gets N. Compute the least positive integer that <u>cannot</u> be N.

I-9. Let $N = 2020^{20}$. Given that $10^{0.3010} < 2 < 10^{0.3011}$, compute the number of digits in the base-10 expansion of N.

I-10. Compute the number of five-term sequences of positive integers a_1, a_2, a_3, a_4, a_5 that satisfy $\frac{a_k}{a_{k-1}} \geq k$ for $2 \leq k \leq 5$ and $a_5 < 250$.

ARML Local 2020 Individual Solutions

I-1. Rewrite $999039 = 1000000 - 961 = 1000^2 - 31^2 = (1031)(969)$. Factor 969 as $969 = 3 \cdot 323 = 3 \cdot 17 \cdot 19$. The number 1031 is prime, so the sum of the four prime factors is $3+17+19+1031 = \boxed{1070}$.

I-2. Taking reciprocals yields $\dfrac{2020}{7} > \dfrac{m+n}{m} > \dfrac{2020}{9}$, which implies $287\frac{4}{7} > \dfrac{n}{m} > 223\frac{4}{9}$. Thus $\dfrac{n}{m}$ can be any integer between 224 and 287 inclusive, so the answer is $287 - 224 + 1 = \boxed{64}$.

I-3. The Pythagorean Theorem implies that $RL = 5$. Because $\triangle LAR \sim \triangle MYL$ and $\triangle MYL \cong \triangle AXR$, it follows that $\frac{LY}{LM} = \frac{RA}{RL}$ and $RX = LY = \dfrac{9}{5}$. Thus $XY = 5 - 2 \cdot \dfrac{9}{5} = \boxed{\dfrac{7}{5}}$.

I-4. The quantity $\sin(x)\cos(x)$ is $\frac{1}{2}\sin(2x)$, so the requested probability is the same as the probability that $\sin(x) < \dfrac{1}{2}$ between 0 and 2π. The only part of that interval where $\sin(x) \geq \dfrac{1}{2}$ is between $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$. Thus the desired probability is $1 - \dfrac{2\pi/3}{2\pi} = \boxed{\dfrac{2}{3}}$.

I-5. Suppose that a student S has been on $T(S)$ teams. Consider the quantity $\sum T(S)$, where the sum is taken over all 21 students S. Each team that has been formed will be counted 3 times by this sum, once in the count $T(S)$ for each student S in the team. Because there are 56 teams, it follows that $\sum T(S) = 3 \cdot 56 = 168$. So the average value of $T(S)$ is $\frac{168}{21} = \boxed{8}$.

I-6. By the reflection principle, the shortest distance $P_1X_1P_2$ is the distance between $P_1 = (1, 0)$ and the reflection of $P_2 = (4, 4)$ across the x-axis which is $P_2' = (4, -4)$. Notice that

$$P_1P_2' = \sqrt{(4-1)^2 + (-4-0)^2} = 5 = 1^2 + 2^2.$$

Similarly, the shortest distance $P_2X_2P_3$ is the distance between $P_2 = (4, 4)$ and $P_3' = (9, -8)$ which is the reflection of $P_3 = (9, 8)$ across the x-axis. Thus

$$P_2P_3' = \sqrt{(9-4)^2 + (-8-4)^2} = 13 = 2^2 + 3^2.$$

It can be verified that

$$P_kP_{k+1}' = \sqrt{((k+1)^2 - k^2)^2 + (1 - (k+1)^2 - k^2)^2} = \sqrt{4k^4 + 8k^3 + 8k^2 + 4k + 1} = k^2 + (k+1)^2,$$

and so

$$\begin{aligned}
\min(P_1X_1P_2X_2P_3 \cdots P_{99}) &= 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + \cdots + 98^2 + 98^2 + 99^2 \\
&= 2(1^2 + 2^2 + \cdots + 99^2) - 1^2 - 99^2 \\
&= 2 \cdot \frac{99 \cdot 100 \cdot 199}{6} - 9802 \\
&= \boxed{646898}.
\end{aligned}$$

I-7. Let $AE = x$ and $DF = y$, in which case $[AEFD] = \frac{x+y}{2}$ and $[BEFC] = 1 - \frac{x+y}{2}$. Then the larger of those two areas being at least $\frac{3}{4}$ corresponds to either $x + y \le \frac{1}{2}$ or $x + y \ge \frac{3}{2}$. Considering the unit square of possible values of (x, y), these two regions have combined area $\boxed{\dfrac{1}{4}}$.

I-8. Suppose that n positive numbers add to 17. Define $a_n = \left(\frac{17}{n}\right)^n$ for natural numbers n, and suppose that $M = \max a_n$ over all natural n. By the AM-GM Inequality, it follows that $N \le M$. Consider the following calculations to establish the value of n for which $\left(\frac{17}{n}\right)^n$ is the greatest.

$\left(\frac{17}{1}\right)^1 = 17,$

$\left(\frac{17}{2}\right)^2 < 9^2 = 81,$

$\left(\frac{17}{3}\right)^3 < 6^3 = 216,$

$\left(\frac{17}{4}\right)^4 = ((4.25)^2)^2 < 20^2 = 400,$

$\left(\frac{17}{5}\right)^5 = ((3.4)^2)^2 \cdot 3.4 < 12^2 \cdot 3.4 = 144 \cdot 3.4 < 490,$

$\left(\frac{17}{6}\right)^6 = \left(\frac{289}{36}\right)^3 > \left(\frac{288}{36}\right)^3 = 8^3 = 512,$

$\left(\frac{17}{7}\right)^7 = \left(\frac{17}{7}\right)^2 \cdot \left(\frac{17}{7}\right)^2 \cdot \left(\frac{17}{7}\right)^2 \cdot \left(\frac{17}{7}\right) = \frac{289}{49} \cdot \frac{289}{49} \cdot \frac{289}{49} \cdot \frac{17}{7} < 5.9 \cdot 5.9 \cdot 6 \cdot \frac{17}{7} < 35 \cdot 6 \cdot \frac{17}{7} = 5 \cdot 6 \cdot 17 = 510,$

$\left(\frac{17}{8}\right)^8 = ((2.125)^2)^4 < 4.6^4 < 22^2 = 484,$

$\left(\frac{17}{9}\right)^9 < 2^9 = 512,$

$\left(\frac{17}{10}\right)^{10} = ((1.7)^2)^5 = (2.89)^5 < 3^5 = 243,$

$\left(\frac{17}{11}\right)^{11} = \left(\frac{17}{11}\right)^{10} \cdot \frac{17}{11} < 243 \cdot 2 = 486,$

$\left(\frac{17}{12}\right)^{12} = \left(\frac{289}{144}\right)^6 < ((2.1)^2)^3 < 5^3 = 125,$

$\left(\frac{17}{13}\right)^{13} = \left(\frac{17}{13}\right)^{12} \cdot \frac{17}{13} < 125 \cdot 2 = 250,$

$\left(\frac{17}{14}\right)^{14} = \left(\frac{17}{14}\right)^{13} \cdot \frac{17}{14} < 250 \cdot 2 = 500,$

$\left(\frac{17}{15}\right)^{15} < \left(\frac{18}{15}\right)^{15} = (1.2)^{15} = ((1.2)^3)^5 < 2^5 = 32,$

$\left(\frac{17}{16}\right)^{16} = \left(\frac{17}{16}\right)^{15} \cdot \frac{17}{16} < 32 \cdot 2 = 64.$

It is clear that for $n \ge 17$, it follows that $\left(\frac{17}{n}\right)^n \le \left(\frac{17}{17}\right)^n = 1$, so $M = a_6$. By the Binomial Theorem, $\left(\frac{17}{6}\right)^6 = \left(3 - \frac{1}{6}\right)^6 = \sum_{k=0}^{6} \binom{6}{k} 3^{6-k} \left(-\frac{1}{6}\right)^k = \sum_{k=0}^{6} (-1)^k \binom{6}{k} 9^{3-k} \left(\frac{1}{2}\right)^k = 729 - 6 \cdot \frac{81}{2} + 15 \cdot \frac{9}{4} - 20 \cdot \frac{1}{8} + 15 \cdot \frac{1}{9 \cdot 16} - \cdots$, which is approximately $486 + 33.75 - 2.5 + 0.1 - \cdots \approx 517.35$, so M is between 517 and 518. Thus 518 cannot be N for any set of positive numbers.

For $0 \le x \le 17$, consider $f(x) = \frac{17^4 \cdot x(34-x)}{6^6}$. Note that f is continuous with $f(0) = 0$ and $f(17) = a_6$. By the Intermediate Value Theorem, for every natural number K strictly between 0 and a_6, there exists x such that $f(x) = K$. Thus every K strictly between 0 and a_6 could be N for some set of positive numbers. Thus $\boxed{518}$ is the minimal value that cannot be N.

Alternate Solution: Define M as above. Consider $F(x) = \frac{\ln x}{x}$ for positive x. Because $F'(x) = \frac{1-\ln x}{x^2}$, it follows that F is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$. Note that the function $f(x) = e^x$ is strictly increasing for all real numbers, and thus the function $e^{F(x)} = x^{1/x}$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$. Note also that the function $g(x) = x^{17}$ is strictly increasing for all real numbers, so the function $G(x) = x^{17/x}$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$.

Now consider that $a_n = \left(\frac{17}{n}\right)^n = \left(\frac{17}{n}\right)^{17\cdot(n/17)} = G\left(\frac{17}{n}\right)$. Because

$$\ldots < \frac{17}{8} < \frac{17}{7} < 2.5 < e < 2.8 < \frac{17}{6} < \frac{17}{5} < \ldots < \frac{17}{1},$$

and because $G(x)$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, \infty)$, it follows that $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$ and $a_7 > a_8 > a_9 > \ldots$. As above, it can be shown that $a_6 > a_7$. Thus $M = a_6$, and the solution follows as above.

I-9. The number of digits in N is $\lfloor \log_{10} N \rfloor + 1$. Suppose that $\epsilon = \log_{10} 1.01$. Note that

$$\begin{aligned}
\log_{10} N &= 20 \log_{10}(2020) \\
&= 20\left(\log_{10} 101 + \log_{10} 2 + \log_{10} 10\right) \\
&< 20\left(2 + \epsilon + 0.3011 + 1\right), \text{ where } 0 \le \epsilon \le 0.01 \\
&= 20 \cdot (3.3011 + \epsilon) \\
&= 66.022 + 20\epsilon < 67.
\end{aligned}$$

Accordingly, N has $\boxed{67}$ digits.

Note: To justify $\epsilon \le 0.01$, note that $\log_{10}(1 + x) < \ln(1 + x) < x$ for positive x.

I-10. Given the increasing sequence of positive integers $\mathcal{S} = (a_1, a_2, a_3, a_4, a_5)$, define the subsequence (a_1, a_2, a_3) to be *left-acceptable* if $a_2 \ge 2a_1$ and $a_3 \ge 3a_2$, and define the subsequence (a_3, a_4, a_5) to be *right-acceptable* if $a_4 \ge 4a_3$ and $a_5 \ge 5a_4$. Then N is the number of integer sequences $(a_1, a_2, a_3, a_4, a_5)$ such that (a_1, a_2, a_3) is left-acceptable, (a_3, a_4, a_5) is right-acceptable, $a_1 \ge 1$, and $a_5 < 250$. Note that $a_3 < 13$, as otherwise, $a_4 \ge 52$ and $a_5 \ge 260$, which violates the requirement that $a_5 < 250$. Similarly, it follows that (a_1, a_2) must be one of $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 4)$, and $a_3 \ge 6$. For convenience, define R_k to be the number of right-acceptable triples (a_3, a_4, a_5) with $a_3 = k$. To count R_k, first observe that if (x, y, z) is right-acceptable, then for $0 < x' < x$, the triple (x', y, z) is also right-acceptable. For $6 \le k < 12$, consider counting $R_k - R_{k+1}$, that is, the number of right-acceptable triples with $a_3 = k$, but where (a_4, a_5) is not one of the R_{k+1} possible ordered pairs for a right-acceptable triple in which $a_3 = k + 1$. For these triples, $4k \le a_4 \le 4k + 3$, and these values of a_4 yield a total of $(250 - 5 \cdot 4k) + (250 - 5 \cdot (4k + 1)) + (250 - 5 \cdot (4k + 2)) + (250 - 5 \cdot (4k + 3)) = 970 - 80k$ possible ordered pairs (a_4, a_5). Thus

$$R_k = 970 - 80k + R_{k+1}. \tag{$*$}$$

Furthermore, note that if $a_3 = 12$, then it follows that $a_4 \ge 48$ and $5 \cdot 48 = 240 \le a_5 < 250$, so $(a_4, a_5) = (48, a_5)$ or $(49, a_5)$. If $a_4 = 48$, then there are 10 possible values for a_5:

$240, 241, \ldots, 249$. If $a_4 = 49$, then there are 5 possible values for a_5: $245, 246, 247, 248, 249$. Thus $R_{12} = 15$. Now consider the possible left-acceptable triples. If (a_1, a_2) is $(1, 2)$, then a_3 must be at least 6, and if (a_1, a_2) is $(1, 3)$, then a_3 must be at least 9. If (a_1, a_2) is either $(1, 4)$ or $(2, 4)$, then a_3 must equal 12. Using $(*)$ to compute the values of R_k, the following calculations give values for R_{11}, R_{10}, R_9, R_8, R_7, and R_6: $R_{11} = 970 - 80 \cdot 11 + 15 = 105$, $R_{10} = 970 - 80 \cdot 10 + 105 = 275$, $R_9 = 970 - 80 \cdot 9 + 275 = 525$, $R_8 = 970 - 80 \cdot 8 + 525 = 855$, $R_7 = 970 - 80 \cdot 7 + 855 = 1265$, and $R_6 = 970 - 80 \cdot 6 + 1265 = 1755$.
It therefore follows that the desired number of sequences is

$$\begin{aligned}
&R_6 + R_7 + R_8 + 2(R_9 + R_{10} + R_{11}) + 4R_{12} \\
&= 1755 + 1265 + 855 + 2(525 + 275 + 105) + 4 \cdot 15 \\
&= \boxed{5745}.
\end{aligned}$$

ARML Local 2020 Collaborative Relay Problems

R1-A. Some of the prime factors of 16005 are single-digit prime numbers. Compute the sum of these single-digit primes.

R1-B. Let T = TNYWR. Compute the number of distinct noncongruent triangles whose vertices are vertices of a regular T-gon with side length 1. *Note that in this problem, consider two triangles T_1 and T_2 noncongruent if and only if the following is true: for any of the 6 ways to label all vertices of T_1 as A_1, B_1, and C_1 and for any of the 6 ways to label all vertices of T_2 as A_2, B_2, and C_2, the triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are noncongruent.*

R2-A. A bus in Springfield has a maximum capacity of 40 passengers. The bus begins its route with N passengers. At the first stop, no one gets on the bus and 4 passengers get off the bus, at which time the number of passengers on the bus is divisible by 7. At the second stop, no one gets off the bus and 2 passengers get on the bus, at which time the number of passengers on the bus is divisible by 5. Compute N.

R2-B. Let T = TNYWR. Let $a_1 = -T$, $a_2 = 20.20$, and $a_n = a_{n-1} + a_{n-2}$ for each $n \geq 3$. Compute the greatest integer N for which a_N is negative.

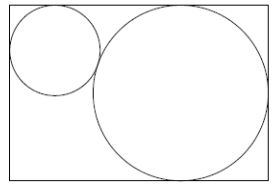
R3-A. Compute the greatest possible value of $x + y$ where x and y are positive integers and $11x + 21y = 2020$.

R3-B. Let T = TNYWR. Three cubes of side length 1, 7, and n cm are to be glued together at their faces. Given that the minimum possible surface area of the resulting solid is $2T$ square cm, compute n.

RA-1. Compute the least positive integer x for which $6x^2 - 7x > 20$.

RA-2. Let T = TNYWR. Consider the geometric sequence $4, -\dfrac{3}{2}, \dfrac{9}{16}, \ldots$. The sum of the infinite series of these numbers is S and the sum of the first T numbers of the sequence is S_T. Compute $S \cdot S_T$.

RA-3. Let T = TNYWR. Two circles inside a rectangle are externally tangent to each other, one with radius 3 and the other with radius $r > 3$. The smaller circle is tangent to two sides of the rectangle while the larger circle is tangent to three sides, as shown. Given that the rectangle has area $11 \cdot T$, compute r.



RB-1. Compute the value of $\dfrac{2020^2 - 17 \cdot 2022 - 4}{2022}$.

RB-2. Let T be the number you will receive. Let $f(n)$ denote the $2n$-digit number that results from adjoining n 20s together. For example, $f(4) = 20202020$. Compute the greatest single-digit divisor of $f(T)$.

RB-3. Let T be the digit you will receive. Compute the number of T-digit positive integers such that each digit from 1 to T inclusive appears exactly once and the sum of any three consecutive digits is a multiple of 3.

R3-1. Line ℓ has a slope of 3, and is tangent to the circle $x^2 + y^2 = 40$ at a lattice point (i.e., a point with integer coordinates) in the second quadrant. Given that this lattice point has coordinates (a, b), compute $a + b$.

R3-2. Let $T = \text{TNYWR}$. The point (T, T) is the midpoint of the line segment whose endpoints are $(-3, 5)$ and (A, B). Compute $A + B$.

R3-3. Let $T = \text{TNYWR}$. Consider the triangle bounded by the coordinate axes and the graph of $3x + 2y = T$. Compute the area of this triangle.

R3-4. Let $T = \text{TNYWR}$. Consider the triangle bounded by the coordinate axes and the graph of $5x + 2y = T + 3$. Compute the area of this triangle.

R3-5. Let $T = \text{TNYWR}$. Compute the distance between the point $(8, 4)$ and the line $3x + 9y = 2T$.

R3-6. Let $T = \text{TNYWR}$. Given that A and B are acute angles such that $\tan A = \dfrac{1}{3}$ and $\tan B = T$, compute $\sin(2A) + \cos(2B)$.

ARML Local 2020 Collaborative Relay Solutions

R1-A. Because $16005 = 3 \cdot 5 \cdot 11 \cdot 97$, the answer is $3 + 5 = \boxed{8}$.

R1-B. Knowing that $T = 8$, consider how many distinct noncongruent triangles can be formed whose vertices are vertices of regular octagon $ARMLYGON$ with side length 1. Note that $AM^2 = 1^2+1^2-2\cdot1\cdot1\cdot\cos 135^\circ = 2+\sqrt{2}$ by the Law of Cosines. Note also that if two triangles are congruent, their longest sides are congruent, so if two triangles T_1 and T_2 are such that their longest sides are not congruent, then T_1 is noncongruent to T_2. Because $AM^2 = 2 + \sqrt{2} > 1^2$, it follows that $\overline{AM}$ is the longest side of $\triangle ARM$. Note that $\text{m}\angle RAM = \text{m}\angle RMA = 22.5^\circ$, so $\text{m}\angle AML = 135^\circ - 22.5^\circ = 112.5^\circ > 90^\circ$, so $\angle M$ is the largest angle of $\triangle AML$ and thus $\overline{AL}$ is the longest side of $\triangle AML$, so $AL > AM$. Note also that $\angle ALY$ is a right angle, so its hypotenuse $\overline{AY}$ is the longest side of $\triangle ALY$ and thus $AY > AL$. Therefore $AM < AL < AY$.

Without loss of generality, consider first all triangles for which one side is $\overline{AR}$. There are three distinct noncongruent triangles for which one side is $\overline{AR}$, namely $\triangle ARM$, $\triangle ARL$, and $\triangle ARY$, and these three are pairwise noncongruent because their longest sides are pairwise noncongruent. By symmetry, $\triangle ARM \cong \triangle RAN$, $\triangle ARL \cong \triangle RAO$, and $\triangle ARY \cong \triangle RAG$, so there are only three distinct noncongruent triangles for which one side is $\overline{AR}$. Now, consider triangles for which the shortest side is $\overline{AM}$ (again without loss of generality). The third vertex of such a triangle must be O or Y or G. Because $AO = AM = MY = \sqrt{2+\sqrt{2}}$ and because $AY = MO$ by symmetry, it follows that $\triangle AMY \cong \triangle MAO$. Note that $AM < AL < AY$ and $AG = AL$. Note also that $\overline{AY}$ is the longest side of $\triangle AMY$ and $\overline{AG}$ is the longest side of $\triangle AMG$, so $\triangle AMY$ is noncongruent to $\triangle AMG$, and both of these triangles are noncongruent to any of $\triangle ARM$, $\triangle ARL$, and $\triangle ARY$ because $AM > AR$. Thus there are two distinct noncongruent triangles of this type.

Now, consider triangles for which the shortest side is $\overline{AL}$. The other side of such a triangle must be a segment with one endpoint at A and at least as long as $\overline{AL}$. There are two such segments, namely $\overline{AG}$ and $\overline{AY}$. In $\triangle ALG$, note that $LG = AM < AL$. In $\triangle ALY$, note that $LY = AR < AL$. This results in a contradiction in both cases, and so there cannot be any triangle whose shortest side is $\overline{AL}$. Note also that $\overline{AY}$ is the longest segment that can be drawn with vertex A and so it cannot be the shortest side of any triangle one of whose vertices is A. Thus there are $\boxed{5}$ distinct noncongruent triangles that can be created by choosing three of the vertices of a regular octagon with side length 1.

R2-A. The problem's conditions imply that $N-4$ is a multiple of 7, or that $N-4 = 7k \rightarrow N = 7k+4$ for some non-negative integer k. Also, $N-2$ is a multiple of 5, which implies $N = 5m+2 \rightarrow 7k+2 = 5m$ for some non-negative integer m. Because the maximum capacity is 40, $k \leq 5$. In the cases where $k = 0$, 1, 2, 3, or 5, m is not an integer, and these are contradictions. In the case where $k = 4$, it follows that $m = 6$, and so $N = 7 \cdot 4 + 4 = 5 \cdot 6 + 2 = \boxed{32}$.

R2-B. Let $k = 20.20$. Then the sequence is $-T, k, -T + k, -T + 2k, -2T + 3k, \ldots$. In general, if F_i is the ith term in the Fibonacci sequence which begins $F_1 = F_2 = 1$, then it follows by induction that $a_n = -F_{n-2}T + F_{n-1}k$ for $n \geq 3$. The problem asks for the greatest integer N for which a_N is negative, which is equivalent to asking for the greatest integer N for which $-F_{n-2}T + F_{n-1}k < 0 \leftrightarrow \frac{F_{N-1}}{F_{N-2}} < \frac{T}{k}$. Of interest, the quotient of consecutive Fibonacci terms converges to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$, with consecutive quotients closer in absolute distance to φ and with $\frac{F_n}{F_{n-1}} < \varphi$ for even $n \geq 2$ and $\frac{F_n}{F_{n-1}} > \varphi$ for odd $n \geq 3$. With $T = 32$, $\frac{32}{20.20} < \frac{8}{5} = \frac{F_6}{F_5} < \varphi$, it follows by induction that $a_N > 0$ (because $\frac{F_{N-1}}{F_{N-2}} > \varphi$) for all even N and for all odd values of $N \geq 7$. Confirming for $N = 5$ that $\frac{F_4}{F_3} = \frac{3}{2} < \frac{32}{20.20}$, the greatest N for which a_N is negative is $N = \boxed{5}$. Note that if one were to wait until receiving T to begin, the sequence would be

$$-32, 20.20, -11.8, 8.4, -3.4, 5, 1.6, \ldots.$$

Because 5 and 1.6 are both positive, every term of the sequence after these will be positive, so the greatest N for which a_N will be negative is 5.

R3-A. To maximize $x + y$ subject to the constraint that $11x + 21y = 2020$, begin by maximizing x and minimizing y. Notice that $21y \equiv 2020 \pmod{11}$, which implies $21y \equiv 7 \pmod{11} \rightarrow -y \equiv 7 \pmod{11}$, and thus the least positive value of y is $11 - 7 = 4$. This implies $x = \frac{2020-21\cdot 4}{11} = 176$. Thus the greatest $x + y$ is $176 + 4 = \boxed{180}$. Note that the slope of the line $11x + 21y = 2020$ is $-\frac{11}{21}$, so the next lattice point will have $y = 4 + 11 = 15$ and $x = 176 - 21 = 155$ so $x + y = 170$. This will continue as one moves along the line in the direction of the vector $\binom{11}{-21}$, with each succeeding lattice point having coordinates such that $x + y$ is $21 - 11 = 10$ less than the preceding lattice point.

Note: In general, to see how one might choose whether to maximize x or y with the constraint $ax+by = k$, rewrite the constraint equation as $y = \frac{k}{b} - \frac{a}{b}x$. To maximize $x+y = \left(1 - \frac{a}{b}\right)x + \frac{k}{b}$, consider $\frac{a}{b}$. If $\frac{a}{b} < 1$, maximize x. If $\frac{a}{b} > 1$, minimize x.

R3-B. Consider three cubes of side length $a \leq b \leq c$ cm that are to be glued together at their faces. The sum of the surface areas of the three cubes is $A = 6(a^2 + b^2 + c^2)$ square cm. Let S be the surface area of the resulting solid after gluing. Obviously $S < A$, but by how much could S be less than A? The cube with side length b cm could be glued at most by one full face to the cube with side length c cm, resulting in subtracting at most twice the smaller cube's face area from A (because one gluing removes that face's area twice). The cube with side length a cm could be glued at most by one full face to each of the other two cubes, resulting in subtracting at most four times its face area from A (because two gluings remove that face's area twice each). Thus $S \geq A - 2b^2 - 4a^2 = 2a^2 + 4b^2 + 6c^2$. Note that in the case where $a + b \leq c$, it is possible to glue the cube with side length b cm by exactly one full face to the cube with side length c cm (by making them share a vertex), and then glue the cube with side length a cm by exactly one full face to each of the other two cubes. In this case exactly $2b^2 + 4a^2$ would be subtracted from A, resulting in $S = 2a^2 + 4b^2 + 6c^2$, so the minimum possible value of S in the case where $a + b \leq c$ would be $2a^2 + 4b^2 + 6c^2$ square cm.

In this problem, the sum of the surface areas of the three cubes is $A = 6(1^2 + 7^2 + n^2) = 300 + 6n^2$.

Suppose $n \leq 1$. Then $n + 1 \leq 7$, so the minimum surface area of the configuration is $300 + 6n^2 - 4 \cdot n^2 - 2 \cdot 1^2 = 298 + 2n^2 = 2T$ square cm, which implies $n = \sqrt{T - 149}$. With $T = 180$, $n = \sqrt{31}$, which is not less than or equal to 1. This is a contradiction.

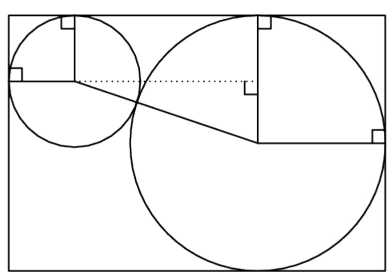
Suppose $n \geq 6$. Then the total surface area of the unglued cubes is at least $6 \cdot (1^2 + 6^2 + 7^2) = 516$ square cm. If the cubes are glued so that the maximum area is removed by gluing, that removed area is bounded above by $4 \cdot 1^2 + 2 \cdot 7^2 = 102$ square cm, and that leaves an exposed surface area of at least $516 - 102 = 414$ square cm. With $T = 180$, this is a contradiction because $414 > 360 = 2 \cdot 180$.

Suppose instead that $1 < n < 6$. Then $1 + n \leq 7$, so the minimum surface area of the configuration is $300 + 6n^2 - 4 \cdot 1^2 - 2 \cdot n^2 = 296 + 4n^2 = 2T$, which implies $n = \sqrt{\frac{T-148}{2}}$. With $T = 180$, the answer is $n = \sqrt{\frac{32}{2}} = \boxed{4}$, which lies in the interval $1 < n < 6$, as needed.

RA-1. Note that $6x^2 - 7x - 20 > 0$ implies $(3x + 4)(2x - 5) > 0$ so $x < -4/3$ or $x > 5/2$. Thus the least positive integer in the solution set is $x = \boxed{3}$.

RA-2. For this sequence, $a = 4$ and $r = -\frac{3}{8}$. The value of S is $S = \dfrac{4}{1 - (-3/8)} = \dfrac{32}{11}$. The sum of the first three terms is $S_3 = 4 - \frac{3}{2} + \frac{9}{16}$, or $S_3 = \frac{49}{16}$. The desired product is $\dfrac{32}{11} \cdot \dfrac{49}{16} = \boxed{\dfrac{98}{11}}$.

RA-3. Draw in segments which connect the center of each circle to the points of tangency, as well as the segment between the centers of the two circles. By the Pythagorean Theorem, the width of the rectangle is divided by these lines into segments of length 3, $\sqrt{(r+3)^2 - (r-3)^2} = \sqrt{12r} = 2\sqrt{3r}$, and r.



It follows that the area of the rectangle is $2r(3 + 2\sqrt{3r} + r) = 2r(\sqrt{3} + \sqrt{r})^2 = 2(\sqrt{3}r + r)^2$. Substituting and simplifying,

$$r + \sqrt{3r} = 7$$

so then $\sqrt{r} = \frac{-\sqrt{3}+\sqrt{31}}{2}$. It follows that $r = \boxed{\dfrac{17 - \sqrt{93}}{2}}$, as desired.

RB-1. Let $x = 2020$. It follows that the given expression is equivalent to

$$\frac{x^2 - 17x - 38}{x+2} = \frac{(x-19)(x+2)}{x+2} = x - 19.$$

The answer is $2020 - 19 = \boxed{2001}$. Similarly, the expression can be rearranged as $\frac{2020^2-4}{2022} - \frac{17\cdot 2022}{2022} = \frac{(2020+2)(2020-2)}{2022} - 17 = 2018 - 17 = \boxed{2001}$.

RB-2. Note that

$$f(n) = 2 \cdot \underbrace{1010\ldots10}_{2n \text{ digits}} = 2 \cdot (10^{2n-1} + 10^{2n-3} + \cdots + 10^1).$$

The sum in parentheses is a geometric series with common ratio 100; its sum is $\frac{10(100^n-1)}{99}$. Note that 8 cannot be a divisor of $f(n)$ for any n because the last three digits of $f(n)$ are always 020, and 8 is not a divisor of 20. Note also that 5 is clearly a divisor of $f(n)$ and that 9 will be a divisor of $f(n)$ if and only if n is a multiple of 9 because the sum of all digits of $f(n)$ equals $2n$ and 2 and 9 are relatively prime. Now, consider whether 7 is a factor of $\frac{10(100^n-1)}{99}$ by considering its residue modulo 7. Noting that 99, 100, and 10 are congruent to 1, 2, and 3 modulo 7, respectively,

$$(99)^{-1}(10)(100^n - 1) \equiv 3(2^n - 1) \pmod 7.$$

For $n = 1, 2, 3$, it follows that $\frac{10(100^n-1)}{9}$ is equivalent to 3, 2, and $0 \pmod 7$, respectively. Therefore if n is a multiple of 3, it follows that $f(n)$ is a multiple of 7. With $T = 2001$, it follows that $f(T)$ is a multiple of 7 and not a multiple of 9, so the greatest single-digit factor of $f(T)$ is $\boxed{7}$.

RB-3. To satisfy the conditions of the problem, it must be that in every group of three consecutive digits of a number, either they are all equivalent to each other modulo 3, or they all have different residues modulo 3. The latter condition is the only relevant one for this problem. To see this, consider that if each group of three consecutive digits is equivalent modulo 3, then all digits of the integer would be equivalent modulo 3, but this is impossible if $T \geq 2$ because 1 and 2 are not equivalent modulo 3.

Note that if a, b, c, and d are four consecutive digits of N (in that order), then $a + b + c$ and $b + c + d$ are both multiples of 3, and therefore $a - d$ is a multiple of 3 which implies $a \equiv d \pmod 3$. This argument implies that the first, fourth, seventh, etc., digits of N have the same residue modulo 3. Similarly, the second, fifth, eighth, etc., digits of N have the same residue modulo 3, and the third, sixth, ninth, etc., digits of N have the same residue modulo 3. This pattern will be important in this solution.

Therefore let N be a T-digit number that satisfies the conditions of the problem, and let S_0, S_1, and S_2 be the sets of digits between 1 and T inclusive that have 0, 1, and 2 as their residue modulo 3, respectively. If $T \in S_1$, then S_1 has one more element than both S_0 and S_2, and therefore N must begin and end with an element in S_1. The second and third digits of N would be in S_0 and S_2 in some order, and then the pattern would repeat. Once a pattern is determined, there are $\lfloor \frac{T}{3} \rfloor!$ ways to arrange the digits of S_0 and S_2 in N, and $\lceil \frac{T}{3} \rceil!$ ways to arrange the digits of S_1 in N, for a total of $2 \cdot \lfloor \frac{T}{3} \rfloor!^2 \cdot \lceil \frac{T}{3} \rceil!$ ways to arrange the digits of N. Similarly, if $T \in S_2$, then N must begin and end with a digit in S_1 and S_2 or vice versa, and there is a total of $2\lfloor \frac{T}{3} \rfloor! \lceil \frac{T}{3} \rceil!^2$ ways to arrange the digits of N. If $T \in S_0$, then the three leftmost digits of N are from three different sets S_0, S_1, and S_2 in any of $3! = 6$ possible orderings, and the pattern repeats, for a total of $6\frac{T}{3}!^3$ ways to arrange the digits of N. With $T = 7 \equiv 1 \pmod 3$, it follows that $T \in S_1$, and so the answer is $2 \cdot 3! \cdot 2! \cdot 2! = \boxed{48}$.

R3-1. The tangent line has a slope of 3, so the radius to which it is drawn has a slope of $-1/3$. This radius "runs" 3 units for every unit it "drops". To land on a lattice point on the circle requires 6 units over and 2 units up from the origin. Thus $a + b = -6 + 2 = \boxed{-4}$.

R3-2. Use the definition of midpoint to arrive at two equations: $T = \frac{A-3}{2}$ and $T = \frac{B+5}{2}$. Therefore $A = 2T + 3$ and $B = 2T - 5$. The desired sum is $A + B = 4T - 2$. Substituting, it follows that $A + B = 4(-4) - 2 = \boxed{-18}$.

R3-3. The triangle has vertices at $(0, 0)$, $(\frac{T}{3}, 0)$, and $(0, \frac{T}{2})$. The area of the triangle is $\frac{1}{2} \cdot \frac{|T|}{3} \cdot \frac{|T|}{2}$, which is $\frac{T^2}{12}$. Substituting, it follows that $\frac{(-18)^2}{12} = \boxed{27}$.

R3-4. The triangle has vertices at $(0, 0)$, $(\frac{T+3}{5}, 0)$, and $(0, \frac{T+3}{2})$. The area of the triangle is $\frac{1}{2} \cdot \frac{T+3}{5} \cdot \frac{T+3}{2}$, which is $\frac{(T+3)^2}{20}$. Substituting, it follows that $\frac{(30)^2}{20} = \boxed{45}$.

R3-5. The line $3x + 9y = 2T$ has slope $-\frac{1}{3}$, so the perpendicular line through $(8, 4)$ that will be used to measure minimum distance has slope 3. That line has equation $y = 3x - 20$. The two lines intersect at $(\frac{2T+180}{30}, \frac{2T-20}{10})$. Substituting $T = 45$, it follows that the intersection point is $(9, 7)$. The distance between the points in question is $\sqrt{3^2 + 1^2} = \boxed{\sqrt{10}}$.

R3-6. Construct a right triangle with acute angle A and legs of length 1 and 3. Then the hypotenuse of this triangle has length $\sqrt{10}$ and so $\sin A = \dfrac{1}{\sqrt{10}}$ and $\cos A = \dfrac{3}{\sqrt{10}}$. Similarly, construct a right triangle with acute angle B and legs of length T and 1; the hypotenuse is of length $\sqrt{T^2 + 1}$ so $\sin B = \dfrac{T}{\sqrt{T^2 + 1}}$ and $\cos B = \dfrac{1}{\sqrt{T^2 + 1}}$. Thus $\sin(2A) + \cos(2B) = 2 \sin A \cos A + \cos^2 B - \sin^2 B = 2 \cdot \dfrac{1}{\sqrt{10}} \cdot \dfrac{3}{\sqrt{10}} + \dfrac{1}{T^2 + 1} - \dfrac{T^2}{T^2 + 1}$. Substituting, the answer is $\dfrac{3}{5} + \dfrac{-9}{11} = \boxed{-\dfrac{12}{55}}$.

ARML Local 2020 Tiebreaker Problem

TB-1. Triangle ABC has side lengths $AB = 8$, $BC = 5$, and $AC = 7$. Point P lies inside $\triangle ABC$ so that $\text{m}\angle APB = \text{m}\angle BPC = \text{m}\angle CPA = 120^\circ$. Compute $AP + BP + CP$.

ARML Local 2020 Tiebreaker Solution

TB-1. Let $AP = x$, $BP = y$, and $CP = z$. Apply the Law of Cosines to $\triangle APB$, $\triangle APC$, and $\triangle BPC$, to obtain $25 = y^2+z^2+yz$, $49 = x^2+z^2+xz$, and $64 = x^2+y^2+xy$. Add these three equations to obtain $138 = 2(x^2+y^2+z^2)+(xy+yz+xz) = 2(x+y+z)^2-3(xy+yz+xz)$. Now note that the areas of $\triangle APB$, $\triangle APC$, and $\triangle BPC$ are $\frac{\sqrt{3}}{4}xy$, $\frac{\sqrt{3}}{4}xz$, and $\frac{\sqrt{3}}{4}yz$, respectively, so if K is the area of $\triangle ABC$, then $K = \frac{\sqrt{3}}{4}(xy+xz+yz)$. By Heron's Formula, $K = 10\sqrt{3}$, so $xy+xz+yz = 40$. Substituting into the equation above yields $138 = 2(x+y+z)^2 - 120$, from which $x+y+z = \boxed{\sqrt{129}}$.

Answers to ARML Local 2020

Team Round:

T-1. 4	T-2. 15	T-3. 37	T-4. 126	T-5. 40
T-6. 20	T-7. 3384	T-8. 16	T-9. $\frac{113088}{73}$	T-10. $\frac{3}{2}$
T-11. 152	T-12. $\sqrt{13}$	T-13. 660	T-14. 181	T-15. $\frac{54\sqrt{77}}{77}$

Individual Round:

I-1. 1070	I-2. 64	I-3. $\frac{7}{5}$	I-4. $\frac{2}{3}$	I-5. 8
I-6. 646898	I-7. $\frac{1}{4}$	I-8. 518	I-9. 67	I-10. 5745

Collaborative Relay Round:

Two-Person Relays

R1-A. 8	R1-B. 5
R2-A. 32	R2-B. 5
R3-A. 180	R3-B. 4

Three-Person Relays

RA-1. 3	RA-2. $\frac{98}{11}$	RA-3. $\frac{17-\sqrt{93}}{2}$
RB-1. 2001	RB-2. 7	RB-3. 48

Six-Person Relay

R3-1. -4 R3-2. -18 R3-3. 27 R3-4. 45 R3-5. $\sqrt{10}$ R3-6. $-\frac{12}{55}$

Tiebreaker:

TB-1. $\sqrt{129}$

Part III

ARML Power Contests

Beating The Odds

The Background

This set of problems will present a number of situations where a team of intelligent players is trying to improve their odds of winning some kind of game. The traditional setting for these puzzles is a team of prisoners who are trying to outwit a warden who loves logic puzzles, and who will release them if they can beat the latest puzzle. The prisoners could simply make random guesses and have some small chance of winning their freedom, but if they are smart they will be able to greatly improve their odds. In a few cases you will be asked to find the winning strategy. In others, you will be presented with a strategy and asked to compute the probability that it will work.

Rule: in the problems, if you are asked to "analyze" a strategy, you should compute the strategy's probability of success and explain why your answer is correct, though a full detailed proof is not necessary.

Let's start with a simple example.

The warden tells Alice and Bob to plan their strategy for the following game. The warden will give Alice and Bob each a coin, and tell them to flip it as many times as they want, writing down the sequence of heads and tails as they go. When they are done, and without communicating with each other, each will write down a positive integer. Let's say Alice writes a and Bob writes b. If the a^{th} entry on Bob's list is heads, and the b^{th} entry on Alice's list is also heads—in other words, if each can guess the position of a heads on the other's list—they go free. Otherwise, they stay in prison. (If a is more than the number of times Bob has flipped the coin, they automatically lose; same if b is more than the number of Alice's flips.)

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. If Alice and Bob each write down some arbitrary positive integer, explain why their probability of winning their freedom is $1/4$. [2 pts]

2. Actually, if they just guess an arbitrary positive integer, their winning probability is a tiny bit less than $1/4$. Explain why. [1 pt]

Now let's say they plan the following simple strategy. Each will flip until he or she has at least one head (otherwise, according to the rules they can't possible win!) and then write down as his or her guess the position of his or her own first head.

3. Explain why their winning probability is now $1/3$. [2 pts]

So in this simple example, with just a little strategy, Alice and Bob can improve their chances of winning. Keep this idea in mind throughout the rest of the puzzles.

In the next puzzle, Alice, Bob, and Charlie can make a strategy to always escape. In this puzzle, the warden lines the three prisoners up, with Alice in front, then Bob, then Charlie. Each can only see someone who is in front of them, so Alice sees no one and Bob only sees Alice. The warden shows them four hats, two white and two black, and randomly places one hat on each person's head. If anyone can name what color hat is on his or her own head, all will go free. (Of course, no one can see the color of hat on his or her own head!)

4. In the winning strategy, Charlie may be able to name his hat color right away, but if he can't and stays silent for an agreed-on length of time (say, 5 seconds) then Bob will know for sure his own hat's color. Describe how either Charlie or Bob will always be able to win this game for the team. [2 pts]

5. The warden instead shows the prisoners two white and *three* black hats. Explain why the prisoners will still be able to win. [3 pts]

If there are three hats of each color, there is no strategy that guarantees success. But a good strategy for the prisoners is as follows. If Charlie sees two hats of one color, he guesses that he is wearing the other color. Otherwise, he keeps quiet. If, after an agreed upon delay (say 5 seconds), Charlie hasn't spoken, Bob will know that his and Alice's hats must be different colors, so he will guess that he is wearing the opposite color from the hat he sees on Alice's head.

6. Analyze this strategy. (See the rule at the beginning of the problem for the meaning of "analyze." If you are able to construct a better strategy, include it on the comments sheet for extra credit! [3 pts]

The warden tires of this game, thinking it is too easy for the prisoners. The next game is a little trickier. He sits Alice, Bob, and Charlie around a table so that each can see the other two. For each, the warden secretly flips a coin to randomly place either a black hat or a white hat on the prisoner's head, then invites the prisoners to speak—either any one singly or in any combination, whatever their strategy dictates is best. They must speak right away and no other communication is permitted. If anyone speaks, and *everyone* who speaks names the correct color hat on their own head, they all go free. Otherwise, back to their cells!

7. A simple strategy is for the prisoners to elect a leader, and the leader (and only that person) will guess what color of hat is on his or her head. Analyze this strategy. [1 pt]

8. A better strategy is for all to agree that anyone who sees that the other two are wearing a hat of the same color will guess that they are wearing the other color, while anyone seeing one hat of each color will keep silent. Analyze this strategy. [2 pts]

9. Devise and analyze a strategy for this problem if there are five prisoners sitting around the table. (Note: maximum of three points if your strategy gives less than 3/4 probability of winning. There is a strategy that wins over 90% of the time!) [4 pts]

The warden likes the idea of using lots of prisoners in the game. This time, ten prisoners are invited, and told to stand in a line (as in problems 4 and 5) so that each can see the hats of every prisoner closer to the front, but not their own hat nor any hat behind them. The warden then goes through the line, flipping a coin to decide whether to place a black or white hat on each head. Every prisoner must guess the color of hat they are wearing, and the warden will release them if there is *at most* one wrong guess. The prisoners will be allowed to speak in any order they wish, and each can hear the others' guesses. As usual, they can plan a strategy before the game begins, but after the game starts the only communication will be the guesses the prisoners make.

10. Analyze the strategy where each prisoner simply guesses the color of hat they are wearing. [2 pts]

11. A vastly superior strategy is for the person at the end of the line to look at all nine hats on the prisoners in front of him or her, and guess "black" if an even number of those nine are black, or guess "white" if an odd number are. Explain why the group should definitely win its freedom. If the rest of the group now works together properly, what is the probability that at least nine prisoners guess correctly? All ten? [2 pts]

The warden catches on to this scheme, and tries to make it even harder on the prisoners. Instead of assigning black and white hats, the warden takes hats of *eleven* different colors—one hat of each color (which we will call "color 1," "color 2," etc.)—and places them randomly on the prisoner's heads. Naturally, one hat will be left over and not used in this game, though only the warden knows which hat that is. All the other rules of the previous game are still in place.

12. Analyze the strategy where each prisoner simply guesses the color of hat they are wearing, but no one guesses a color that someone else has previously guessed (since there is only one hat of each color, there is no chance that two prisoners could have the same color hat, and at least one of them would automatically be guessing incorrectly). [3 pts]

13. This group of prisoners knows something about math. The strategy they use is that the person at the end of the line will speak first, and will guess by adding up the colors of the hats of the front nine, and announcing the total modulo 11. If the rest of the group now works together properly, what is the probability that at least nine prisoners guess correctly? [1 pt]

14. Determine, with explanation, the probability that *all ten* prisoners will guess correctly using this strategy. [2 pts]

The warden again catches on, and throws in an extra twist. One rule is added: if any two prisoners guess the same color, they all automatically lose. Now the strategy of the last prisoner in line naming the sum of the other hats won't always work, because sometimes the sum modulo 11 is also the number of one of the hats of the first nine prisoners. Since the prisoner last in line has already mentioned the hat color of one of the other prisoners, he or she has already guessed wrong. But now for the prisoner actually wearing that hat, he or she can't guess right without repeating that color, but a wrong guess would be the second wrong guess for the group.

15. In light of this new rule, analyze the strategy of the last prisoner in line naming the modulo 11 sum of the hats he or she sees. [2 pts]

Is there a way for the prisoners to recover? Of course there is! Use the idea of *parity*. Given a list of numbers, you may put them in order by swapping pairs. For instance, 45231 $\longrightarrow$ 15234 $\longrightarrow$ 12534 $\longrightarrow$ 12354 $\longrightarrow$ 12345. You might have swapped pairs in a different sequence, but no matter what sequence of swaps you choose, you must always use an *even* number of swaps. Similarly, no matter what sequence of swaps you use to order 25143 it will always be an odd number of swaps. It is a fact, which may be used without proof (a proof will be provided in the solutions) that *every* list is either even or odd—that no matter what sequence of swaps is used to put it back in order, the number of swaps is always even or it is always odd, and no list can have one even-length sequence and one odd-length sequence. You may also assume that exactly half of the possible lists are in an odd order, and half are in an even order (by swapping the first two numbers, you get a one-to-one correspondence between even-ordered lists and odd-ordered lists).

To see the idea of the prisoners' strategy, let's temporarily reduce to three prisoners and four hat colors. If we write a string of numbers like "3124" we mean the first person in line is wearing hat color 3, the second person is wearing color 1, the person at the end of the line wears color 2, and color 4 is unused. So the prisoner at the end of the line sees 31?? and knows the pattern is either 3124 or 3142. One of these patterns requires an even number of swaps to put in proper order (3124 $\longrightarrow$ 1324 $\longrightarrow$ 1234) but the other pattern requires an odd number of swaps (3142 $\longrightarrow$ 1342 $\longrightarrow$ 1243 $\longrightarrow$ 1234). In fact, no matter which two hat colors the last person in line sees, there are always exactly two possible hat sequences—if the last person knows the sequence is *ab*?? then the sequence must be either *abcd* or *abdc*, and one of the sequences is odd while the other is even (because a single swap of the last two hats would change from one to the other).

16. Assume the last person in line always guesses that the sequence is even. So in our 31?? example, the last person would guess they themselves are wearing color 2. What is the probability that they guess right? [1 pt]

17. Now once the person in the middle of the line hears the person behind them guess 2, explain why they can now always figure out the color of hat they are wearing—whether or not the person behind them guessed correctly. [1 pt]

18. Explain why the person in the front of the line can now always determine their hat color—again whether or not the person at the end of the line was correct. [1 pt]

19. Extend this strategy to the ten-person-eleven-hat situation, and explain why at worst the last person in line will guess wrong, while everyone else guesses correctly. [3 pts]

One last game. This time, the warden takes five prisoners, and puts a large stack of hats on each person's head. Each hat is either black or white, chosen at random. The warden guarantees that everyone is wearing at least one black hat. Everyone can see the others' hats, but not their own. Each person must guess an integer, and then looks at that number hat on his or her own head. If everyone names a number where their own hat is black, they go free. Otherwise, back to jail.

Obviously, if each just guesses some integer, the probability that the hat at that position on their own head is black is $1/2$, so the chance all five guess correctly is $(1/2)^5 = 1/32$. But a relatively simple strategy brings their probability of winning up to $1/6$.

20. Find and analyze such a strategy. [2 pts]

Author's Note: Several teams found small loopholes in the instructions, allowing them to solve the problems more easily than had been intended. The following extra details are meant to close those loopholes.

Before problem #4, there should be an additional instruction that if anyone names an incorrect color, all prisoners will be returned to their cells. This prevents a strategy like Charlie simply naming Alice's hat color, after which she will certainly guess correctly!

Before problem #9, the rule about speaking immediately should be relaxed if the win probability is to rise to 90% or more. The theory of error correcting codes disallows any win probability above 26/32 otherwise.

Before problem #20, the rule that the prisoners must speak simultaneously need to be reinstated. Otherwise, the first person will make a guess like $2^a3^b5^c7^d$ where a is the lowest black hat on the second person's head, b is the lower black hat on the third person's, and so on. Now the first person has a 1/2 chance of being correct, but with a little factoring everyone else will be able to guess correctly after the first person has already spoken, so their probability of winning as a group is 1/2.

The Solutions

1. When Alice writes down a number, there is a 50-50 chance that Bob's coin came up heads on that flip. So her probability of being correct is $1/2$. The same is true for Bob. Since they both must be correct to win their freedom, the probabilities are multiplied, so the final probability is $1/2 \cdot 1/2 = 1/4$.

2. The previous answer doesn't take into account what happens if Alice or Bob, or both, guess numbers too large. For if Alice guesses, say, 10 and Bob only flipped his coin five times, they automatically lose. It is unlikely that both will guess large numbers, so this only makes the chances a tiny bit less than $1/4$.

3. The gist of it is that if they both guess the same number, it is a guaranteed win. To be more precise, there are four equally likely possibilities: both have heads as their first toss, Alice has heads but Bob has tails on the first toss, Alice has tails but Bob has heads, or both toss tails on their first toss. In the first case they will both guess that the other has a heads on their first toss, and win. In the second Alice will guess wrong (so it doesn't matter what Bob does). In the third case, Bob will guess wrong. But in the fourth case, *neither* will guess that the other has a heads on the first toss, as neither does themselves; so this case is actually irrelevant; when it happens both will toss again and guess a higher number. So there are really only three possibilities, one of which is a win, so their winning probability is $1/3$. Note that this theme, of getting everyone to guess right or getting everyone to guess wrong will show up in several of the later problems.

4. If Charlie sees two hats of the same color (say, black) he will know instantly that his own hat must be the other color (white) and will guess correctly right away. But if Charlie hesitates, Bob knows that Charlie sees one hat of each color—one black and one white. Thus, his hat must be the opposite of the color that he sees on Alice's head. So either Charlie will guess correctly right away, or hesitate and Bob will then be able to guess correctly.

5. The strategy is slightly more involved this time. First, if Charlie sees that the other two are wearing white hats, he will instantly and correctly guess that he is wearing one of the black hats. Otherwise, he hesitates. If he hesitates, Bob knows that Charlie must see at least one black hat on his and Alice's heads. So if Alice's hat is white, he knows his hat is black and can guess correctly. But if Alice is wearing black, Bob can't be sure whether his own hat is black or white, and must also hesitate. Then Alice will know her hat is black.

6. It is probably easier to analyze when this strategy will lose. It can only lose if all three are wearing the same color hat, for then Charlie will guess wrong right away and lose. In any combination where there are two hats of one color and one of the other, the trio will win—either Alice and Bob wear matching hats and Charlie will guess correctly right away, or Alice and Bob have different color hats, Charlie will hesitate, and Bob will be able to guess correctly. So what is the probability that they win? Well, there are $\binom{6}{3} = 20$ ways to choose the three hats to be placed on the trio's heads, and exactly two of these (the all-white and all-black choices) have all three hats the same color. They lose 2 of out 20 combinations, so their probability of winning must be $9/10$.

7. The leader is basically making a 50-50 guess. He or she doesn't know anything special, so they are just guessing the outcome of a coin flip. The probability of a win is 1/2.

8. Similar to problem 6, the only way they can lose is if all three are wearing the same color hat. In that case, *all three* will see that color on the other two heads, and wrongly guess the opposite color. In other words, all three will guess wrong. But if two of the hats are one color and one is the other, *only one* prisoner will speak—the one wearing the odd-color hat. For the other two will see one hat of each color and keep quiet, while the odd one out will see two matching hats, and guess the opposite color, which will be correct!

   In this game, there are eight possible hat combinations (each of the three can wear either black or white, $2^3 = 8$). Two of them—the all white and the all black—are losers. So they win with probability $6/8 = 3/4$.

   The trick to improving the win probability is to concentrate the losses. In the two losing cases, all three prisoners will guess wrong. In the six winning cases, only one prisoner will guess at all, and guess correctly. So if you consider all possible arrangements of hat colors, the total number guesses the prisoner will make is 12, six of which are right and six of which are wrong. So on average each person guesses wrong half of the time. We have just smartly concentrated all the wrong guesses together and spread out the correct guesses to cover more cases.

9. A number of strategies could work. A simple one is that if anyone sees at least three hats of one color, he or she should guess the other color. This strategy will win any time there are three hats of one color and two of the other, as the prisoners wearing the 3-hat will see two of each of the other colors and keep silent, while the other two will both speak and name their correct color. Now there are $2^5 = 32$ combinations of hat colors, of which $\binom{5}{3} = 10$ have three black and two white hats, and another ten have three white and two black hats. The strategy loses if there are at least four hats of one color, as then everyone wearing that color will guess wrong. So the probability of winning is $20/32 = 5/8$.

   An extremely simple way to achieve 75% success rate is for two prisoners to keep silent and for the other three to ignore them and their hats completely, and play the game as in problem 8 amongst just the three of them. As shown above, their win probability is $3/4$.

   The probability can be improved if the warden allows the prisoners a moment of hesitation as in problems 4 and 5. Combining all the ideas so far, the strategy will be for anyone who sees four hats of a single color to instantly guess the other color, while anyone seeing at least one hat of each color hesitates. If everyone hesitates then revert to the strategy of the previous paragraph. Now if all hats are the same color, everyone will guess wrong right away. But if the hats are divided four and one, the odd one out will guess correctly while everyone else remains silent. And if the hats are divided three and two, everyone will hesitate, and the prisoners will win as described in the previous paragraph. So the prisoners only lose if all five hats are the same color, which happens twice out of the 32 combinations. They win with probability $15/16$.

   Actually, with hesitation tactics the prisoners can increase their win probability to $31/32$. The can put themselves in some order (in fact they can line up so that they can only see people in front of them) and proceed as follows. If the first person sees all black hats, he or she guesses at random as to his or her own hat color. Thus they will lose 1/32 of the time—1/2 of the 1/16 of the time when all four other prisoners are wearing black. But if he or she sees any white hat, hesitate. Then the rest of the group *knows* at least one of them is wearing white. Go to the next person in line. If he sees only black hats, he knows his hat must be white and can guess correctly. Otherwise, he sees at least one white hat and also hesitates. Then the remaining three know that at least one of their hats is white. Continue down the line—anyone seeing all black hats knows their own hat must be white and guesses correctly, otherwise hesitate. If the last person is reached, that hat must be white. So unless the very first person guessed wrongly, 1/32 of the time, the group will win.

Without the delay tactics, the theory of error correcting codes prevents the prisoners from being freed in more than 26 of the possible 32 distributions of hats. It seems unlikely that there is a scheme that beats the 3/4 achieved above, though.

10. Since there are ten prisoners, there are $2^{10} = 1024$ possibilities for hat combinations, or for guesses. The prisoners escape only if nine or all ten guess correctly. There is only one combination where all ten guess correctly, and there are exactly ten combinations where exactly nine guess correctly—any one of the ten prisoners may be the wrong-guesser. So there are a total of 11 winning combinations of guesses. The probability that the prisoners are freed is 11/1024.

11. Once the prisoner at the end of the line makes a "guess" all of the other prisoners know whether there are an even or an odd number of black hats among them. Say, for the sake of argument, that it is even. Then the next prisoner in line will count the remaining black hats. If it is also even, she will know her hat is white, while if it is odd then her hat must be black. Of course, these are reversed if the first prisoner's "guess" had told the group that an odd number of black hats were among them. So the second-to-last prisoner in line can always guess correctly her own hat color. The remaining eight can take that information plus the knowledge whether the total number of black hats in the final nine is even or odd, and working their way down the line each prisoner can correctly deduce the color of their own hat. Thus the probability that at least nine prisoners guess correctly is 1, and the tenth prisoner will be right half of the time, so the probability that all ten guess correctly is 1/2.

12. The random guessing strategy isn't very effective in this case! Since there are eleven different hats, there are 11! different arrangements of the hats (including the hat that is not being worn by anyone). Since no two prisoners are going to guess the same color, the only time it is possible to win when someone guesses the wrong color is if he or she guesses that they are wearing the color that is actually not being worn by anyone. For if (say) Alice were to guess the color of hat the is actually being worn by Charlie, then Charlie—who cannot guess this same color—must also guess wrong, so there will be at least two wrong guesses and the prisoners will lose. So there are eleven winning combinations of guesses—each of the ten could guess the unworn color while the other nine guess correctly, or all ten could guess correctly. So the probability of the prisoners winning this game is $11/11! = 1/10! = 1/3628800$.

13. Much like in problem 11, each of the other nine prisoners will be able to correctly determine the color of hat he or she is wearing. For after the last in line has announced the modulo 11 total of the hats he sees, the second-to-last in line can see all those same hats except one—the one she is wearing. So she simply subtracts the total she sees from the total the last-in-line gave, modulo 11, and that is the color of hat she is wearing. Proceeding forward in line, each prisoner knows the modulo 11 totals of all the hats of the front nine, of the hats in front of him or her, and (since everyone behind him or her said the correct color of their hat) of the hats in back of him or her. Subtracting, he or she can now correctly figure out his or her own hat color. So the probability that at least nine prisoners guess correctly is 1.

14. Since we know that nine of the prisoners will definitely guess correctly, the answer is the probability that the last person in line guesses correctly as well. That can only happen if the hat he is wearing is the modulo 11 sum of the hats of all the other prisoners.

    Now the sum of all the hats' colors is 66, which is 0 (or 11) modulo 11 of course. Let the last prisoner in line be wearing hat color a and let the unused color be b. Then in order for the last person in line to guess correctly, the sum of the colors he or she sees must be a. That means that $a + a + b$ must add to zero (or eleven) modulo 11. In other words, the last prisoner in line will guess correctly if and only if $b \equiv -2a \pmod{11}$. Now be careful! The colors a and b must be different. So if $a = 11$ we could get a correct guess only if $b = 11$, which is impossible. On the other hand, for any other choice of a, there are only *ten* possible different choices for b, and one of them will lead to a correct guess. So the probability of a correct guess is (probability of wearing color 11)(probability of guessing correctly if wearing color 11) + (probability of not wearing 11)(probability of guessing correctly if not wearing 11) $= \frac{1}{11} \cdot 0 + \frac{10}{11} \cdot \frac{1}{10} = 1/11$.

    Note that the naive analysis, where the last prisoner guesses correctly if $b \equiv -2a \pmod{11}$, so one out of the eleven possible choices for b allows a correct guess, gives the correct answer, but is an incorrect analysis!

15. The probability that at least nine guess correctly with the new rule in place is 2/11.

    A sneaky trick will allow us to derive this with no more work. Consider the solution to the last problem where we concluded that the probability that all ten guess correctly is 1/11. In that solution, the last prisoner guessed correctly when his hat color was a, the unworn hat color was b, and $2a + b \equiv 0 \pmod{11}$. But we only need nine of the prisoners to guess correctly, and as long as the last prisoner names the total he sees, all other nine prisoners will guess correctly. That means that in the combinations of a and b that allowed all ten to guess correctly, we could swap the last prisoner's hat with the unworn hat. That would leave the total of all the seen hats unchanged, and the last prisoner would announce this total so all the others could then guess correctly. The last prisoner would now guess incorrectly, guessing the unworn hat's color instead of his own, but since at least nine guessed correctly the prisoners win. Thus we simply double the previous answer!

16. The probability is 1/2. There are two possible sequences, and since a single swap would take us from one to the other, one must be even and the other odd. Either is equally likely.

17. Since the middle person sees the 3 in front of them, and hears the last person guess 2, he knows he cannot be wearing either of those colors. From his perspective, the line of hats must be 3124, 3142, 3412, or 3421. But from either 3412 or 3421, the middle person knows that the last person would have guessed 1, not 2, because he knows the last person is assuming the order is even, and 3421 is odd. Thus 3412 and 3421 are ruled out, and the order of hats must be 3124 or 3142. So regardless of whether the last person in line was correct in their guess or not, the middle person must have hat number 1.

18. The prisoner in the front of the line knows that the middle person named the correct color, and that since the last person guessed 2, that is not his hat color either. So he knows the order must be 3124, 3142, 4123, or 4132. But if the last person in line had seen 41??, his proper guess would have been 3, not 2, as 4132 is even while 4123 is odd. So 4123 and 4132 are ruled out, leaving 3124 and 3142. In either event, he knows he is wearing hat color 3.

19. As before, the person at the end of the line sees the nine hats in front of him, and there are only two possible orders for the eleven hats. One is even and the other is odd, so he guesses it is even and names the hat color that he would be wearing if the order were even. He may be right or wrong. Then the second-to-last person in line sees the eight hats in front on her, and knows there are six possible orders for the hats she doesn't see. Let's say those colors are a, b, and c. These can occur in six possible orders: abc, bca, cab, bac, acb, and cba. Three of these will be even, and the other three will be odd. Let's say that abc is even. Then bca and cab will also be even, and the other three will be odd. If the last person guesses, say, color c the second-to-last person knows that orders cab and cba are ruled out (the last person obviously won't guess a color of hat he already sees on someone else's head). Orders abc and acb are also ruled out because if the last person had seen color a on the second-to-last person, he would have guessed he was wearing b, not c. That means the second-to-last person must be wearing hat color b.

    Now inductively, we may assume that everyone who has spoken, with the possible exception of the last prisoner in line, has named the correct color of their own hat. So when it is the next person's turn to guess, there are only three unknows hat colors—their own, the one on the last person's head, and the unworn one. Of the six possible orders of these three hats, two are ruled out by the last person having already named c, so the next person cannot be wearing color c. Two more are ruled out by the even-order-guess rule. The remaining two possibilities always have the next person to speak wearing the same color—there is one where the last person guessed right and one where the last person guessed wrong. So the next person can always guess correctly, and only the last person in line might guess wrong.

20. The simplest strategy is for each person to search upward in everyone else's stack of hats until they find a layers where all four others are wearing a black hat. Then they name that position as where they, too, have a black hat.

    In any given layer, there are $2^5 = 32$ possible combinations of hat colors among the five prisoners. But we can ignore any layer where there are two or more white hats. That leaves only layers where there are at least four black hats. In that case, there are six equally likely distributions of hats among the five prisoners: BBBBW, BBBWB, BBWBB, BWBBB, WBBBB, and BBBBB. In the first five, the person with the white hat will guess wrong. But in the sixth, all five prisoners will guess right. So their probability of winning this game are $1/6$.

Do you see the connection between this strategy and that of problem 3?

It turns out that there are even better strategies, but they start to become ridiculously complicated.

Now, as promised, here is a proof that any order must be either even or odd.

For any order, define the "disorderliness" of the order to be the total number of times larger number occur before smaller numbers. For instance, the disorderliness of 71284536 is $6+0+0+4+1+1+0 = 12$. This is because the 7 occurs before six smaller numbers, neither the 1 nor the 2 come before anything smaller than they are, the 8 before four smaller numbers, the 4 and the 5 each come before one smaller number, and neither the 3 nor the 6 come before anything smaller than they are. Then for any swap, the disorderliness changes by an odd number. In fact, it changes by exactly twice the number of numbers that are between both positionally and numerically the swapped numbers, plus one (because of the swapped numbers themselves). For instance if we change 71284536 to 75284136, there are two numbers (2 and 4) both numerically and positionally between the 1 and the 5, so the disorderliness changes by $2 \cdot 2 + 1 = 5$. You can easily check that the disorderliness of the new arrangement is $6 + 4 + 1 + 4 + 2 = 17$. The disorderliness increases if a larger number brought toward the beginning and a smaller number pushed toward the end, and decreased if the swap is in the opposite direction.

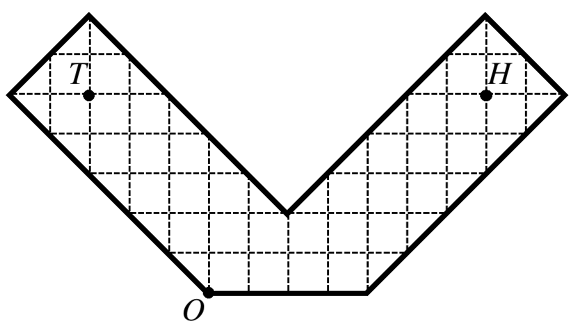
Now the disorderliness of the usual arrangement 123… is zero. If some order has one route to putting it in order that requires an even number of swaps, then its disorderliness must be even, for an even number of odd changes to the disorderliness changes it by an even number. But the disorderliness changes to zero, so it must have been even to start with. Similarly, an odd number of odd changes can only be odd, so if some arrangement allows an odd number of swaps to return to the proper order, its original disorderliness must be odd. And no even number of swaps can ever put it in order, since an even change cannot make an odd number become zero. So every arrangement must be either even or odd, and swapping the first two number creates a one-to-one correspondence between even and odd arrangements, meaning exactly half of the arrangements must be even and half odd, as claimed.

Mini-Golf

The Background

Many of you have played a game called "miniature" or "putt-putt" golf. This game is played with a regular golf ball and club (usually the putter—the club used by golfers to hit the ball very accurately over short distances) and consists of several rounds, called "holes." The goal is to start with the ball in a designated starting place called the "tee" and hit it into a designated location (the "hole"). In real golf, there are very long distance involved, plus there may be trees, water, or other "hazards" in the way, and the hole may not be in direct line of sight from the tee. In miniature golf, the distances are usually short—a few meters at most—and the challenge is that the hole may be around a corner, or through a narrow tunnel so that only a very accurate shot can make it into the tunnel, or the path between the tee and the hole may have unusual slopes, or there may be moving objects in the path between the tee and hole requiring that the shot be well-timed as well as properly aimed. We are going to play a mathematically simplified version of the game.

In mathematical mini-golf, each problem will be a geometric shape. The tee and the hole will each be a specific point in the shape, marked "T" and "H" respectively. To make it easier to specify points within the shape, the origin will be located at point "O" and the tee's and hole's coordinates will be specified relative to O. For instance the following might specify a layout for a problem:



Tee: $(-3, 5)$
Hole: $(7, 5)$

The grid inside the shape can be ignored for the sake of playing the game—it is there for visual reference only. All things may be assumed to be exactly as they appear; segments are straight, and will begin and end at integer-coordinate points unless stated otherwise. If part of the shape is a curve, the curve will be specified precisely (e.g., "portion of circle of radius two centered at the origin that lies in the first quadrant").

The rules are simple. You will take a number of shots, starting from the tee and hopefully ending at the hole. To take a shot, first give your shot number, then name the current position of the ball (which may be "tee"). Then aim your shot by naming any point on or within the boundary of the shape to which the ball could travel in a straight ray that does not intersect the boundary at any other point before your named point. You may also aim at the hole if there is a direct shot. If desired, name a stopping point (see below for how far the ball will travel otherwise). The ball will always stop at the hole as soon as it is reached. The ball will travel from its current location, directly toward the named point, bouncing off the boundary as explained below, and stop if it reaches the hole, the named stopping point, or the fourth time it reaches the boundary.

For example, one way to play the example hole is:

- Shot 1. Start: tee; Toward: $(2, 1)$; Stop at: $(2, 1)$.
- Shot 2. Start: $(2, 1)$; Toward: hole.

This play initially hits the ball from the tee to the point $(2, 1)$ and then takes the straight shot to the hole.

Part of the fun of mini-golf is that you can sometimes get a better shot by bouncing the ball off of the boundaries. If you specify a shot with no stopping point, or whose stopping point is not reached before the ball hits the boundary, the ball will bounce off the boundary. This happens so that the angle of incidence and the angle of reflection are equal. In less fancy language, draw the perpendicular to the boundary (or to its tangent line, if the boundary is curved), called the "normal line." The angle between the ball's approach to the boundary and the normal line will be equal to the angle at which the ball leaves from the normal, with approach and departure on opposite sides of the normal but the same side of the boundary (or tangent). The ball now continues in its new direction until it reaches the named stopping point or bounces off another wall. The ball may bounce off the boundary up to three times, but will automatically stop when it reaches the fourth even if it has not reached the named stopping point—you are not allowed to hit the ball hard enough that it can bounce more times.

Do not aim at the places on the boundary such as corners, where two different pieces join together; your ball will automatically stop if it reaches such a point even if that is the first time it reaches the boundary. (The physics of reflections from such points depends on more factors than we can consider here, and the idealized version of the physics we are actually using in the problem breaks down due to the discontinuities involved.)

Here are some more combinations of shots that will complete the example problem:

- Shot 1. Start: tee; Toward $(-2, 4)$; Stop at: hole.

A hole-in-one! The "Toward" could have also been any of the points $(-1, 3)$, $(0, 2)$, $(1, 1)$, $(2, 0)$, or even $(1/2, 3/2)$, but try not to be that difficult! Since the ball hasn't reached the stopping point by the time it gets to $(-2, 4)$ it continues, until it bounces off the wall at a $45°$ angle at the point $(2, 0)$, and then continues directly toward the hole.

Another possibility:

- Shot 1. Start: tee; Toward: $(-1,5)$; Stop at: $(0,0)$.
- Shot 2. Start: $(5,5)$; Toward: hole; Stop at: hole.

This time, the first shot bounced off the wall at $(-1,5)$, then bounces again at $(-1,1)$. It misses the point $(0,0)$, so it doesn't stop—in fact, the stopping point could have been omitted altogether since it is irrelevant. The shot continues to $(5,1)$ and bounces again, heading straight upward now. When it reaches $(5,5)$ it hits its fourth wall and stops. Now a quick tap puts the ball in the hole.

One more hole-in-one:

- Shot 1. Start: tee; Toward: $(-4,6)$.

This trickster starts by aiming the ball toward the wall behind the tee. It bounces off and then heads along the path given for the first hole-in-one above. This golfer was so sure they had a correct shot that they didn't even bother to note that the ball would stop at the hole!

Warnings! Be very careful measuring and naming your points!

1. If your "toward" point is not in direct sight of the ball's current location, your shot will be ruled illegal and the problem will be marked wrong. For example, the first shot in the example problem cannot be toward the hole—even though it is the same direction as the legal shot toward $(-1,5)$—because the walls prevent the hole from being in sight.
2. If you accidentally name a stopping point not in the actual path of the ball, the ball will just continue until it hits four walls, a corner, or the hole.
3. The ball always stops as soon as it reaches the hole or a corner. You may take another shot after reaching a corner, but once you reach the hole that problem is over. You may not always want to reach the hole as quickly as possible, because some problems will ask for "trick shots" that hit a specified number of walls before reaching the hole—and if you hit the hole too soon you will not have solved the problem!
4. If the "Start" point of a shot is not actually where the ball is (did you forget to name a stopping point in the previous shot?), it will be ignored and the current location of the ball will be substituted. If the resulting shot is illegal, then the problem will be marked wrong.

Test your readiness! If you can answer the following questions, you are ready for the contest. If not, make sure you understand what happens before trying the real contest questions.

1. In the example problem, where will the ball lie after the shot: Start: tee; Toward: $(-3,3)$, assuming no stopping point is given?
2. In the example problem, where will the ball lie after the shot: Start: tee; Toward: $(-3/2,3/2)$. assuming no stopping point is given?

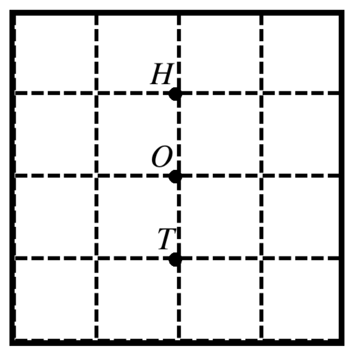
Answers

1. The ball will bounce at $(-3, 3)$, then at $(1, 3)$, then straight off the bottom wall at $(1, 0)$ and stop when it hits the wall at $(1, 3)$, since the ball always stops if it hits the boundary a fourth time.
2. The ball will bounce at $(-3/2, 3/2)$ which will reflect it toward $(2, 0)$. By symmetry, it will bounce a third time at $(11/2, 3/2)$ and then head to the hole—a three-bounce hole-in-one!

<u>**The Problems**</u>

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. Consider the very simple square region shown below.



Tee: $(0, -1)$
Hole: $(0, 1)$

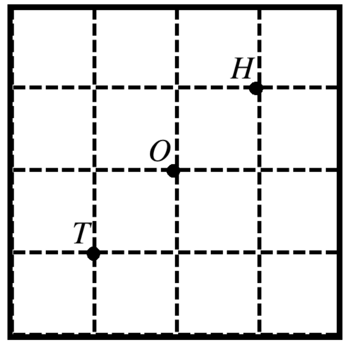
Clearly, any shot that you might take that aims straight upward will result in a hole-in-one. Your first shot could be toward the hole, or toward $(0, 2)$ or even $(0, 4)$ (as long as you don't name a stopping point before the hole!) and you would get a hole-in-one without bouncing off a single wall. This hole is so boring that to entertain ourselves we will insist on hitting at least one wall before going into the hole.

(a) List all points *on the boundary* toward which you can aim the shot and get a hole-in-one after exactly one bounce off of the boundary. [2 pts]

(b) List all points *on the boundary* toward which you can aim the shot and get a hole-in-one after exactly two bounces off of the boundary. [2 pts]

(c) List all points *on the boundary* toward which you can aim the shot and get a hole-in-one after exactly three bounces off of the boundary. [2 pts]

2. Let's move the hole and tee a little and try the same questions:



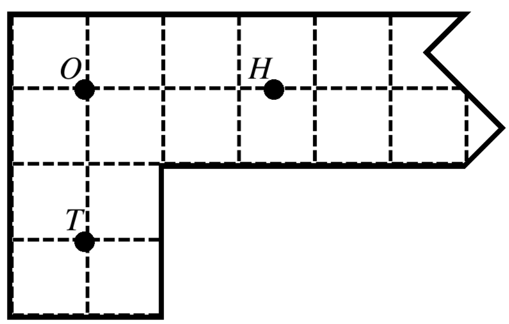
Tee: $(-1,-1)$
Hole: $(1,1)$

   (a) List all points *on the boundary* toward which you can aim the shot and get a hole-in-one after exactly one bounce off of the boundary. [2 pts]

   (b) List all points *on the boundary* toward which you can aim the shot and get a hole-in-one after exactly two bounces off of the boundary. [2 pts]

   (c) List all points *on the boundary* toward which you can aim the shot and get a hole-in-one after exactly three bounces off of the boundary. [2 pts]

3. Look at the region below:



Tee: $(0,-2)$
Hole: $(x,0)$

Here, the zig-zag on the right is intended to indicate that the shape goes on indefinitely, and there is no end-wall for the ball to bounce off of at the right. The hole is at some position $(x,0)$ with $x \geq 0$. Obviously, if $x < 2$ there is a straight path from the tee to the hole that doesn't touch any walls. When $x = 2$ exactly, the straight shot would pass through the corner at $(1,-1)$ and—according to the rules—instantly stop there, so we can't hit the hole with a straight shot. So with no bounces, the *hole-in-one interval* for x is $[0,2)$ (don't include negative x's).

   (a) Determine the hole-in-one interval for x allowing exactly one bounce off of the top boundary. [1 pt]

(b) Determine the hole-in-one interval for x allowing exactly one bounce off of the left-hand boundary. [1 pt]

(c) Determine, with explanation, the interval(s) for x that allow a hole-in-one with exactly two bounces, which may be off of any part(s) of the boundary. [3 pts]

(d) Determine, with explanation, the interval(s) for x that allow a hole-in-one with exactly three bounces, which may be off of any part(s) of the boundary. [3 pts]

4. If the region is an equilateral triangle, it does not matter where the hole and tee are located within the triangle—there are always exactly three points, one on each side of the triangle, where we could aim the ball to get a hole-in-one with exactly one bounce.

   (a) Design a mini-golf hole, including the location of the tee and the hole, whose boundary is an obtuse triangle, and for which there are only two points on the boundary where you could aim to get a one-bounce hole-in-one. [1 pt]

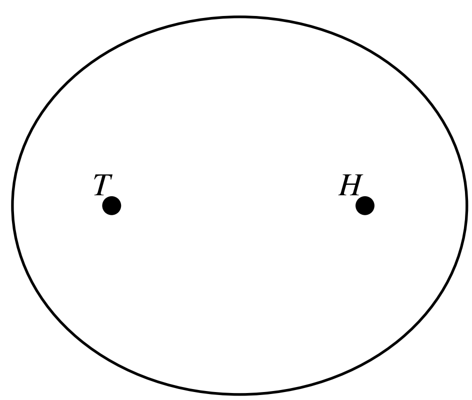
   (b) Prove that for any right or acute triangle there are exactly three points on the boundary where you could aim to get a one-bounce hole-in-one. [2 pts]

5. If the hole in problem 3 is far enough to the right—if x is large enough—there is no hole-in-one shot requiring only one bounce. Create a mini-golf problem where the hole is in sight of the tee (so there is a no-bounce hole-in-one), but there are no one-bounce holes-in-one. [1 pt]

6. Now create a problem where there are exactly five one-bounce holes-in-one. The shape should be a polygon. [1 pt]

7. Let the shape be a circle with the tee at the center and the hole at any other point. In addition to the direct shot there is exactly one one-bounce hole-in-one by aiming directly away from the hole. Explain why there are no two- or three-bounce holes-in-one. [1 pt]

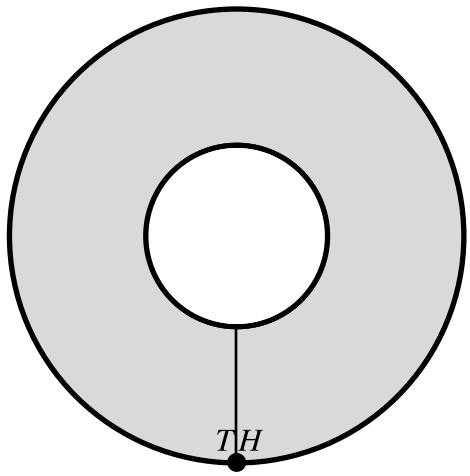
The figues below would be a fun hole to play. The shape is an ellipse with the tee and the hole at the foci. Unless you called for the ball to stop before it reached the hole, *every* shot is a hole-in-one! This is because of the reflecting property of an ellipse: rays emanating from one focus reflect off of the ellipse heading toward the other focus. This allows the creation of such interesting phenomena as whispering galleries.



8. Except for the straight shot, every hole-in-one in the ellipse is a one-bounce hole-in-one. Create an example where infinitely many of the shots are *two*-bounce holes-in-one. [2 pts]

9. Create an example where infinitely many of the shots are three-bounce holes-in-one. [2 pts]

10. From your work on problem 3, you know that if you create an "L"-shaped region with one side long enough compared to the other, there are no holes-in-one, but there certainly are ways to complete the hole in two shots. Prove that we can do something similar for any number of shots. That is, prove that for every positive integer n you can create a shape where you can complete the hole in n shots but not fewer than n. [2 pts]



The puzzle shown consists of two concentric circles. The outer one has radius 1, the inner has radius r. There is an infinitely thin wall running from the outer circle to the inner, and the tee and hole are on opposite sides of this wall at the very bottom of the circle. There is no direct line-of-sight from the tee to the hole, since they are on opposite sides of the wall. So any hole-in-one will require at least one bounce off of a wall. A one-bounce hole-in-one would require the ball to first hit the boundary at the exact top of the outer circle in order to bounce back to the very bottom, and this point is not in sight either because the inner circle is in the way, so there are no one-bounce holes-in-one.

11. (a) Determine, with justification, the interval of r that allows a two-bounce hole-in-one. [2 pts]

    (b) Determine, with justification, the interval of r that allows a three-bounce hole-in-one. [2 pts]

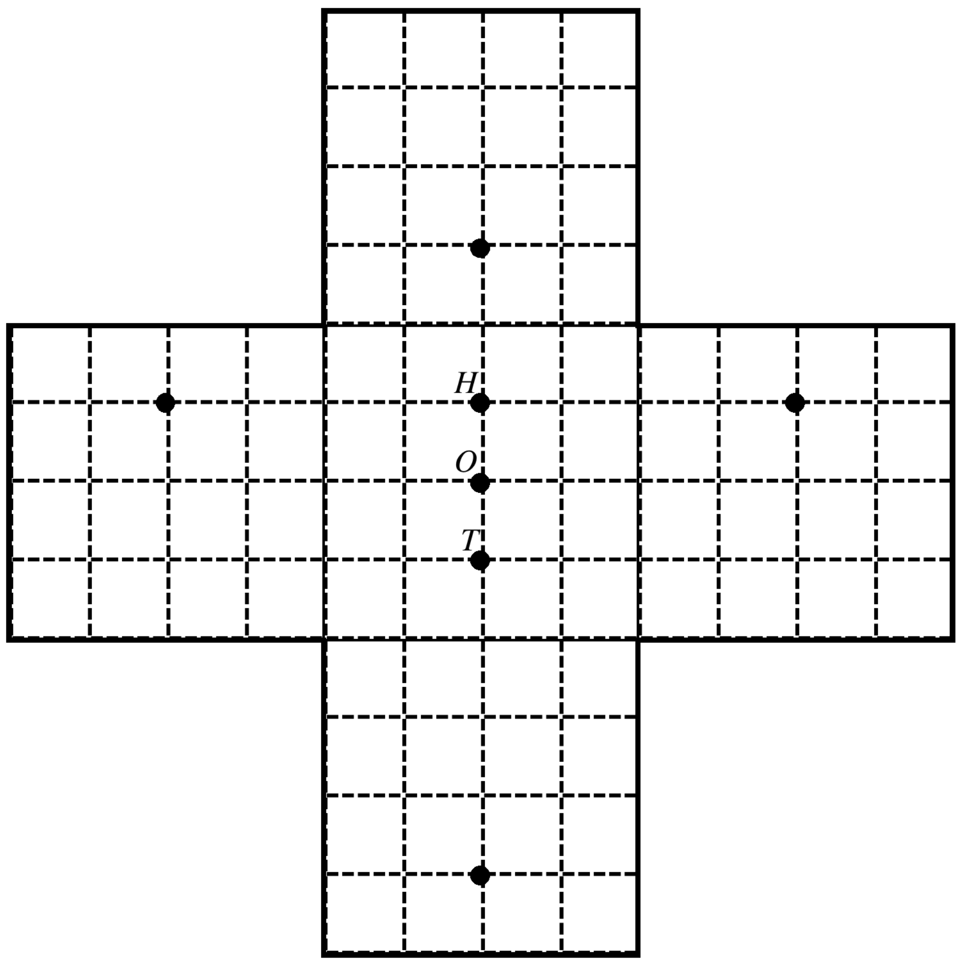
    (c) Find, with explanation a formula for the minimum number of shots that must be taken to reach the hole starting from the tee, if the radius of the inner circle is r. You may use the greatest integer function $\lfloor x \rfloor$. [2 pts]

    (d) If the tee and hole are moved to the opposite end of the wall, at the very bottom of the inner circle instead of their current location, prove that there are *no* holes-in-one regardless of the inner radius r. [2 pts]

The Solutions

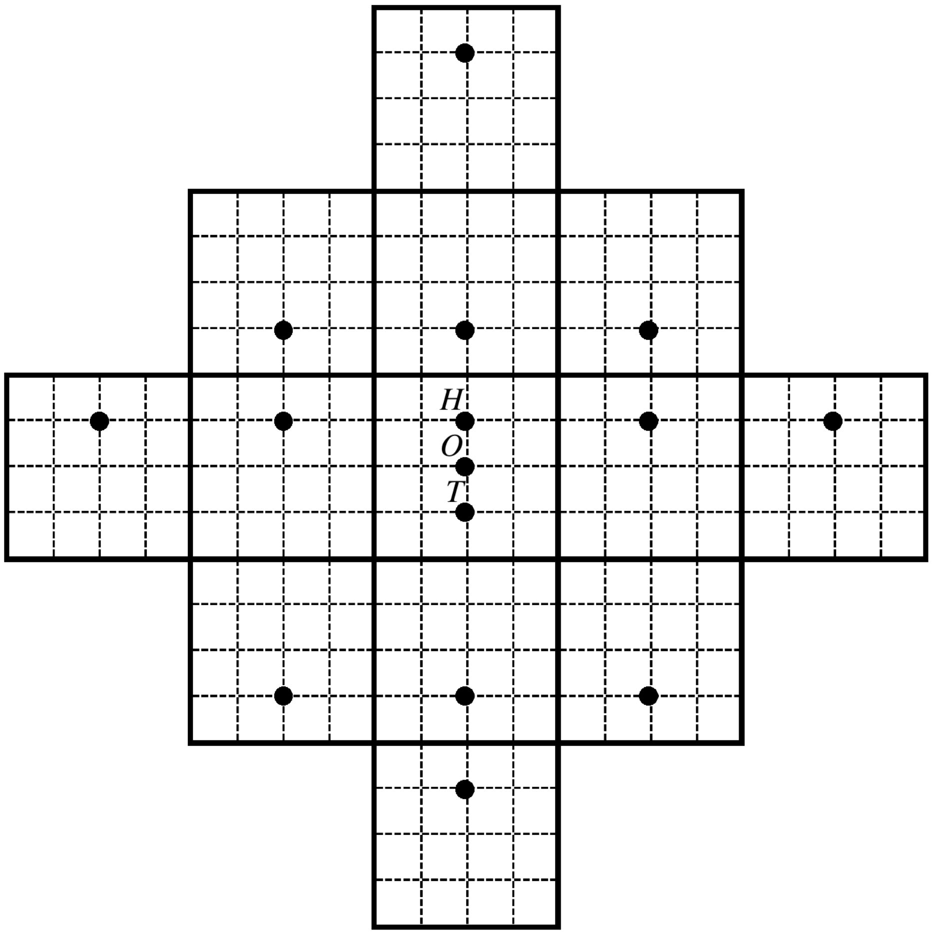
1. We could obviously just list solutions to the parts, but it might be wise to look for more comprehensive solutions instead. We will use the *reflection principle* which says that instead of bouncing the ball against a wall, we will pretend the wall is a mirror and then aim at the spot where we see the image of the hole. In fact, instead of a mirror, we can just pretend the wall is an invisible barrier between us and a reflected copy of the hole we are playing, and we will aim for the reflected hole.

   (a) Here's what we get by reflecting the hole over its walls:



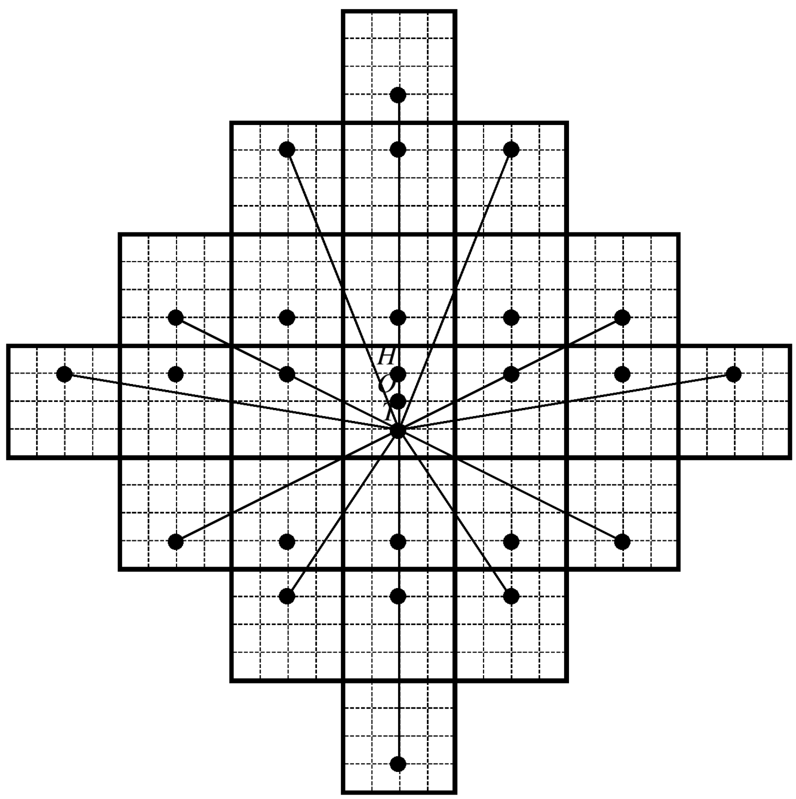
   We can now simply draw straight lines from the tee to the holes in the reflected areas. These lines pass through $(2, 0)$, $(-2, 0)$, and $(0, -2)$ on the boundary. We can't get to the hole in the reflected area above the original area, because the straight line shot to that hole passes through the original hole, and would stop there.

(b) Reflecting everything twice yields the following picture:

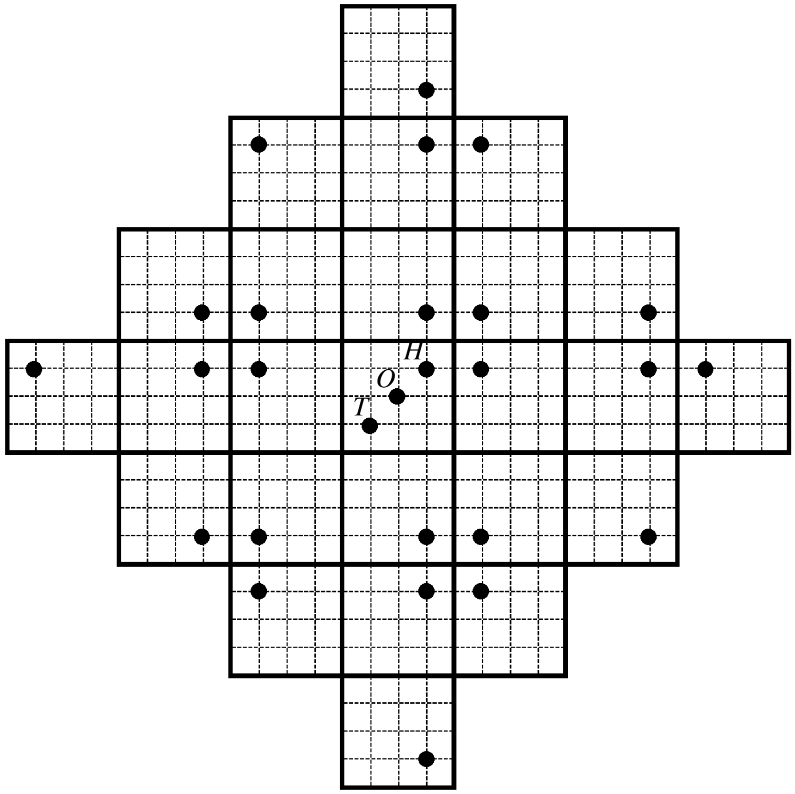


We now have to get the ball into a hole in one of the second reflections. The holes in the very top and very bottom reflections cannot be reached without hitting another hole (or reflected hole) first, but we can reach the other six holes, by aiming at $(2, 1)$, $(2, -1/2)$, $(1, -2)$, $(-1, -2)$, $(-2, -1/2)$, or $(-2, 1)$.

(c) This time, we have drawn the lines on the diagram to represent the shots aimed at the holes three reflections away. As usual, the holes in the very bottom and very top reflections cannot be reached. But this time, the same is true about the hole in the reflections at "2 o'clock" and "10 o'clock." Also, to hit the hole in the reflections at "4 o'clock" and "8 o'clock" you would have to aim at one of the corners, but if the ball hits a corner it stops immediately. So this time, six of the twelve new reflections are not reachable. The remainder can be reached by aiming at $(6/5, 2)$, $(2, -2/3)$, $(2/3, -2)$, $(-2/3, -2)$, $(-2, -2/3)$, and $(-6/5, 2)$.



2. Here is the thrice-reflected hole for this problem.



We will simply find where the lines from the tee to the various reflected holes hit the boundary of the original hole, excluding any results where we would hit a corner or an earlier-reflected hole before reaching the desired hole.

(a) We can reach the hole in the first-reflected areas by aiming at $(2, 1/2)$, $(1/2, 2)$, $(-2, -1/2)$, and $(-1/2, -2)$.

(b) To reach the holes in the second-reflections, we can aim at any of the six points $(-2/5, 2)$, $(2, -2/5)$, $(0, -2)$, $(-2/3, -2)$, $(-2, -2/3)$ or $(-2, 0)$.

(c) The hole in the third-reflections can be reached by aiming at $(-1/2, 2)$, $(1/5, 2)$, $(2, 1/5)$, $(2, -1/2)$, $(3/2, -2)$, $(-1/3, -2)$, $(-5/6, -2)$, $(-5/3, -2)$, $(-2, -5/3)$, $(-2, -5/6)$, $(-2, -1/3)$, and $(-2, 3/2)$.

3. Because the shape is not convex, it will be harder to reflect is this time, so we will supply descriptions instead.

(a) We could easily reflect the shape through the top boundary only. We see if we shoot the ball straight up toward the origin it would bounce straight back toward us, while aiming toward the left at all can only get to holes with negative x-coordinates, which we are ignoring. This time, the hole *cannot* be at the origin, because to reach the hole with just one bounce from the top boundary would require a shot straight upward which would fall into the hole before reaching the top boundary. But if the hole is slightly to the right of the origin, we can easily bounce off the top boundary and hit the hole. How far to the right can we go? Well, if we just miss the corner at $(1, -1)$ the ball will hit the top boundary just to the left of $(3, 1)$ and then rebound, passing $y = 0$ just before $x = 4$. So the hole-in-one interval for one bounce off the top boundary is the interval $(0, 4)$.

(b) If we aim at $(-1, -5/3)$ or lower, the ball will bounce off of the left boundary and then hit the right boundary of the vertical section and never reach the hole. If we aim just a bit higher, the ball will pass just to the left of $(4, 0)$. On the other hand, by aiming at $(-1, -1)$ we can have the ball pass through the origin. So the hole-in-one interval for one bounce from the left boundary is the interval $[0, 4)$.

(c) Let's start at the origin and see which points on the x-axis we can reach after two bounces. Can we hit the origin? Sure! If our first shot is toward $(1, 1/2)$ the ball will bounce there, then at $(-1, 3/2)$, and then it will pass through the origin. If we aim a little lower, the ball will bounce off of the right wall, then the left, and then pass through the x-axis to the right of the origin. But if we aim too low, after bouncing off the right then left walls, the ball will hit the right wall again. The lowest we can aim is just above the point $(1, 1/4)$ so that after bouncing off the right and then left walls, the ball will pass just above the corner, and will cross the x-axis just to the left of $x = 5$. So the interval is at least $[0, 5)$. Can we get any points further to the right?

Yes! If our first bounce is from the left boundary and the second from the top we can actually get quite far to the right. If our first shot is toward $(-1, 0)$ the ball will bounce there, then at $(-1/2, 1)$, and then pass through the origin. The lower we aim, the farther to the right the ball will pass through the x-axis after two bounces. For instance, aiming at $(-1, -1)$ will get a bounce at the point, then at $(1, 1)$, and hitting a hole at $(2, 0)$. We can aim as low as $(-5/3, -1)$ (well, just above this point) as then the ball will hit this point, pass just above the corner, hit the top boundary just to the left of $(7, 1)$ and drop into a hole just to the left of $(10, 0)$. So with these kinds of shots we can get the whole interval $[0, 10)$.

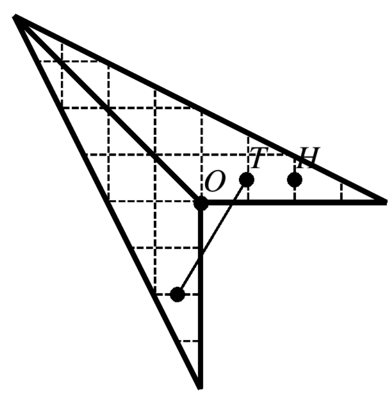
A nice way to see that we can't get any points beyond this is to play the hole backward. Could we start at the hole and reach the tee after two bounces? If the first bounce is from the top wall and the second from the left wall, we are in the case of the previous paragraph. If the first bounce is from the top wall and the second from any wall other than the left, the ball will not be going in the correct direction to head back toward the tee. If the first bounce is off the bottom wall of the horizontal corridor, the second would have to be from the top wall or the ball would have upward movement after two bounces, and could not be heading toward the tee. It is not hard to check that the hole can only be at positions $x < 6$ for this pair of bounces to return to the tee. Finally, if the first bounce if from the left wall, the second must be from the right wall and we are in the position of the first paragraph. So $[0, 10)$ is the hole-in-one interval for two bounces.

(d) With three bounces, it might not be immediately obvious that you can hit a hole at the origin. The trick is to aim *downwards* at the point $(-1/3, -3)$ and you can get bounces at the point, then at $(-1, -1)$ and finally at $(-1/3, 1)$ before falling into the hole at the origin. By aiming a little further to the left and hitting the bottom edge first, you can reach holes along the x-axis for x up to, but not including 4.

We can hit a hole at $(4, 0)$ by aiming first at $(-1, -1)$. The second bounce would be at $(1, 1)$, the third at $(3, -1)$, and then the ball hits the hole at $(4, 0)$. By aiming lower, we can hit holes farther to the right. In fact, aiming just above $(-1, -5/3)$ the second bounce will be just to the left of $(7, 1)$, and the third just to the left of $(13, 1)$. The ball will reach a hole just to the left of $(16, 0)$. So the hole-in-one interval is at least $[0, 16)$.

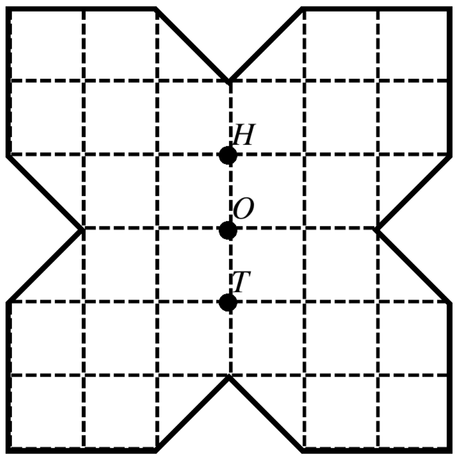
We can't do better than this. Our best try would be to aim toward the right wall with a small slope. If we aim just above $(1, -9/5)$, the second bounce is from the left wall just above $(-1, -7/5)$. The ball then passes just above the corner, finally hitting the top wall just left of $(11, 1)$ and making it back to the x-axis just left of $(16, 0)$, but we've already made it that far. No other combinations of bounces can get us farther. So our interval is $[0, 16)$.

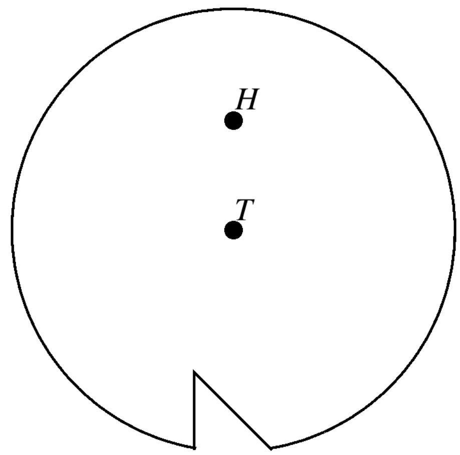
4. (a) Any triangle where the tee and hole are very close to one side of the obtuse angle will allow for only two one-bounce holes-in-one. Using the reflection principle, there can be at most one shot bouncing off of each of the three sides that gives a hole-in-one, because for each side there is only one line from the tee to the reflection of the hole in that side. But if an angle is obtuse, reflecting in one of the sides of the obtuse angle will produce a figure that is not convex, and the straight line from the tee to the reflected hole would pass outside of the figure.



(b) Following the lead of the answer to the previous part, we simply note that we can reflect the triangle in each of the three sides, and since the resulting figure will be convex as long as there are no obtuse angles in the triangle, the line from the tee to the reflected hole will stay inside the figure. There is one such line for each of the three sides of the triangle, thus exactly three shots that will give one-bounce holes-in-one.

5. There are a wide variety of possible solutions to this problem. One idea is to take a shape with some one-bounce solutions, and then alter the boundary near where the ball would need to bounce to direct the bounce away from the hole—or put a corner there. Another idea would be to put the tee at the center of a circle—so that all shots are bounced directly back toward the tee—except the shot directly away from the hole that should bounce back toward the hole should be modified to send the ball elsewhere. Two pictures are below.

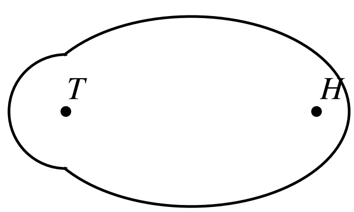


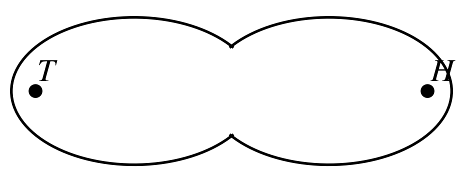


6. A simple solution is to make the shape a regular pentagon, and place the tee and hole near the center. Then, per the reflection principle, there is one one-bounce hole-in-one bouncing off of each side of the pentagon.

7. As mentioned in the solution to problem 5, if the tee is at the center, any shot is reflected directly back at the tee, and then passes through the center again. So the shot bounces back and forth along the diameter it started on. If this diameter doesn't contain the hole, then the ball will never reach the hole no matter how many bounces are allowed. If the diameter does include the hole, you either took a direct shot, resulting in a no-bounce hole-in-one, or you aimed directly away from the hole which gives a one-bounce hole-in-one. The ball never gets a chance at a second bounce because it already went into the hole.

8. There are many ways to do this. One is to cut off part of an ellipse and replace it with a circular piece centered at the tee. That way, shots from the tee toward the circle bounce off the circle, back through the tee, and then again off the ellipse and into the hole. Another is to glue two ellipses together so that they share a focus, with the tee at the other focus of the first ellipse and the hole at the other focus of the second. Then a shot from the tee bounces off the first ellipse, through the common focus, off the second ellipse, and into the hole. Diagrams are below.

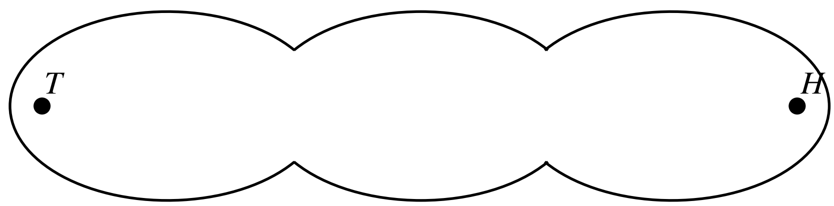




Another possibility is to have two parabolas that share the same axis, and open toward each other. The tee is at the focus to one and the hole at the focus of the other. Rays from the focus of a parabola reflect off the parabola parallel to the axis, so the shot would go from the tee to the first parabola, rebound parallel to the common axis of the parabolas, and reflect off the second parabola toward the hole.

There are many other possibilities as well!

9. Taking our cue from the second solution to problem 8, we simply chain together three ellipses where each pair share a focus, and the tee and hole are at the unshared foci of the ellipses on the ends of the chain.



Note that in both this and the previous problem there is no need for the ellipses to share an axis. And like that problem, this one admits many other solutions relying on the reflection properties of conics (and other shapes).

10. As shown in problem 3, we can make the horizontal section of the "L" so long and place the hole so far along the x-axis so that no shot can hit the hole. For instance, we can place the hole at $(100, 0)$. This requires at least two shots, and the play

- Shot 1. Start: tee; Toward: $(0, 0)$; Stop at: $(0, 0)$
- Shot 2. Start: $(0, 0)$; Toward: hole; Stop at: hole

shows that two shots are indeed possible. So there are holes that can be completed in two shots but not in one shot.

(Of course, there are holes that can be completed in one shot but not in zero—any hole that allows a hole-in-one is such a hole.)

Now the reason we put the hole so far away, instead of, say, $(16, 0)$ which as we know cannot be reached in one shot either, is that we can't reach *any* points—not just those on the x-axis—in the range $x \in [99, 101]$ with just one shot. That is, it takes two shots to reach any point in that range.

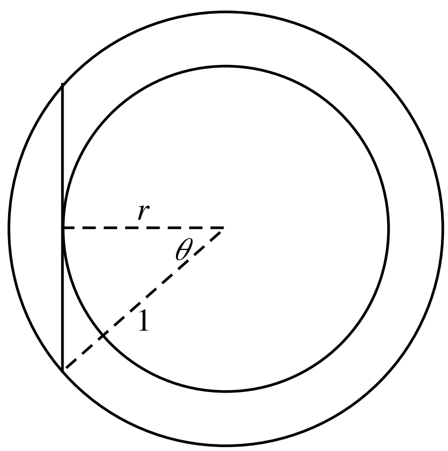
Then we can just put another long corridor—long enough so that no starting position that can't see the end wall of this corridor can reach within two units of that end wall—attached at right angles to this corridor. We can't get very far into this corridor in two shots, but we can get anywhere in it with three shots. Inductively, we can continue to attach long corridors at right angles to the previous corridor. If these corridors are long enough, we can't get to the end of the nth one in fewer than $n + 1$ shots, but since there are only n of them we can do it in exactly $n + 1$ shots.

There are many other acceptable solutions based upon not being able to get "far enough" along part of a path and then inductively attaching another path of the same type to obtain a solution. Note that problem 11(c) also leads to a solution, though since the rules do not allow referring to later questions, the team would have to solve 11(c) here and then refer back to it for their solution to that problem.

11. (a) The two-bounce shot will follow an equilateral triangle from the tee, bouncing off the outer circle twice, into the hole. Thus the inner circle must be smaller than the inscribed circle of this triangle. Since the outer circle has radius one, the circle inscribed in the equilateral triangle has radius $1/2$, so the allowable interval is $(0, 1/2)$. A radius of exactly $1/2$ would have the ball graze the inner circle, creating an extra, undesired bounce.

    (b) This time, we need the circle inscribed in the square that is inscribed in the outer circle. It has radius $\sqrt{2}/2$ so the required interval is $(0, \sqrt{2}/2)$. Note: it should be clear in both these problems that it never helps to bounce the ball from the inner circle, so that we only need to consider the bounces on the outer circle.

    (c) If the radius of the inner circle is r, then if we aimed a shot that just barely grazed the inner circle the shot would first bounce off the outer circle at an angle 2θ away, where θ is as shown in the diagram:



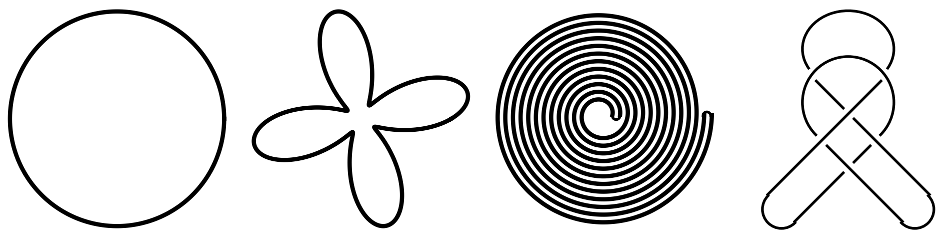
    From right triangle trigonometry, we know that $\theta = \text{Arccos}(r)$. Since the ball is allowed to bounce three times before coming to a stop at the place where it hits a wall for the fourth time, the ball can travel an angle of 8θ before coming to a stop. To be accurate, the ball travels an angle just less than 8θ because we don't want to graze the inner circle, we want to just barely miss it. So, to require n shots, we need $8(n-1)\theta \le 2\pi < 8n\theta$. Thus, we need n to be the first integer strictly greater than $\pi/(4\,\text{Arccos}(r))$, which is $\left\lfloor \dfrac{\pi}{4\,\text{Arccos}(r)} + 1 \right\rfloor$. (Correct answers using degrees rather than radians are acceptable.)

    (d) Any shot that starts at the inner circle will first hit the outer circle, and then *must* hit the inner circle for its second bounce. This is simply by symmetry with respect to the line through the center of the circles and the point where the ball first hits the outer circle. Furthermore, this point of second bounce is at an angle of at most $2\,\text{Arccos}(r)$ from the tee, less if the shot was not aimed tangent to the small circle. After the second bounce, the ball will continue (by symmetry again) by bouncing off the outer circle, and then stopping the fourth time it hits a wall, which will be at the inner circle, having covered at most $4\,\mathrm{Arccos}(r)$ by this time. The only way this could be as large as the 2π needed to make it all the way around to the hole is if $\mathrm{Arccos}(r) = \pi/2$ which requires $r = 0$. But this would mean there is no inner circle at all, so we conclude that we can never get a hole-in-one with this hole.

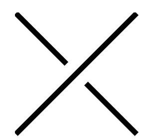
All Tied Up!

<u>The Background</u>

We're going to study knots. A knot is a loop of string in space with its ends connected together. We're going to see which knots can be transformed into which other knots without the string having to be cut or pass through itself. That is, we are going to be allowed to move the string around as long as we don't break the loop. The simplest knot is just a circle, and it is called the *unknot*. Since we can move the string around, all the pictures below are pictures of the unknot.



Notice that in the last picture we will have to slide parts of the loop past other parts before everything untangles. More on exactly how to do that shortly. Basically, the idea is to make a knot with a real piece of string, join the ends of the string together, and draw a sketch, making sure to be able to tell which part of the string goes over and which goes under anywhere they cross.

To be precise, a *crossing* is just a small piece of a picture that looks like this: , possibly rotated. A *knot diagram* consists of a bunch of crossings together with curves joining the "spokes" of the crossings to each other. These curves cannot intersect each other. One spoke can connect to a different spoke on the same crossing, or to any spoke on another crossing. Each spoke is connected to exactly one other spoke.

A *strand* is any one curve in a diagram, from where is starts as an underpass to where it ends in another underpass. It may be an overpass in a number of crossings (including where it starts or ends). For example, the right-most diagram of the unknot at the top of this page has six strands.

<u>The Problems</u>

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. Explain why it is impossible to have a knot diagram where one spoke is directly connected to the spoke diametrically opposite to it on the same crossing without the connecting curve passing through another crossing. [2 pts]

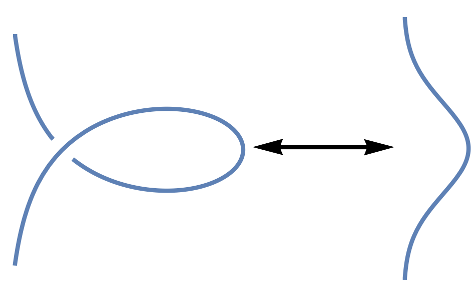
2. How many different knot diagrams are there with just a single crossing? Remember that we are allowed to move the string around as much as we like, and rotations of the picture do not count as different diagrams. [2 pts]

All one-crossing knot diagrams are pictures of the unknot. This is not true with two-crossing diagrams.

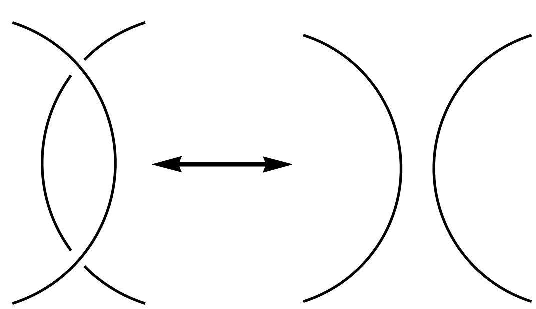
3. Including the unknot, there are three essentially different arrangements of string that can be drawn with exactly two crossings. Draw a two-crossing diagram that displays:
   (a) an unknot; [1 pt]
   (b) two unknots that are linked together; [1 pt]
   (c) two unknots that are not linked together. [1 pt]

For more complicated diagrams, it will be essential to have precise rules for how we can tell when one diagram represents the same knot as another. There are three simple rules for doing so. They are called the *Reidemeister moves*.

The first Reidemeister move allows you to twist or untwist an isolated loop. If you have a piece of string as shown on the right, you may add one crossing by creating a loop as shown on the left. If you have the loop on the left, you can remove the crossing by untwisting it to create the untwisted curve on the right. You may twist either direction—so instead of having the piece of string that comes from the bottom go over the piece that comes from the top, as shown, you could have the part from the top go over the part from the bottom. This moves adds or removes one crossing in the diagram.

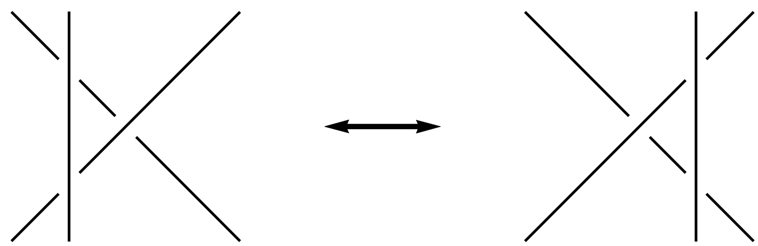


The second Reidemeister move allows you remove or create a pair of crossings where one piece of a string will cross over another piece, and then cross back over it, as shown below.



Note that one string must pass over the other at both crossing. If it goes over and then under, the two loops are linked together and you can't separate them without breaking the string, possibly changing the kind of knot you have.

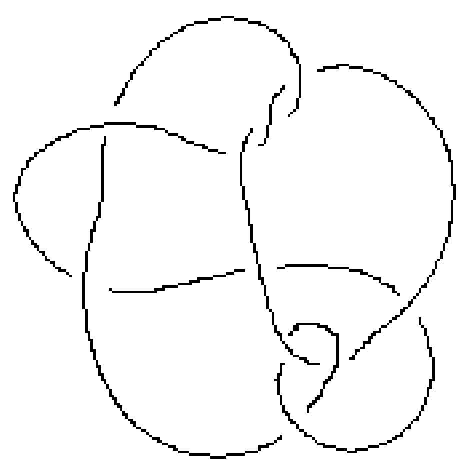
The third kind of Reidemeister move doesn't change the number of crossings, but allows you to pass from one arrangement to another by moving a string from the right side of a crossing to the left or vice versa.



Note that if you tried to simply slide the string from the left to the right or vice versa, when it passes the crossing in the lower pair of strings you would have a three-way crossing which is not allowed in a knot diagram. The third Reidemeister move allows you to move the string around and "skip over" this problem. It should also be noted that instead of thinking that we are moving the topmost string from the left to the right, we could think of this move as sliding the bottom string down and to the left.

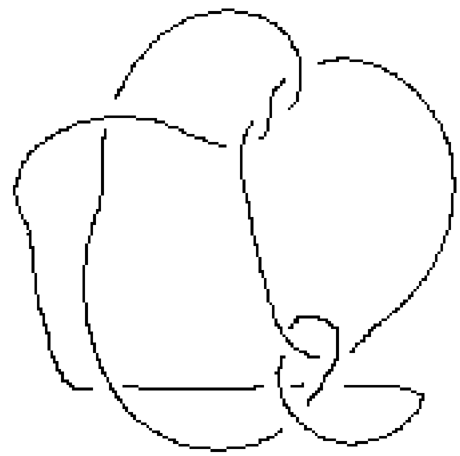
It should be fairly obvious that if you have a picture of a real knot tied with real string, then these three ways of changing the picture would allow you to move the string around in space without having to break or cut the string, so that the new picture is a picture of the same knot just sitting differently in space. (The precise proof of this is quite technical, so we won't get into it.)

Consider the following amazing knot diagram, discovered by Ken Millett:



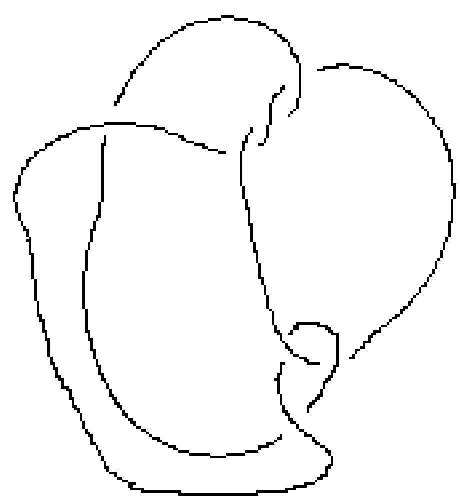
4. Explain why no single Reidemeister move applied to this diagram can reduce the number of crossings. [2 pts]

In the middle of the Millett diagram is a single horizontal string that passes underneath three vertical strands. Imagine you start to slide this string downward, keeping it horizontal. As this string slides beneath the place in the diagram where the top and bottom loops wrap around each other, you get a picture that looks like the following:



5. As the horizontal string passed those two crossings between the top and bottom loops, how many of each kind of Reidemeister move were used to change the diagram? [2 pts]

If you continue to slide this string downward past the other parts of the diagram, you end up with this picture:



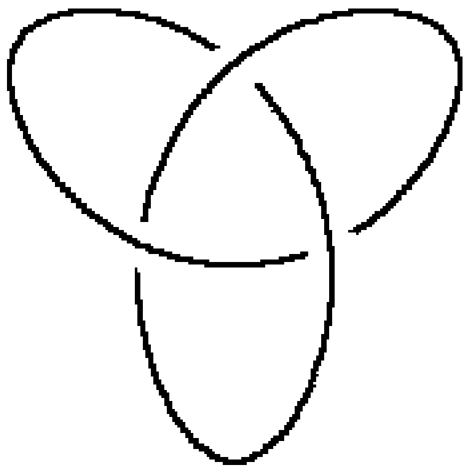
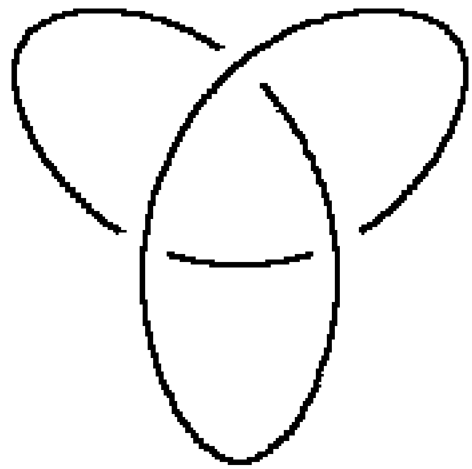
6. How many of each type of Reidemeister move were involved in this last slide? [2 pts]

From this point, you should be able to see how the diagram can keep unwinding until you end up with just a loop—a diagram for the unknot. So Millett's diagram is a complicated diagram for the unknot, one which (by problem 4) you can't make simpler in terms of number of crossings until you first make it more complex!

Coloring

As Millett's diagram shows, you can't easily tell when two knots are the same just by looking at their diagrams, and there might not be a simple way to see how to get from one diagram of a knot to another diagram of the same knot. We need more tests to see how to tell two knots apart.

Definition: A diagram is *3-colorable* if you can color each strand with one of three colors so that the whole diagram isn't all the same color, but at each crossing either all the strands are the same color or the three strands that meet there are three different colors.

The knot shown above left is called a *trefoil*. It is 3-colorable. Color each of the three strands a different color, and you have colored the diagram so that it is not all the same color, and since all three strands meet at all three of the crossings, each crossing has strands of three different colors. On the other hand, the diagram on the right is a sneaky variation of the unknot—after a type-2 move that undoes the two bottom crossings, you have a diagram with a single crossing which is an unknot.

7. Explain why the diagram of the unknot above *cannot* be 3-colored. [2 pts]

Reidemeister moves respect 3-colorability. That is, if a diagram is 3-colorable and you perform any Reidemeister move, the resulting diagram is also 3-colorable. That means that *all* diagrams of a knot are 3-colorable, or *none* of them are. So 3-colorability is actually a property of the knot, not just of any particular diagram.

8. Prove that type-1 moves respect 3-colorability. That is, if a diagram is 3-colorable and you apply a type-1 move, then so is the resulting diagram. Remember that a type-1 move might either create or remove a twist. [2 pts]

9. Prove that type-2 moves respect 3-colorability. [2 pts]

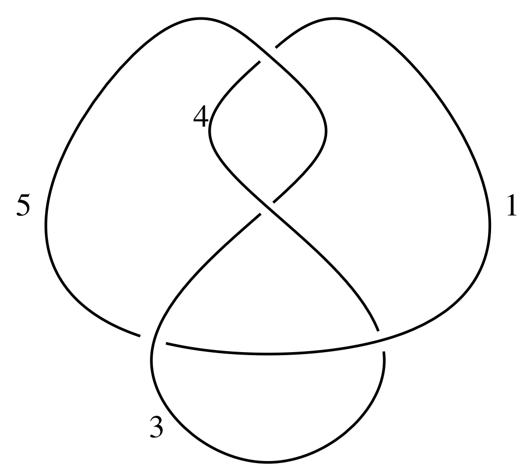
10. Prove that type-3 moves respect 3-colorability. [3 pts]

One diagram for the unknot is just a circle, which certainly can't be 3-colored since there is only one strand to color, so you can't use at least two colors! Since one diagram of the unknot cannot be 3-colored, *no* diagram of the unknot can be 3-colored. On the other hand, because one diagram of the trefoil can be 3-colored, *every* diagram of the trefoil can be 3-colored. So the trefoil and the unknot are fundamentally different knots—one cannot be stretched or twisted into the other without breaking the string—because no diagram for one is also a diagram for the other.

11. For each knot other than the one called "3_1" (which is a trefoil) in the "Table of Small Knots" on the last page of this contest, determine whether the knot is 3-colorable. [3 pts]

We won't actually use colors, but instead we will use the numbers 1, 2, or 3 to indicate which color each strand is painted. We can simply place the number near the middle of each strand to indicate the color.

We can extend the idea of colorability to other numbers of colors. We will say that a knot diagram is k-colorable if it can be colored with at least two, but no more than k colors, and the sum of the two overpass colors equals the sum of the two underpass colors modulo k at each crossing. (The two overpass colors are, of course, the same as the strand can't change color mid-crossing!) The figure below demonstrates that the knot called 4_1 in the table can be 5-colored. Note that the colors do not all have to be used—the color "2" is missing in this coloring.



12. Explain why the knot 4_1 cannot be 3-colored. (Thus, it is a different knot than the trefoil, which can be 3-colored.) [2 pts]

13. Find a 7-coloring of the knot labeled 5_2 in the table. [1 pt]

14. Prove that no knot (consisting of just a single knotted string—not allowing two or more linked loops!) can be 2-colored. [2 pts]

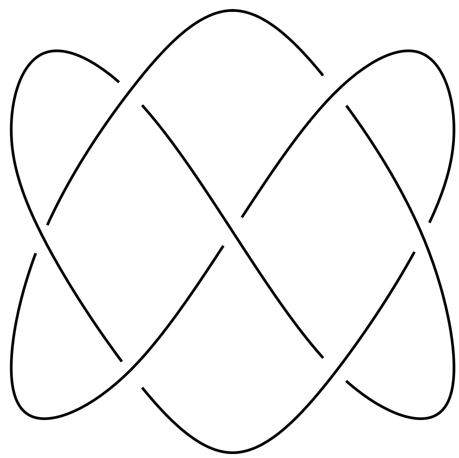
15. Determine the smallest k for which each knot is k-colorable:

    (a) 5_1 [1 pt]

    (b) 6_2 [1 pt]

    (c) 6_3 [1 pt]

16. Show that the knot below can be both 3-colored and 5-colored. [2 pts]



In the picture of the trefoil and the unknot on page 389, the only difference between them is that in the lower-left crossing we have swapped the underpass and the overpass. So even though the trefoil and the unknot are different knots, we can convert the trefoil into the unknot by changing just one crossing. We say that the trefoil has *unknotting number one.* In general, the unknotting number of a knot is the smallest number of crossings that must be changed to convert it into the unknot. For instance, the knot 5_1 in the table has an unknotting number of 2, because no single crossing change will let it completely unravel, but two crossing changes certainly will.

17. Find the unknotting number of:

    (a) 6_1 [1 pt]

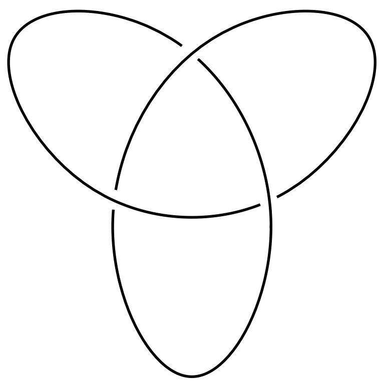
    (b) 6_2 [1 pt]

    (c) 6_3 [1 pt]

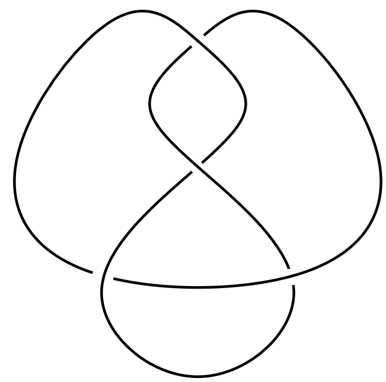
    (d) the knot in problem 16 [1 pt]

18. Draw a knot whose unknotting number is 3. [1 pt]

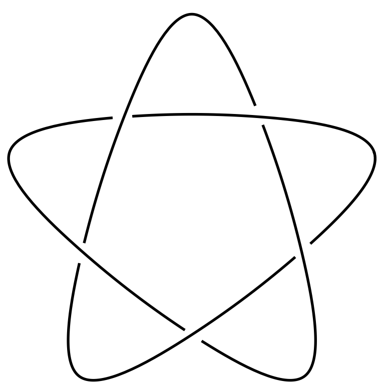
Table of Small Knots



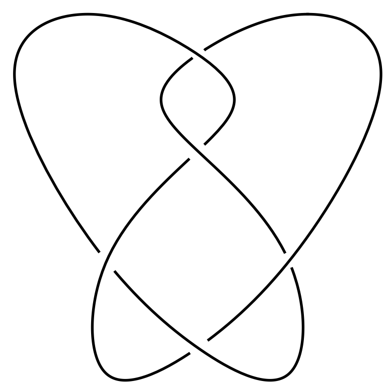
3_1



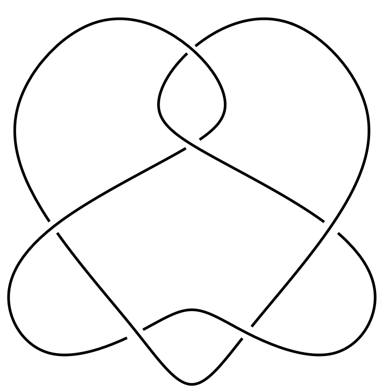
4_1



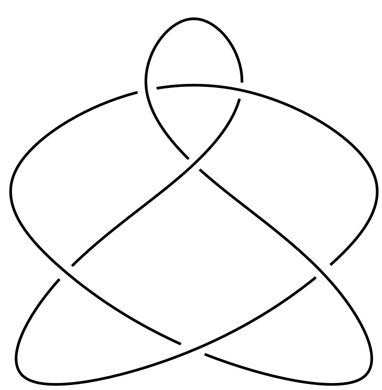
5_1



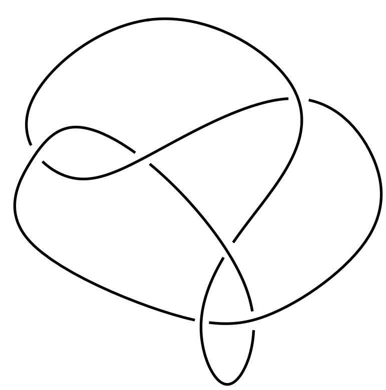
5_2



6_1



6_2

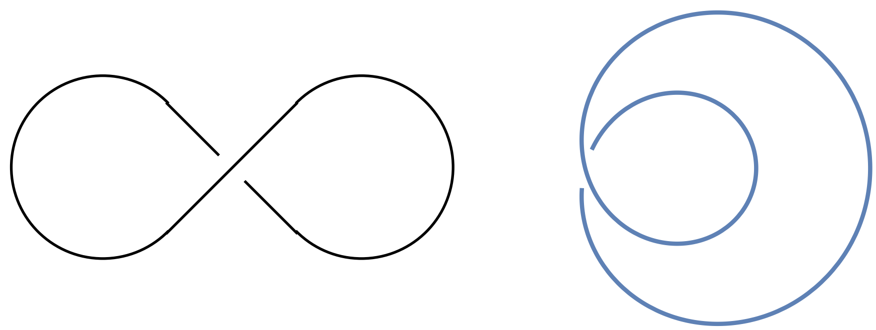


6_3

The Solutions

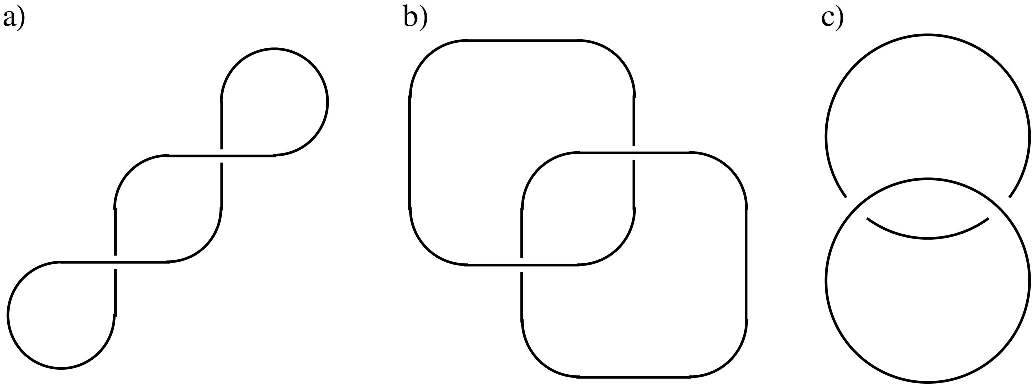
1. If diametrically opposite spokes are connected to each other, the strand that connects them together with the crossing itself would form a closed loop. Exactly one of the other two spokes on this crossing is on the inside of this loop. There may also be some other crossings inside this loop, but the problem is that there will always be an odd total number of spokes inside the loop, and since spokes are connected in pairs by strands, there will be some unconnected spoke inside the loop, violating the condition that each spoke in the diagram is connected to exactly one other spoke.

2. Because of problem 1, we know that adjacent spokes must be connected. There are essentially two different ways to do this. One is a figure-of-eight arrangement, the other has one loop enclosing the other. If you consider how string could move around in space, we could "flip" the enclosing loop to the other side to obtain the figure-of-eight, so both pictures are really of the same knot—the unknot. (If explained like this, 1 will be accepted as a correct answer.)



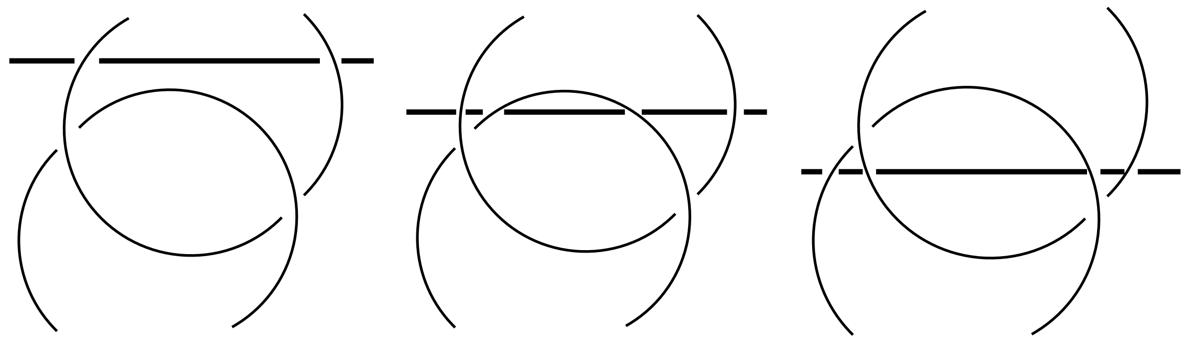
3. Of course, there are multiple arrangements of the string that will satisfy each of these conditions. Here is one set of possibilities:

a) b) c)



4. The only moves that can reduce the numbers of crossings are the first two types. Since there is no place where the string makes a simple twisted loop, there are no moves of the first type available to reduce the number of crossings. And there is no place where one strand crosses over another and then back, so no type-two moves are available either.

5. As the horizontal string moves downward, we get this progression of diagrams near the place where the top and bottom loops wrap around each other:



Between the first and second picture there is a type-II move, and then a type-III move takes us to the third picture. As the string continues its motion downward, another type-III move occurs as it passes the lower crossing, and then a final type-II move occurs as it leaves this portion of the picture entirely. So a total of two type-II and two type-III moves are used.

6. As the string continues to slide downward, it first must cross behind a crossing of two other strings—a type-III move. Then it slides past the bottom of the rest of the diagram, which is a type-II move. But this creates a kink in the string as it slides, which must be removed by a type-I move. So there is one move of each type used in this last slide.

7. Follow the strand that leaves the bottom-right spoke of the top crossing. It runs through the bottom-left crossing from top to bottom, then the bottom-right crossing from bottom to top, the through the top crossing, and finally ends at the bottom-right crossing. This strand must all be one color. That means that three of the spokes of the top crossing have the same color, so this crossing cannot have all three different colors meeting there. So since a legally-colored diagram has each crossing with either all three or only one color, this top crossing must have only one color. That means that the strand that makes up the top-left lobe of the picture is the same color as the first strand that we considered. The same argument applies to the bottom-right crossing. So all three strands are the same color, and we have failed to 3-color the diagram.

8. Let's say that a diagram is 3-colorable and you use a type-I move to add a twist. Since you are working with a single strand, this new twist creates one new crossing, and all the strands that meet at this new crossing come from the same original strand of the knot, so they are all the same color. Since the original diagram was 3-colorable, another color must be used somewhere else in the diagram, so the new diagram with the new twist also has another color in it somewhere, and thus satisfies the requirements for 3-colorability. Conversely, if you use a type-I move to remove a twist, the problem that might show up is that the incoming and outgoing strands from the twist might be different colors, so we wouldn't be able to undo this twist to leave a single strand which can only be one color. But at least three of the spokes of the crossing being removed are parts of the same strand, so it is not possible for all three colors to meet at this crossing. Thus, to be a legal 3-coloring, all of the strands that meet

here are the same color, and thus the incoming and outgoing strands are the same color after all. Since the original diagram was 3-colorable, there is another color in use somewhere else in the diagram, so after untwisting this twist we are left with a legally 3-colored diagram.

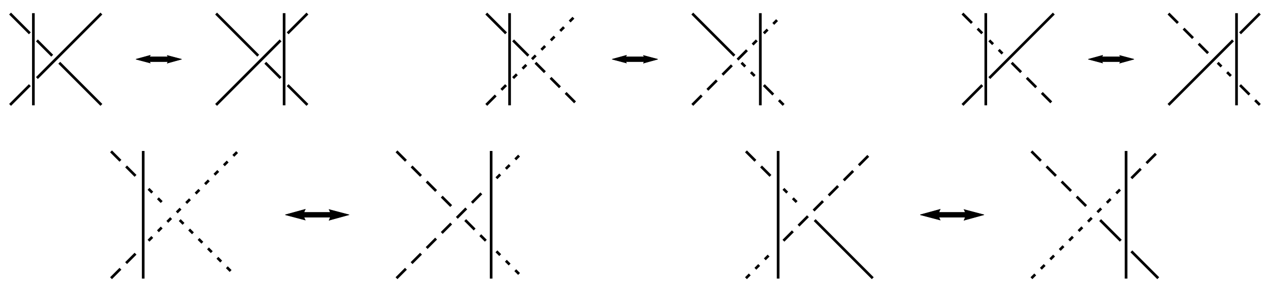
9. In a type-II move, we either move one strand over another to create two new crossings, or we remove two crossings by undoing this operation. The creation of two new crossings starts with two strands, A and B, which are initially separated from each other, and then moves A to cross over B and then cross back. When it does so, a third strand, C, is created to connect the two underpasses of the two new crossings.

   If, initially, A and B are the same color, then we can paint C this same color, and (since another part of the diagram must have a different color, so we continue to meet the at-least-two-colors condition) the new diagram is legally 3-colored. On the other hand, if A has color a and B has color b, then we can paint C with the third color and we have a new 3-colored diagram.

   Conversely, let's say we want to remove two crossings using a type-II move. Let A be the double-overpass strand, with color a. If one of the incoming underpasses also has color a, then three of the spokes at this crossing have color a, so the fourth must also. Thus the double-underpass strand has color a, and three of the spokes at the other crossing will have the same color. So the fourth spoke—the other incoming underpass—must again be this color. In other words, the whole diagram is the same color, and we can pull the strands apart, leaving them this color. Since another color must have been used somewhere else in the diagram, the new diagram without these two crossings will also be properly 3-colored.

   Finally, if one of the incoming underpasses has color b, then at this crossing there are at least two colors. That requires the third color to also be present, so the double-underpass strand will be colored c. Then at the other crossing there are at least two colors in use, so the remaining strand must be colored b to get all three colors at this crossing. That is, *both* incoming underpass strands have the same color. So we can pull the strands apart, removing these two crossings, leaving the strand that was the double-overpass colored a and the other strand b and we have a legitimate 3-colored diagram.

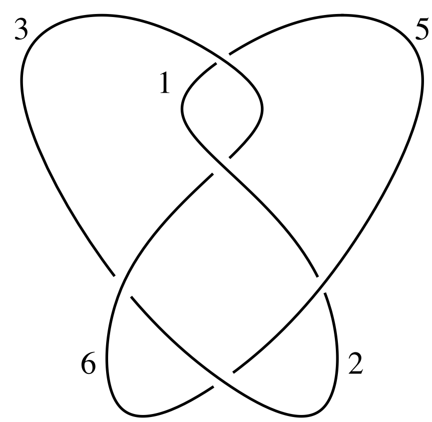
10. Let's assume that the vertical strand that is a double-overpass and that moves from left-to-right or right-to-left over the third crossing in a type-III move has color a. Once you know the colors of the strands coming in from the left, the color of every other strand in the type-III diagram is determined. Note that when the overpass color and one of the underpass colors is known, the other underpass color can be determined from the 3-colorng condition; it is the same color as the others if they are the same color, or it is the third color if they are not. The figure shows the five possibilities. In each case, the colors of the strands on the left determine the remaining colors in the diagram, and the strands exiting on the right have the same colors after the move as they had before it. So in each case the type-III move can be made and the colors of the rest of the diagram do not need any adjustment and the coloring remains valid. In the figure, a solid line represents color a, a dashed line is color b, and a dotted line is color c. (There are actually nine possibilities of the two strands coming in from the left being any of the three colors, but colors b and c may be swapped throughout the diagram, so each of the diagrams with more than one color actually represents two possibilities.)



11. Through trial-and-error, only the knot called 6_1 (along with the original 3_1) in the table of small knots is 3-colorable. To see this, start at any crossing, and assume the overpass is colored a. If one of the underpass strands is also colored a, then so is the other. These three strands can now be traced around to meet at other crossings, and eventually the entire knot would be colored a, which is not allowed in a 3-coloring. But if one of the underpasses is colored b, the other must be c, and then the three strands can be traced through the other crossings, determining the colors of the other strands, until some strand is forced to be two different colorings. This process rules out all but 6_1, where the process yields a legitimate 3-coloring.

12. Oops! This problem was supposed to go *before* the previous one, since the explanation in the previous answer certainly applies here!

13. One example is shown below. Several others are possible.



14. Since the two ends of an overpass are the same color, their sum is zero modulo 2. Then the two underpass colors must also be the same. That means as you follow the string around the knot, it always remains the same color as you cross an overpass or go under an underpass. So the whole string is the same color—you can't use more than one color so you cannot get a legitimate 2-coloring.

15. It might help to notice that colorings can be "multiplied" by a number c, in the sense that if you have a k-coloring and you multiply the color of each strand by c, you now have a legitimate ck-coloring. The process works in reverse in two ways. First, if you have a ck-coloring and

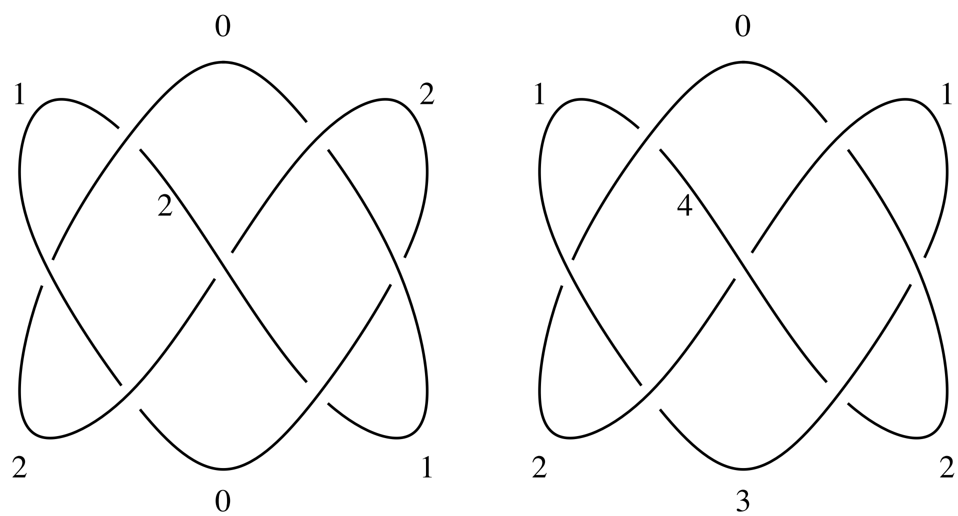
each strand is colored with a multiple of c, then you can divide everything by c to obtain a k-coloring. But if *not* every color is a multiple of c, then you can take each color *modulo* c and you now have a legitimate c-coloring. For example, if you have a $21-coloring$ of some knot, and each color is a multiple of 3, then dividing everything by 3 yields a 7-coloring. But if some strand of the 21-coloring is not colored by a multiple of 3, you can take everything modulo 3 and you receive a 3-coloring of the knot. These two ways of dividing, taken together, mean that if a knot can be k-colored, then it can be p-colored, where p is some prime divisor of k. So the smallest coloring number for any knot must be a prime!

(a) Since, from our earlier work, 5_1 is not 2- or 3-colorable, we check the next prime and find that 5_1 is indeed 5-colorable.

(b) We have already ruled out 2 and 3, and it will only take a moment to rule out 5 and 7 and rule in 11.

(c) This time, the first prime number that works is 13.

16. The figures below show one way to color the diagram with 3 colors and one way with 5 colors. There are other possibilities as well.



17. It is usually quite difficult to determine the unknotting number of a given knot, especially since different diagrams might make the raveling and unraveling a lot more (or less!) clear. But in these cases, the answer can be easily determined.

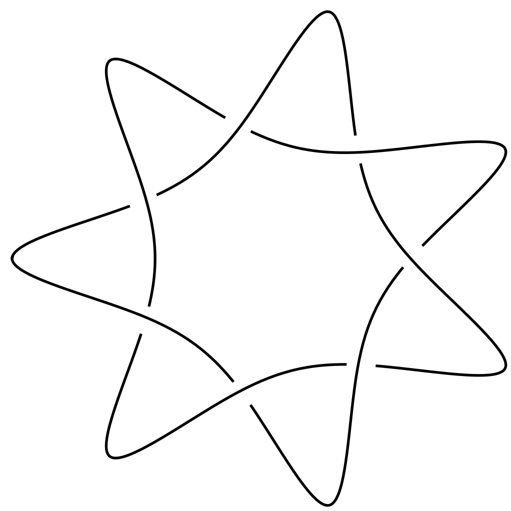
(a) The unknotting number is one. Change either of the two upper crossings. Then separating the two upper loops with a type-II move shows the remaining string to be just a four-times twisted unknot.

(b) Changing the upper-middle crossing will allow the entire knot to unravel, so the unknotting number is again one.

(c) This time, the crossing in the upper-right can be changed to allow the knot to unravel, so this knot also has unknotting number one.

(d) Proving that there is no way to unknot this with a single crossing change is actually fairly advanced—remember you have to consider changing a single crossing in every possible diagram that could be obtained from the one shown using every possible combination of Reidemeister moves—but by experimenting it should be fairly clear that it can't be done with a single crossing change. But it is easy to do with two crossing changes. Starting with the central crossing, you can easily unravel the remains into a diagram with three crossings—the trefoil—which we know can be unknotted with one more crossing change.

18. There are many possibilities. The simplest is probably the trefoil-like knot, but with seven twists instead of three, as shown below.



Note: Problem 12 was supposed to come before Problem 11, but the transposition was not noticed before the publication of the contest. The solutions to the contest took note of this, hence the "Oops!" comment in Solution 12. ARML decided to publish the contest and solutions in the form originally seen by contestants.

Cryptarithms

The Background

A *cryptarithm* is an arithmetic puzzle where digits are replaced by letters and the puzzle is to figure out which digit each letter stands for. One of the best known examples is a classic puzzle by one of the greatest puzzle creators of all time, Henry Dudeney.

```
   S E N D
 + M O R E
 ---------
 M O N E Y
```

Dudeney's puzzle is a special kind of cryptarithm called an *alphametic,* where the letters actually spell words that fit together in a sentence. The rules of cryptarithms are: 1) each letter stands for exactly one digit, and it is the same digit each time the letter is used, 2) no digit is represented by two or more different letters, and 3) the leading digit of a number cannot be zero.

Dudeney's puzzle can be solved as follows. (It is highly recommended that you get out some scratch paper and work along here!) First, a four-digit number is less than 10000, so `MONEY`, being the sum of two four-digit numbers, must be less than 20000. But since the leading digit `M` cannot be zero, it must be a one. Then the number `MORE` is less than 2000, so `MONEY` is less than 12000. Thus, the letter `O` must represent a 0 or a 1. But it cannot represent 1, since `M` already does, so it must represent 0.

Next, look at the hundreds column where it tells us that $\texttt{E} + 0 = \texttt{N}$. Since `E` and `N` must represent different digits, the only way that could happen is if there had been a carry from the addition in the tens column. So we know that `N` is one more than `E` (or that `E` is 9 and `N` is zero, but that is ruled out because we already know that `O` is 0). It also means that there is no carry from the hundreds place to the thousands place. Then we know that $\texttt{S} + 1 = 10$ so `S` is 9.

The table below summarizes what we know so far:

0	1	2	3	4	5	6	7	8	9
O	M								S

Additionally, we know that the pair `EN` represents 23, 34, 45, 56, 67, or 78. You will later be asked to finish this puzzle.

In problems that ask you to solve a cryptarithm, the answer should be given as in the table above. That is, if a problem asks you to *solve* a cryptarithm (as opposed to explaining something, etc.)

then the answer is simply a table whose first row is the digits zero through nine (do not leave any out, even if they don't appear in the answer to the puzzle) and whose second row is the letter that represents the digit above it. You can earn partial credit for partial solutions. For example, you might earn one point out of three for the above table, even if you aren't able to figure out the values of the remaining letters.

In problems where a *restoration* is called for (problems 9–11) you will have a multiplication problem where the digits have been replaced by letters. Your answer should show the entire original multiplication (not just the factors and product!). In other words, show the entire long multiplication with the digits restored to their original positions.

One last thing to note. Many cryptarithms admit multiple solutions, while others have none at all. For example, `YOU` + `ME` = `LOVE` has no solution at all, while you can check that both

0	1	2	3	4	5	6	7	8	9
N	F	M	R	P	E	T	H	A	O

and

0	1	2	3	4	5	6	7	8	9
N	M	F	R	P	E	T	H	A	O

are solutions to `MOTHER` + `FATHER` = `PARENT`. (The roles of the `M` and `F` are switched.) In the puzzles for this contest, unless otherwise stated, *every puzzle will have a unique solution.*

Enjoy the puzzles!

<u>**The Problems**</u>

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. Let's explore why there are no solutions to the puzzle given in the reading:

```
  Y O U
+   M E
-------
L O V E.
```

(a) Fill in the blanks with the smallest possible integers: Even if we were allowed to have different letters representing the same digit, the three-digit number `YOU` is at most ________ and the two-digit number `ME` is at most ________, so their four-digit sum `LOVE` is at most ________ . [1 pt]

(b) Continue filling in the blanks with the only digits that follow logically from your work in part (a): Thus, the only hope for an answer to this puzzle is if `L` represents the digit ________ and `O` represents ________ . [1 pt]

(c) Now, remembering that we *cannot* have two different letters representing the same digit, explain why there cannot be any solution to this puzzle. [2 pts]

2. Complete the solution to Dudeney's puzzle. Remember to give your answer in the proper format! [2 pts]

```
   S E N D
 + M O R E
 ---------
 M O N E Y
```

3. Solve the astronomical alphametic below. [4 pts]

```
   S A T U R N
 + U R A N U S
 -------------
 P L A N E T S
```

4. A *doubly-true* alphametic occurs when the words spell out a true mathematical sentence. Try this one: [3 pts]

```
 F O R T Y
     T E N
 +   T E N
 ---------
 S I X T Y.
```

5. Why should addition have all the fun? The following multiplication problem is clearly not a doubly-true alphametic! [3 pts]

```
    T W O
*   T W O
---------
T H R E E
```

6. There's nothing special about English, either. The following is doubly-true *auf Deutsch.* [3 pts]

```
  E I N S
  E I N S
  E I N S
+ E I N S
---------
  V I E R
```

7. Get out your best factoring tricks to solve this one: [3 pts]

$$9(\texttt{B I G T O P}) = 4(\texttt{T O P B I G}).$$

8. Here is another old classic, created by master puzzle creator Alan Wayne. Solve: [3 pts]

$$\texttt{K I S S} = \sqrt{\texttt{P A S S I O N}}.$$

9. The following isn't strictly a cryptarithm, but it seems to be in the same spirit. In the following multiplication problem, each digit has been replaced with the same letter, X. And each digit in the original problem was a prime. There is only one solution. For your answer, restore the multiplication, that is, reproduce the entire multiplication, restoring as many digits as you can. [3 pts]

```
    X X X
*     X X
---------
  X X X X
X X X X
---------
X X X X X
```

10. In the variant below (again, this isn't strictly a cryptarithm), each time the letter `E` occurs it should be replaced by an even digit (though not necessarily always the same digit), while each `O` is to be replaced by an odd digit (again, not necessarily the same odd digit each time) to make a correct multiplication. Can you restore all the digits? [3 pts]

```
      E E O
*       O O
---------
    E O E O
    E O O
---------
O O O O O
```

11. This one is another odd/even restoration, only a little bit harder: [4 pts]

```
   O E E
*    E E
-------
 E O E E
 E O E
-------
 O O E E.
```

12. There's nothing special about base 10. Determine, with proof, all bases in which the following alphametic has at least one solution. [5 pts]

```
  K Y O T O
  K Y O T O
+ K Y O T O
----------
  T O K Y O
```

The Solutions

1. (a) 999, 99, 1098

   (b) 1, 0

   (c) Since the value of ME is less than 100, the value of YOU must be more than 900 for their sum to be a four-digit number. Since we know from part (b) that O must represent 0, and a digit can't be represented by more than one letter, the value of YOU is at most 908. Since the digit 9 is already used, we know that the value of ME is less than 90. So the sum YOU + ME is less than 998 so cannot be a four-digit number. So the puzzle has no solution.

2. So far we know that O, M, and S represent 0, 1, and 9 respectively, and that the pair EN represents 23, 34, 45, 56, 67, or 78. Note that the addition in the tens- and hundreds-places reverses these two digits. So in all of the possible cases, NE − EN = 9. Since the digit 9 is already represented by S, we see that R must be 8 and there must be a carry from the units-place. That also rules out the possibility that N represents 8 (so E cannot be 7).

   We go through the remaining possibilities. If E is 2, then D would have to be 8 or 9 to cause a carry, but both those digits are already taken. If E is 3, a D of 7 would cause a carry, but then Y would be 0, which is already taken. Likewise, with an E of 4, D could be 6, 7, 8, or 9 to cause a carry, but 8 and 9 are already used, while 6 and 7 cause Y to be 0 or 1, which are already taken.

   That leaves E as either 5 or 6. Check that 6 doesn't work (D can't be 0, 1, 2, or 3 because they cause no carry, 4 or 5 because they cause Y to be an already used value, 6 because that is the value of E, 7 because that is the value of N if E is 6, and of course 8 and 9 are ruled out). So the only hope is to try E representing 5; then N would be 6. D can then represent 7 and Y would be 2. The sum would be $9567 + 1085 = 10652$ which is true. So the solution is

0	1	2	3	4	5	6	7	8	9
O	M	Y			E	N	D	R	S

.

3. Start with the obvious: since the only possible carry into the largest position can be a one, P represents 1. And since there cannot be a carry into the units digit, we learn that N represents 0. Now look in the hundreds-place, where adding 0 to U resulted in E, meaning there had to be a carry and E is one larger than U (or U is 9 and E is 0—this doesn't happen since we already know that N is 0).

   In the hundred-thousands-place, we have A + R = A and the digit 0 is already taken, so R must represent 9 and there must be a carry coming into this position. Then, from the tens-place we learn that T is one less than U. Since U is not 9, there will be no carry from the hundreds-place into the thousands-place. Thus, the digits T and A must add to 10.

Since we know that `U` is one more than `T` and `E` is one more than `U` and the digits 0, 1, and 9 have already been taken, the trio `TUE` represents one of 234, 345, 456, 567, or 678. But `T` can't be 4, or `A` and `E` would both have to be 6. Similarly, if `T` were 5 then `A` would have to be 5 also.

Let's just try the remaining possibilities. Setting `T` to be 2 and `U` to be 3 causes a problem in the hundred-thousands-place where, even with the carry from the ten-thousands-place, the largest we could make `S` is 8 (9 is already taken!) which would force `L` to be 0, 1, or 2 (or there to be no carry into the millions place), all of which are already taken. A similar problem happens by setting `T` to be 3. The only remaining possibility is to set `T` to be 6. This leaves three choices for `S`, but two of them lead to `L` being 0 or 1. The remaining choice, 5, leads to the correct addition $546790 + 794075 = 1340865$ and the solution

0	1	2	3	4	5	6	7	8	9
N	P		L	A	S	T	U	E	R

.

4. The units-place of this sum tells us that $\mathtt{Y} + \mathtt{N} + \mathtt{N} = \mathtt{Y}$, perhaps with a carry. So `N` is either 5 or 0. The tens-place of the sum tells us that there can't be a carry (because $\mathtt{E} + \mathtt{E} + 1$ cannot add to zero or ten), so `N` must be 0. Then `E` will be 5.

   There must be carries from both the hundreds-place and the thousands-place. The three digits in the hundreds-place can only add to numbers in the teens or twenties, so the carry into the thousands-place must be a 1 or a 2. If it is a 1, that would make `O` a 9 since we need a carry from this column also. But if `O` is 9 and the carry is only one, that would make `I` be 0, which has already been used. That means that the carry must be 2 and `O` must be 9, and `I` is 1.

   In the tens-place, $\mathtt{T} + 5 + 5 = 10 + \mathtt{T}$, so the carry into the hundreds-place is a 1. So, since the carry into the thousands-place is 2, we know that $1 + \mathtt{R} + 2\mathtt{T}$ is at least 20. In fact, since 0 and 1 are already taken, `X` must be at least 2, so $1 + \mathtt{R} + 2\mathtt{T}$ is at least 22. Since `R` is not the 9, this last sum requires `T` to be either 7 or 8.

   Trying 7 for `T` means `R` must be 8, and `X` will be 3. The only digits now left are 2, 4, and 6, but `F` and `S` must be consecutive, so this scenario is impossible. Therefore `T` must be 8.

   We still must have $1 + \mathtt{R} + 2\mathtt{T}$ be at least 22, so `R` will now be either 6 or 7. But choosing it to be 6 again requires `X` to be 3, which has the same problem as above. The only choice left is `R` is 7, `X` is 4, and we can finish the puzzle with `F` being 2, `S` being 3, and `Y` being 6. The complete solution is

0	1	2	3	4	5	6	7	8	9
N	I	F	S	X	E	Y	R	T	O

.

5. The number `THREE` is a square. The squares of numbers from 200 and up are either six-digit numbers, or do not begin with the same digit as the number of which they are the square. So the `T` represents the digit 1. Further, the only squares that end in a repeated digit are those that end in -00 or -44. Since `O` and `E` can't both be 0, the former is ruled out. The numbers less than 200 whose squares end in -44 are $112^2 = 12544$ (but `T` and `W` can't both represent the digit 1), $138^2 = 19044$ (which is the answer), $162^2 = 26244$ (which doesn't begin with a 1), and $188^2 = 35344$ (ditto), so we have the answer

0	1	2	3	4	5	6	7	8	9
R	T		W	E				O	H

.

6. First, the digit `E` must be 1 or 2. Leading digits can't be 0, and larger values make the sum larger than 10000.

   Now look at the hundreds-place. We need four copies of `I` plus any carry from the tens-place to add to a number that ends in `I`. That is, we need $4\texttt{I} + \text{carry} \equiv \texttt{I} \pmod{10}$. The carry can be 0, 1, 2, or 3. So we learn that the value of `I` can be 0 (if the carry is 0), 3 (if the carry is 1), 6 (if the carry is 2), or 9 (if the carry is 3).

   Let's look again at the value of `E`. If it is 2, then `I` cannot be 6 or 9, or the sum will be more than 10000. If `I` is 3, we know that the carry from the tens-place must have been a 1. Looking in the tens-place, then, we see that if c is the carry from the units-place, we have $4\texttt{N} + c = 12$. Since the carry c can only be 0, 1, 2, or 3, the only solution is that `N` is 3. But we can't have both `N` and `I` representing 3, so if `E` is 2 then `I` being 3 is ruled out.

   So if `E` is 2, the only possibility left is that `I` is 0. This requires zero carry from the tens-place, so `N` is 0, 1, or 2. Since 0 and 2 are already taken, `N` would have to be 1. But then the addition in the tens-place will not work out (the carry from the units-place can't be larger than 3). We conclude that `E` represents 1.

   Looking at the tens-place we find that `N` must be 0 or 5 (with a carry of 1 from the units-place), or 2 or 7 (with a carry of 3 from the units-place). Now we examine the possibilities one by one. First, if `I` represents 0, then there can be no carry from the tens-place, which would also require `N` to be 0. Nor can `I` represent 6, because now `EINS` is between 1602 and 1698, so the sum of four copies of `EINS` would add to a number between 6408 and 6792, requiring `V` to also represent 6.

   Trying 9 for `I` means that the carry from the tens-place had to have been 3. That means that `N` would have to be 7. But now `EINS` is between 1970 and 1978, so quadrupling it leads to a number between 7880 and 7912, thus requiring `V` to also represent 7.

   The only remaining possibility is to have `I` represent 3. As we have seen, this requires a carry of 1 from the tens-place, which means that `N` will have to represent 2. The tens-place now requires a carry of 3 from the units-place, meaning `S` represents either 8 or 9. We rule out 8 because that would result in `R` representing 2 which is already represented by `N`. So `S` represents 9, and we have the sum $1329+1329+1329+1329 = 5316$. The answer grid is thus

0	1	2	3	4	5	6	7	8	9
	E	N	I		V	R			S

.

7. Let m be the three-digit number represented by `BIG` and n the number represented by `TOP`. Then the equation tells us $9(1000m + n) = 4(1000n + m)$, or $8996m = 3991n$. We divide each side by the common factor of 13 are we are left with $692m = 307n$. The coefficients are relatively prime, so we must have $m = 307$ and $n = 692$, giving us the answer

0	1	2	3	4	5	6	7	8	9
I		P	B			T	G		O

.

8. First, notice that `K` must be 1, 2, or 3, or the square of `KISS` will have eight digits, not seven.

   Let's concentrate on the digit represented by `S`. It can't be 0, because then $(\texttt{KISS})^2$ would also end in 0, but `N` can't represent 0 if `S` does. The same is true for 1, 5, and 6.

   Now notice that the hundreds-digit of both numbers is `I`. So let's see what happens when we square a number ending in `ISS`.

   If `S` is 2, then $(\texttt{ISS})^2$ ends in $(4\texttt{I}+4)84$. In other words, whatever digit `I` represents, quadruple it and add four to get the hundreds-place of the square. Or rather the units-digit of this number is the hundreds-place of the square. The only choice for `I` for which this results in the same digit is 2, but we are already assuming `S` is 2, so this is ruled out, and `S` cannot be 2. In a similar way, no choice of `I` works if `S` is 4 or 7. If `S` is 8, note that the square of any number ending in 88 will itself end in 44, but the last two digits of `PASSION` are not the same, so 8 is also ruled out. That implies `S` is either 3 or 9.

   When `S` is 3, several choices of `I` will work out. If `I` is 0, `K` can be 1 or 2, and we try $1033^2 = 1067089$ and $2033^2 = 4133089$. Notice that the first of these does not have the form `PASSION` because the hundreds- and hundred-thousands- places are the same. But the second does work. Just to be complete, we check that the other possibilities do not work.

   If `S` is 3, `I` could be 2 and `K` could be 1, leading to $1233^2 = 1520289$. If `I` is 4, `K` could be 1 or 2, but $1433^2 = 2053489$ and $2433^2 = 5919489$. We could also try `I` as 6, but $1633^2 = 2666689$ and $2633^2 = 6932689$. The other possibilities occur when `S` is 9, and then `I` must be 6, leading to trying $1699^2 = 2886601$ and $2699^2 = 7284601$. Of all these possibilities, only one had the form of the word `PASSION`, so that must be our number, and our final answer is

0	1	2	3	4	5	6	7	8	9
I	A	K	S	P				O	N

.

9. So that we can refer to the individual digits in this problem more easily, let's rewrite the problem as `abc` $\times$ `de` $=$ `fghi` $+$ `jklm0` $=$ `nopqr` where each letter stands for a digit, digits can be repeated, and 0 has been inserted into the second addend to account for the left shift in the multiplication process.

   Keep in mind that each digit must be prime, so each is 2, 3, 5, or 7. Also note that the product of pairs of these are among the numbers 4, 6, 10, 14, 9, 15, 21, 25, 35, and 49. The only ones of these that end in a prime digit are 15, 25, and 35. This tells us that at least one of our multipliers must end in a 5 and the other is odd (it could also be 5, but only one needs to be 5).

   Expanding the logic of the previous paragraph, if the three-digit multiplier doesn't end with a 5, then the two-digit multiplier must be 55. Once you know that digit `e` is 5, then `a` can't be 2 or 3 (else `f` is 1) leaving very few possibilities for `abc`. In fact, the only two possibilities for `abc` that leave all prime digits in `fghi` are 555 and 755. But multiplying $555 \times 55 = 30525$ and $755 \times 55 = 41525$, and neither of these products consists of all prime digits.

   Now look at the digit `e`. It certainly can't be 2 (or else `i` would be 0). It can't be 7, because then the carry in computing `h` would be 3, which would require `b` to be 7, and then the carry into the computation of `g` would be 5, and no number would work for `a` to keep all the digits of `fghi` prime.

   We've already seen that if `e` is 5 the only possibilities for `abc` are either 555 or 755. We quickly check the products of these two with 25, 35, 55, and 75 and conclude that `e` cannot be 5. Thus, `e` must be 3.

   Once we know that `c` is 5 and `e` is 3, we learn that `b` must be either 2 or 7 if we want digit `h` to be prime. Note that `b` can't be 2, because then there would be no carry into the hundreds digit, making it impossible to make both `f` and `g` prime (alternately, multiplying each of 225, 325, 525, and 725 by 3 either produces only a three-digit number or a number that does not have all prime digits). Hence, the digit `b` must be 7. By multiplying 275, 375, 575, and 775 by 3, we then learn that `a` must be 7. So we have found that the number `abc` is 775.

   Now we just need to investigate the digit `d`. Quickly checking 775 times 2, 3, 5, and 7 leads us to the conclusion that the only possible digit for `d` is 3.

   So our multiplication is $775 \times 33 = 25575$. Written out, the complete answer is:

```
    7 7 5
*     3 3
---------
  2 3 2 5
2 3 2 5
---------
2 5 5 7 5.
```

10. Let's again rewrite this as `abc` $\times$ `de` $=$ `fghi` $+$ `jkl0` $=$ `mnopq`. Since `abc` is greater than 200, and `abc` $\times$ `d` $=$ `jkl` which is less than 1000, we find that `d` must be less than 5. Note that `d` cannot be 1, since then `abc` and `jkl` would be equal, but `b` is even while `k` is odd. So the only odd number left for `d` is 3.

    This in turn tells us that `a` must be 2, since all larger even numbers lead to a four-digit product with `d`. So `abc` is less than 300, and its product with `e` will be less than 3000, so `f` must be 2. Since `f` $+$ `j` must produce a carry, we learn that `j` $= 8$. This forces `b` to be 8 (neither $267 \times 3 = 801$ nor $269 \times 3 = 807$ has the form `EOO`) and then we quickly learn that the only choice for `c` that works is 5. So `abc` is 285.

    Lastly, since $285 \times 7 = 1995$ is too small to be `fghi`, we learn that `e` is 9. Putting all the pieces together yields the solution:

```
    2 8 5
*     3 9
---------
  2 5 6 5
  8 5 5
---------
1 1 1 1 5.
```

11. Once again, we rename the unknown entries, so that the problem becomes `abc` $\times$ `de` $=$ `fghi` $+$ `jkl0` $=$ `mnop`. Since `d` is at least 2, we find that `a` is either 1 or 3. But if `a` is 1, then `abc` is less than 200, and its product with `e` is less than 2000, but since `f` is even, we have `fghi` is at least 2000, a contradiction. So `a` is 3.

    Now since `abc` is between 300 and 400, and its product with the even digit `d` is less than 1000, we know that `d` must be 2. Also, since `abc` is less than 400 but its product with `e` is larger than 2000, we know that `e` is larger than 5. Since `e` must be even, it will be 6 or 8.

    The digit `b` must be 0, 2, or 4 (otherwise, doubling `abc` would lead to a number between 700 and 800, but we know its first digit must be even). But it can't be 0, because then `abc` is between 300 and 310. Multiplying those numbers by 6 yield numbers between 1800 and 1860, while multiplying by 8 gives 2400 to 2480, so neither choice of `e` would give an acceptable result for `fghi`.

Likewise, `c` must be 6 or 8 to allow the digit `k` to be odd. So we have that `abc` is one of 326, 328, 346, or 348, while `de` is either 26 or 28. It's not hard to try the eight combinations, and learn that only one makes all the rest of the digits have the correct parities. The answer is:

```
  3 4 8
*   2 8
-------
2 7 8 4
6 9 6
-------
9 7 2 4.
```

12. The only base in which the alphametic has a solution is base 9. We can prove this as follows.

Let b be a base in which the problem has a solution. Since there are 4 different letters which appear in the alphametic, there are at least 4 different digits in base b, so $b \geq 4$. Number the columns in the addition problem from left-to-right, so that column 1 reads `K + K + K = T` and so forth.

Consider the fifth column, where we have $3\mathtt{O} = \mathtt{O}$, perhaps with a carry. We conclude that `O` is either 0 or $b/2$. The carry from the fourth column is either 0, 1, or 2, so if `O` is $b/2$ then `K` is $b/2 + c$ where c is this carry. This will always be larger than $b/2$ unless $b = 4$ and the carry is 2, in which case `K` would have to be 0, which is impossible because leading zeroes are forbidden. But if `K` is larger than $b/2$ then the first column addition would produce a carry, which it does not. So we conclude that `O` is 0.

Now we know that `K` is just the carry from the fourth column, and is not 0, so `K` is either 1 or 2. There can be no carry from the third column into the second, so the second column tells us that $3\mathtt{Y}$ is either 0, b, or $2b$. It can't be 0 because `Y` is not 0. So it is either b or $2b$. Either way, we learn that b is divisible by 3. Also, the carry from the second column into the first is either one or two. Since `K` is known to be 1 or 2, we find there are four possible values for `T`: 4, 5, 7, or 8.

Now we plow through the possibilities. If `K` is 1 and `T` is 4, the third and fourth columns tell us that $b + \mathtt{Y} = 12$. Since $0 < \mathtt{Y} < b$ and b is divisible by 3, the only possibility here is $b = 9$ and the sum is $13040 + 13040 + 13040 = 40130$ which is true in base 9.

If `K` is 1 and `T` is 5, we find that $b + \mathtt{Y} = 15$. So either $b = 9$ and $\mathtt{Y} = 6$ with solution $16050 + 16050 + 16050 = 50160$ which is true in base 9, or we try $b = 12$ and $\mathtt{Y} = 3$. But then the addition in the second column fails, so this is not a possibility.

Next, try $\mathtt{K} = 2$ and $\mathtt{T}$ being 7. Then the third and fourth columns give us $2b + \mathtt{Y} = 21$. Since $\mathtt{Y}$ is still a base-b digit and is not 0 and b is divisible by 3, the only solution here is $b = 9$ and $\mathtt{Y} = 3$. This leads to the (correct in base 9) equation $23070 + 23070 + 23070 = 70230$.

Finally, with $\mathtt{K}$ set to 2 and $\mathtt{T}$ set to 8, the third and fourth columns give $2b + \mathtt{Y} = 24$. The only hope for a solution this time is $b = 9$ and $\mathtt{Y} = 6$, which does work to give us $26080 + 26080 + 26080 = 80260$ which is also a solution.

So, in every case, the only base in which we could find a solution is 9, and there were four solutions.

Funny Factorials and Slick Sums

The Background

WARNING! In several of the problems, you may be tempted to use an ellipsis ("...") to shorten arguments or to indicate that a pattern continues. DO NOT do this. All proofs must be complete. Ellipsis probably means you need to use induction or recursion.

Let $f(x)$ be any function whose domain is the set $\mathbf{R}$ of real numbers. Create a new function $\Delta f(x)$ which is defined to be $\Delta f(x) = f(x+1) - f(x)$. So if, for instance, $f(x) = x^2$ then $\Delta f(x) = (x+1)^2 - x^2 = 2x + 1$. We will treat Δ as an operation that takes in functions and gives out functions—a kind of superfunction whose domain and range are regular functions.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. For each function $f(x)$ below, compute and simplify $\Delta f(x)$.
   (a) $f(x) = c$, where c is a constant [1 pt]
   (b) $f(x) = ax + b$, a linear function [1 pt]
   (c) $f(x) = x^3$ [1 pt]
   (d) $f(x) = 2^x$ [1 pt]

As part (c) of this problem demonstrated, Δ doesn't play particularly nicely with powers of x. But there is a variation of factorials that Δ does treat nicely. For any positive integer n, define $x^{\underline{n}}$ by the formula $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$. It's kind of a cross between the nth power and a factorial. The underline is meant to remind you that the things you are multiplying together keep decreasing. You can pronounce this as "*x to the nth falling*" or "*the nth falling factorial of x*". It will be convenient to define $x^{\underline{0}} = 1$. There is a similar *rising factorial* $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$.

2. Expand and simplify each of the following.
   (a) $x^{\underline{3}}$ [1 pt]
   (b) $9^{\underline{4}}$ [1 pt]
   (c) $\Delta x^{\underline{2}}$ [2 pts]

3. In keeping with the warning at the beginning of the problem, write a proper recursive definition for $x^{\underline{n}}$ for all nonnegative integers n. [2 pts]

We can make a few observations about falling and rising factorials. For instance, if n is a nonnegative integer, then $n^{\underline{n}} = 1^{\overline{n}} = n!$, while if $m > n$ are both nonnegative integers then $n^{\underline{m}} = 0$. Also, $x^{\underline{n}} = (-1)^n(-x)^{\overline{n}}$ where the exponent on the (-1) is a regular power. While all these appear more-or-less obvious, if you want to use one of these facts in your solutions, you will—for completeness—need to prove it as a lemma.

We will treat falling and rising factorials like regular exponents with regard to order of operations. So, for instance, $(-5)^{\underline{3}} = (-5)(-6)(-7)$ while $-5^{\underline{3}} = -(5^{\underline{3}}) = -(5 \cdot 4 \cdot 3)$.

4. Prove the factorial law of exponents: $x^{\underline{m+n}} = x^{\underline{m}} \cdot (x-m)^{\underline{n}}$ for all nonnegative integers m and n. [2 pts]
5. Prove the symmetry law $(-x)^{\underline{n}} = (-1)^n(x+n-1)^{\underline{n}}$ for all nonnegative integers n. Note that the exponent on the (-1) is a regular exponent, not a factorial exponent! [2 pts]

You might have noticed that $\Delta x^{\underline{2}}$ from problem 2(c) was less messy than Δx^2 or Δx^3 that we calculated earlier. Falling factorials were created to make nice formulas with Δ.

6. Prove that for any nonnegative integer n, $\Delta x^{\underline{n}} = nx^{\underline{n-1}}$. [2 pts]

We can define falling (and rising) factorials with *negative* exponents in a manner similar to the definition of negative powers. Note that to get from $x^{\underline{3}}$ to $x^{\underline{2}}$ we divide by $(x-2)$. Then we would have to divide by $(x-1)$ to drop the exponent to $x^{\underline{1}}$. And we divide again by $x = (x-0)$ to lower the exponent to $x^{\underline{0}}$. It seems that we should now divide by $(x+1)$ to get to $x^{\underline{-1}}$. That is, it looks like we should define

$$\begin{aligned} x^{\underline{-1}} &= \frac{1}{x+1}, \\ x^{\underline{-2}} &= \frac{1}{(x+1)(x+2)}, \\ x^{\underline{-3}} &= \frac{1}{(x+1)(x+2)(x+3)}, \end{aligned}$$

and so forth.

7. If n is a positive integer, provide a recursive definition for $x^{\underline{-n}}$. [1 pt]
8. (a) If n is a positive integer, prove $x^{\underline{-n}} = \dfrac{1}{(x+n)^{\underline{n}}}$. [2 pts]
   (b) Prove that the law of exponents as in problem 4 continues to hold for arbitrary integers m and n. [2 pts]
   (c) Prove that the Δ-rule from problem 6 continues to hold for negative integers. [2 pts]

Let n be a positive integer and $f(x)$ a function. Define an operation S_n by

$$S_n f(x) = f(0) + f(1) + \cdots + f(n-1).$$

Be careful to note that there are exactly n terms in this sum, running from 0 to $n-1$. For example, if $f(x) = 2^x$ then $S_5 f(x) = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 31$.

9. Define $S_n f(x)$ recursively. [2 pts]

10. Compute $S_{1000} x$. [1 pt]

11. Prove that for any positive integer n and any function $f(x)$, that $S_n(\Delta f(x)) = f(n) - f(0)$. [2 pts]

You may assume—it is not difficult to prove—that both Δ and S_n, for any n, are *linear*. That is, given functions $f(x)$ and $g(x)$ and constants a and b, that $\Delta(af(x) + bg(x)) = a\Delta f(x) + b\Delta g(x)$ and $S_n(af(x) + bg(x)) = aS_n f(x) + bS_n g(x)$.

12. Prove that $S_n x^{\underline{k}} = \dfrac{n^{\underline{k+1}} - 0^{\underline{k+1}}}{k+1}$, where k is any integer other than -1. [2 pts]

13. Compute the values of constants a, b, and c so that $x^3 = x^{\underline{3}} + ax^{\underline{2}} + bx^{\underline{1}} + cx^{\underline{0}}$. [2 pts]

14. Use the result of the previous problem to compute $0^3 + 1^3 + 2^3 + \cdots + (n-1)^3$ in terms of falling factorials. [2 pts]

15. Expand the terms of your solution to the previous problem to express $0^3 + 1^3 + \cdots + (n-1)^3$ as a polynomial in n. [1 pt]

It is similarly possible to easily obtain expressions for the sums of squares, fourth powers, or any other (positive integer) powers of x, as long as there is a convenient way to convert back and forth between polynomials and falling factorials. To do that efficiently, you need *Stirling numbers*, which might be a topic for a future ARML Power Contest.

16. Note that $\Delta a^x = a^{(x+1)} - a^x = (a-1)a^x$. Use this to obtain a formula for the sum of a finite geometric series $1 + a + a^2 + a^3 + \cdots + a^{n-1}$. [2 pts]

17. Prove that the falling factorials satisfy the binomial theorem: for nonnegative integers, n, $(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$. [3 pts]

<u>The Solutions</u>

1. We apply the definition of Δ to each function.

   (a) $f(x+1) - f(x) = c - c = 0$.

   (b) $f(x+1) - f(x) = a(x+1) + b - (ax+b) = a$.

   (c) $f(x+1) - f(x) = (x+1)^3 - x^3 = x^3 + 3x^2 + 3x + 1 - x^3 = 3x^2 + 3x + 1$.

   (d) $f(x+1) - f(x) = 2^{x+1} - 2^x = 2 \cdot 2^x - 2^x = 2^x$. The exponential 2^x is *invariant* under the operation of Δ.

2. Again, we simply apply the definitions.

   (a) $x^{\underline{3}} = x(x-1)(x-2) = x^3 - 3x^2 + 2x$.

   (b) $9^{\underline{4}} = 9 \cdot 8 \cdot 7 \cdot 6 = 3024$.

   (c) $\Delta x^{\underline{2}} = \Delta(x^2 - x) = (x+1)^2 - (x+1) - (x^2 - x) = x^2 + 2x + 1 - x - 1 - x^2 + x = 2x$.

3. Recall that recursive definitions must have a base case and a recursive formula that computes each value in terms of previous values. So we define:

$$x^{\underline{n}} = \begin{cases} 1 & n = 0 \\ x^{\underline{n-1}} \cdot (x - n + 1) & n > 0 \end{cases}.$$

4. We use induction. If $n = 0$ then $x^{\underline{m+n}} = x^{\underline{m}} = x^{\underline{m}} \cdot 1 = x^{\underline{m}} x^{\underline{0}} = x^{\underline{m}} x^{\underline{n}}$.

   Then, inductively assuming the formula is correct for $n-1$, we compute $x^{\underline{m+n}} = x^{\underline{m+(n-1)+1}} = x^{\underline{m+(n-1)}}(x - (m + (n-1)))$ by the recursive definition. Then the inductive assumption tells us this equals $x^{\underline{m}}(x-m)^{\underline{n-1}}(x-(m+(n-1)))$. The last term in this product can be rewritten as $((x-m) - n + 1)$, and $(x-m)^{\underline{n-1}}((x-m) - n + 1) = (x-m)^{\underline{n}}$ by the recursive definition. Substituting this into the expansion for $x^{\underline{m+n}}$ then gives the desired result.

5. We use induction again. When $n = 0$ then both sides of the proposed formula evaluate to 1 so the proposition is verified in the base case.

   Now, inductively assume the formula is true for $n - 1$. Then

$$(-x)^{\underline{n}} = (-x)^{\underline{1+n-1}} = (-x)^{\underline{1}}(-x-1)^{\underline{n-1}}$$

   by the result of problem 4. From the base case, we know that the first factor is simply $-x$, while the inductive assumption tells us the second factor is $(-1)^{n-1}(x + 1 + n - 2)^{\underline{n-1}}$. Multiplying these together gives $(-1)^n(x + n - 1)^{\underline{n-1}} \cdot x$. Of course, x can be rewritten as $(x + n - 1) - n + 1$. Substituting this into the previous expression and using the recursive definition of $(x + n - 1)^{\underline{n}}$, we obtain the desired result.

   (There are certainly other ways to arrive at this result. A direct induction, not using the factorial law of exponents will work, though the notation gets even messier than the above. Using the relationship between falling factorials of x and rising factorials of $-x$ is also a good approach—*if* you remembered to properly define rising factorials and prove that relationship as the questions packet noted.)

6. Actually, as stated the proposition is not quite true when $n = 0$. That is because $x^{\underline{-1}}$ has not (yet) been defined, so technically $0 \cdot x^{\underline{-1}}$ is also undefined. (If you wrote this, you will receive full credit for this problem; you will also earn full credit for the following.) On the other hand, whatever $x^{\underline{-1}}$ is, clearly multiplying it by zero will produce zero. Of course, since $x^{\underline{0}} = 1$ is constant, problem 1(a) tells us that $\Delta x^{\underline{0}}$ is zero also, so the proposition is proven if $n = 0$.

   Now if $n > 0$, we can directly compute $\Delta x^{\underline{n}} = (x+1)^{\underline{n}} - x^{\underline{n}} = (x+1)^{\underline{1+n-1}} - x^{\underline{n-1+1}}$. Now apply the factorial law of exponents to the first term and the recursive definition to the second term to obtain $(x+1)^{\underline{1}}(x+1-1)^{\underline{n-1}} - x^{\underline{n-1}}(x-n+1)$. Since $(x+1)^{\underline{1}} = x+1$ and $x+1-1 = x$ we can simplify this expression to $(x+1)x^{\underline{n-1}} - (x-n+1)x^{\underline{n-1}}$ and factoring out the $x^{\underline{n-1}}$ leaves the desired result.

7. We mirror the definition from problem 3:

$$x^{\underline{-n}} = \begin{cases} \dfrac{1}{x+1} & n = 1 \\[2ex] \dfrac{x^{\underline{-(n-1)}}}{x+n} & n > 1 \end{cases}.$$

8. (a) If $n = 1$ then the statement is true by the definition of $x^{\underline{-1}}$. Now assume the statement is true for some positive $n - 1$. Then by definition, $x^{\underline{-n}} = x^{\underline{-(n-1)}}/(x+n)$. Using the inductive assumption this becomes $\dfrac{1}{(x+n-1)^{\underline{n-1}}(x+n)}$. Writing $(x+n) = (x+n)^{\underline{1}}$ and applying the factorial law of exponents then shows this last simplifies to $1/(x+n)^{\underline{n}}$ as required.

   Of course, the statement remains true if n is negative or zero, it just requires a few more steps to prove.

   (b) First, the proposition is already proven if neither m nor n is negative. So we'll show the cases where both are negative or where just one is negative.

   So let $m = -a$ and $n = -b$ both be negative, so that a and b are positive. Then $x^{\underline{m+n}} = x^{\underline{-(a+b)}} = \dfrac{1}{(x+a+b)^{\underline{a+b}}}$ by part (a). As proven in problem 4, we can write this as $\dfrac{1}{(x+a+b)^{\underline{b}}(x+a)^{\underline{a}}}$. Using part (a) again gives us $(x+a)^{\underline{-b}}x^{\underline{-a}} = (x-m)^{\underline{n}}x^{\underline{m}}$.

   Next, let m be positive, n be negative, and let $m = -n+k$ where $k \geq 0$ so that $m+n = k$. Then the left-hand side of the proposed identity simplifies to $x^{\underline{k}}$. The right-hand side is $x^{\underline{-n+k}}(x+n-k)^{\underline{n}}$. Since both $-n$ and k are positive, we can apply the result in problem 4 to split the first term into two parts, yielding $x^{\underline{k}}(x-k)^{\underline{-n}}(x+n-k)^{\underline{n}}$. Applying the previous part of this problem to the middle factor in this expression turns that factor into $\dfrac{1}{(x+n-k)^{\underline{n}}}$ and this cancels the third factor leaving only the first term in the product, $x^{\underline{k}}$, as desired.

   The proof when m is positive (or zero), n neagitve but larger in absolute value than m is nearly identical, by writing $n = -m - k$ where $k \leq 0$. Now the left-hand side of the formula is $x^{\underline{-k}}$. The right-hand side is $x^{\underline{m}}(x-m)^{\underline{-m-k}}$. Then we apply the rule for negative falling exponents from part (a) to the second term, and the result of problem 4 to the resulting denominator. Applying the rule in part (a) and the result of problem 4 to the terms in the denomintor will now give the desired result, just as in part (a).

(c) We proceed in a manner similar to the proof in problem 6. $\Delta x^{\underline{-n}} = (x+1)^{\underline{-n}} - x^{\underline{-n}} = \dfrac{1}{(x+n+1)^{\underline{n}}} - \dfrac{1}{(x+n)^{\underline{n}}} = \dfrac{x+1}{(x+n+1)^{\underline{n}}(x+1)} - \dfrac{x+n+1}{(x+n+1)(x+n)^{\underline{n}}}$. Both denominators simplify to $(x+n+1)^{\underline{n+1}}$ so the whole expression becomes $\dfrac{-n}{(x+n+1)^{\underline{n+1}}} = -nx^{\underline{-n-1}}$.

9. Similar to the other recursive problems we've seen, we need

$$S_n f(x) = \begin{cases} f(0) & n = 1 \\ S_{n-1}f(x) + f(n-1) & n > 1 \end{cases}.$$

10. This is $0 + 1 + 2 + \cdots + 999$. Using the summation formula for an arithmetic series yields $999 \cdot 1000/2 = 499500$.

11. We proceed by induction. Since $\Delta f(x) = f(x+1) - f(x)$, when $n = 1$ we obtain $S_1\Delta f(x) = f(0+1) - f(0) = f(n) - f(0)$.

    Now assume that $S_{n-1}\Delta f(x) = f(n-1) - f(0)$. Then $S_n\Delta f(x) = S_{n-1}\Delta f(x) + \Delta f(n-1) = f(n-1) - f(0) + f(n) - f(n-1) = f(n) - f(0)$.

12. While we could do a full induction, if we are smart we can use the result of problems 6 and 7(c) to note that $x^{\underline{k}} = \frac{1}{k+1}\Delta x^{\underline{k+1}}$ and then our result follows from problem 11 and linearity (to factor the constant $\frac{1}{k+1}$ out of the S_n).

13. One approach to this problem would be to multiply out all the falling factorials and set up a system of equations to find a, b, and c. In this case, though, a simpler approach is to substitute various values for x to obtain the equations. For instance, inserting $x = 0$ makes $0^3 = 0^{\underline{3}} = 0^{\underline{2}} = 0^{\underline{1}} = 0$ so we learn immediately that $c = 0$. Trying $x = 1$ gives $1^3 = 1$, $1^{\underline{3}} = 1^{\underline{2}} = 0$ and $1^{\underline{1}} = 1$ so $b = 1$. Trying $x = 2$ makes $2^3 = 8, 2^{\underline{3}} = 0, 2^{\underline{2}} = 2^{\underline{1}} = 2$ so $a = 3$. Thus, $x^3 = x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$.

14. We are being asked to compute $S_n x^3$. Using the result of problem 13, this becomes $S_n(x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}})$. Now we can use problem 12 and linearity to evaluates this as $\frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}}$. (Note, all the falling factorials evaluated at zero yield zero, so are omitted.)

15. This is just brute force arithmetic on the previous answer: $\frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} = \frac{1}{4}(x(x-1)(x-2)(x-3) + 4x(x-1)(x-2) + 2x(x-1) = \frac{1}{4}(x^4 - 6x^3 + 11x^2 - 6x + 4x^3 - 12x^2 + 8x + 2x^2 - 2x) = \frac{1}{4}(x^4 - 2x^3 + x^2)$, or, if you prefer it in factored form, $\frac{1}{4}x^2(x-1)^2$.

16. We are asked to find $S_n a^x$. From the formula for Δa^x we divide by $a - 1$ to discover that $a^x = \frac{1}{a-1}\Delta a^x$. Applying S_n to this yields $S_n a^x = \frac{1}{a-1}S_n\Delta a^x = \frac{1}{a-1}(a^n - a^0) = \dfrac{a^n - 1}{a-1}$.

17. One last induction! The statement is clearly true when $n = 0$. So assume the statement true for $n-1$. Then for n we have

$$\begin{aligned}
(x+y)^{\underline{n}} &= (x+y)^{\underline{n-1}}(x+y-n+1) \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{\underline{k}}y^{\underline{n-1-k}}(x+y-n+1) \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{\underline{k}}y^{\underline{n-1-k}}((x-k)+(y-n+1+k)) \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}(x^{\underline{k}}(x-k)y^{\underline{n-1-k}} + x^{\underline{k}}y^{\underline{n-1-k}}(y-n+1+k) \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}(x^{\underline{k+1}}y^{\underline{n-1-k}} + x^{\underline{k}}y^{\underline{n-k}} \\
&= \sum_{k=0}^{n}\left(\binom{n-1}{k-1}+\binom{n-1}{k}\right)x^{\underline{k}}y^{\underline{n-k}}
\end{aligned}$$

where, as per the usual convention, $\binom{n-1}{-1} = \binom{n-1}{n} = 0$. Of course, binomial coefficients satisfy the recurrence $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ which gives exactly the sum we are looking for.

Alcuin's Sequence

The Background

Alcuin of York, who lived in the latter half of the eighth century, was a scholar, poet, and teacher. As a teacher, he wrote a collection of story problems called *Propositiones ad acuendos juvenes* ("Problems to Sharpen the Young"). One of these problems can be generalized to lead to a very interesting sequence. Let's investigate.

> A man died and left as an inheritance to his three children 30 ornate flasks, of which 10 were full of oil, another 10 were half-full, while the remaining 10 were empty. Divide the flasks among the children so that each receives an equal number of flasks and an equal amount of oil.

Since this is a story problem and not meant to mimic reality, we will assume that each flask is equally valuable as each other flask, each holds exactly the same amount of oil, "half-full" means *exactly* half, there is no way to transfer oil from one to another, and all the other unreasonable assumptions that story problems assume. Our only goal is to divide the flasks among the children so that both flasks and oil are distributed fairly.

One way of dividing the flasks is for one child to receive five full flasks and five empty flasks, a second child to receive four full, four empty, and two half-full flasks, while the last child receives one full, one empty, and eight half-full flasks. We summarize in a table:

F	H	E
5	0	5
4	2	4
1	8	1

where F indicates the number of full flasks, H the half-full flasks, and E the empty flasks in one share of the inheritance. Since the shares are (with our unreasonable assumptions) exactly equal in value, we will not distinguish the recipients, and focus only on the number of ways to divide the estate into three fair shares.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. There are four other ways to divide the inheritance fairly. Create four tables similar to the above, each showing a different way of dividing the inheritance. [2 pts]

We can generalize the problem. Assume there are n full, n half-full, and n empty flasks. How many different ways are there to divide the inheritance fairly? Call this number $a(n)$. As we have just seen, $a(10) = 5$, the one given and the four others you were asked to find. We can clearly see that $a(0) = 1$—no one receives any flasks! But $a(1) = 0$ because with one flask of each type there is a total of $3/2$ flasks of oil, but there is no way to divide the three flasks so that each person gets $1/2$ flask of oil. As another example, $a(4) = 2$. This is because the divisions can be (two shares of 2 full and 2 empty, plus one share of 4 half-full) and (one share of 2 full and 2 empty, plus two shares of 1 full, 1 empty, and 2 half-full), and there is no other way to fairly divide the twelve flasks fairly.

2. Complete the table of values for $a(n)$ for small n below (make sure to copy your answer onto your solution page—don't just leave it here!). [4 pts]

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$	1	0			2						5		

3. Explain why in any fair division of the flasks and oil, for any n, the number of full flasks and the number of empty flasks in a share must be the same. [2 pts]

From this problem we learn that the number of each kind of flask in a share is determined entirely by the number of full flasks in the share. This is because there are $3n$ total flasks, so each child receives n of them. But if they receive f full flasks, they must also receive f empty ones, and thus $n - 2f$ half-full flasks.

With this, we can specify a fair division by giving an ordered triple, (a, b, c), which tells the number of full flasks each child receives. Since we are not distinguishing among the children, we will list the number of full flasks in non-decreasing order, $a \leq b \leq c$. Thus, the original division of ten flasks of each sort would be notated $(1, 4, 5)$. The two divisions where there are four flasks of each type would be $(1, 1, 2)$ and $(0, 2, 2)$. Notice that since we are listing all the full flasks, we learn that $a + b + c = n$.

4. Prove that for any fair division $a + b \geq c$. [2 pts]

The inequality in the last problem might be called the *weak triangle inequality.* This is because for a triangle with sides a, b, and c, the regular triangle inequality tells us that $a + b > c$, with a strict inequality. So we might want to look at triangles!

Let $[a, b, c]$ represent a triangle whose sides have lengths a, b, and c. We'll only consider triangles different if they are not congruent, so order of the sides won't matter and we might as well arrange the sides from shortest to longest, $a \leq b \leq c$. As we have already noted, the triangle inequality tells us $a + b \geq c$. We will deal only with triangles whose sides have integer lengths. Note that $a + b + c$ is the perimeter of the triangle.

Let $t(n)$ be the number of incongruent integer triangles with perimeter n. Obviously, $t(1) = t(2) = 0$ because the smallest integer triangle is the equilateral triangle with sides of length one. Thus $t(3) = 1$. But $t(4) = 0$ because $[1, 1, 2]$ violates the triangle inequality so does not represent a triangle. Two other values are $t(7) = 2$ and $t(13) = 5$.

5. Find the triplets $[a, b, c]$ for the two triangles of perimeter 7 and for the five triangles of perimeter 13. [3 pts]

6. Complete the table of values for $t(n)$ for small n below (make sure to copy your answer onto your solution page—don't just leave it here!). [4 pts]

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$t(n)$	1	0			2						5		

The tables in problems 2 and 6 should look suspiciously similar. When we see similarities like this in mathematics, we look deeper to see if there are any connections. In this case, in fact, $a(n) = t(n+3)$. We will prove this presently. **Hint:** since you now know these tables should be the same, with a shift in n, you can check each one from the other. But if your answers are different and you aren't sure which one is right, don't just guess and make them both the same. It is better to have one right answer and one wrong then it would be to guess wrong and make them both the same wrong answer!

7. Prove that if a, b, and c are positive integers then $a+b \geq c$ if and only if $(a+1)+(b+1) > (c+1)$. [2 pts]

8. Prove that there is a one-to-one correspondence between solutions to the flask problem with n flasks of each type and integer triangles of perimeter $n + 3$. [3 pts]

The rest of the problems are devoted to developing some very simple and interesting formulas for $t(n)$.

For integers n, define $p_3(n)$ to be the number of ways to write n as a sum of exactly three positive integers, without regard to order. So, for example, $p_3(4) = 1$, as the only way to write 4 as a sum of three positive integers is $2 + 1 + 1$. But $p_3(7) = 4$, with the sums $5 + 1 + 1$, $4 + 2 + 1$, $3 + 3 + 1$, and $3 + 2 + 2$. We'll write such a triple of numbers as $[[a, b, c]]$ with $a \geq b \geq c$. So the triples that add to seven are $[[5, 1, 1]]$, $[[4, 2, 1]]$, $[[3, 3, 1]]$, and $[[3, 2, 2]]$.

Similarly, define $p_2(n)$ to be the number of ways to write n as a sum of exactly two positive integers. We could define $p_1(n)$ similarly, but, rather trivially, $p_1(n)$ is one if n is positive and zero otherwise.

9. Prove that $p_2(n) = \lfloor n/2 \rfloor$ for nonnegative n, where $\lfloor x \rfloor$ is the floor function—the greatest integer less than or equal to x. [2 pts]

10. Prove that if $[a, b, c]$ represents an integer triangle of perimeter $2n$, then the correspondence $[a, b, c] \leftrightarrow [[n - a, n - b, n - c]]$ is a one-to-one correspondence between integer triangles of perimeter $2n$ and sets of three positive integers that add to n. [3 pts]

11. Prove the recurrence formula $p_3(n + 6) = p_3(n) + n + 3$ for all nonnegative n. (**Hint:** consider a triple $[[a, b, c]]$ that adds to $n + 6$. Try subtracting two from each term if possible. Try cases depending on the smallest of the three numbers, c.) [3 pts]

12. Define $\|x\|$ to be the integer closest to x. Prove $p_3(n) = \left\|\dfrac{n^2}{12}\right\|$. [3 pts]

13. Prove that $n^2/12$ is never exactly halfway between two consecutive integers. (Thus, there was no ambiguity in the previous problem.) [1 pt]

14. Prove that if $a + b + c$ is odd, then the correspondence $[a, b, c] \leftrightarrow [a + 1, b + 1, c + 1]$ is a one-to-one correspondence between integer triangles of odd perimeter n and those of even perimeter $n + 3$. [2 pts]

Putting all these pieces together yields both explicit and recursive formulas for $t(n)$. If n is even, problem 10 shows that $t(n) = p_3(n/2)$, for which problem 11 provides a recursive formula and problem 12 shows is equal to $\|n^2/48\|$. On the other hand, if n is odd, then problem 14 yields $t(n) = \|(n + 3)^2/48\|$.

One more nifty fact about Alcuin's sequence, just for fun:

15. Prove that if $[a, b, c]$ is an integer triangle, then there exists unique nonnegative integers α, β, and γ so that $a = 1 + \beta + \gamma$, $b = 1 + \alpha + \beta + \gamma$, and $c = 1 + \alpha + \beta + 2\gamma$. Furthermore, if α, β, and γ are any three nonnegative integers, the $[a, b, c]$ obtained from these formulas is an integer triangle. [3 pts]

16. Prove that $t(n)$ equals the number of ways to write $n - 3$ as a sum of 2's, 3's, and 4's. For example, we know that $t(13) = 5$ and $13 - 3 = 10$ can be written as $4 + 4 + 2$, $4 + 3 + 3$, $4 + 2 + 2 + 2$, $3 + 3 + 2 + 2$, and $2 + 2 + 2 + 2 + 2$. [1 pt]

Note: There have been a number of articles on Alcuin's and related sequences. The material for this contest was inspired by the following two articles, especially the second one:

Raymond A. Beauregard and Vladimir A. Dobrushkin, Finite Sums of Alcuin Numbers, *Mathematics Magazine,* **86** (2013) 280–287.

Donald J. Bindner and Martin Erickson, Alcuin's Sequence, *American Mathematical Monthly,* **119** (2012) 115–121.

The Solutions

1. Here are the other four distributions of full, half-full, and empty flasks:

F	H	E
5	0	5
5	0	5
0	10	0

F	H	E
5	0	5
3	4	3
2	6	2

F	H	E
4	2	4
4	2	4
2	6	2

F	H	E
4	2	4
3	4	3
3	4	3

2. To avoid having listing out the details of the many possibilities, we will use the facts discussed later on that show we really only need to display how many full flasks each person gets, because they will get the same number of empty flasks as full, then enough half-full to make up their fair share of flasks. So, for example, with eleven flasks of each type, the triple $(2, 4, 5)$ will mean that one person gets two full, two empty, and seven half-full flask, while the second person gets four full, four empty, and three half-full, and the last person gets five full, five empty, and one half-full flask. (This is the same notation used later in the problem itself.)

   So the possibilities for the numbers of flasks within the range of the table are:

   3 flasks: (1, 1, 1),
   4 flasks: (1, 1, 2) and (0, 2, 2),
   5 flasks: (1, 2, 2),
   6 flasks: (0, 3, 3), (1, 2, 3), and (2, 2, 2),
   7 flasks: (1, 3, 3), and (2, 2, 3),
   8 flasks: (0, 4, 4), (1, 3, 4), (2, 2, 4), and (2, 3, 3),
   9 flasks: (1, 4, 4), (2, 3, 4), (3, 3, 3),
   10 flasks: five total possibilities, as given by the answer to problem 1,
   11 flasks: (1, 5, 5), (2, 4, 5), (3, 3, 5), and (3, 4, 4),
   12 flasks: (0, 6, 6), (1, 5, 6), (2, 4, 6), (3, 3, 6), (2, 5, 5), (3, 4, 5), and (4, 4, 4).

   Therefore, we end up with

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$	1	0	1	1	2	1	3	2	4	3	5	4	7

3. If there are $3n$ flasks to distribute, each person must receive n flasks. But there are also $3n/2$ total flasks' worth of oil, so each person receives $n/2$ flasks' worth of oil. So the average amount of oil per flask they receive is $1/2$. Thus, each full flask must be balanced by an empty flask to make the average come out to $1/2$. In terms of arithmetic, if the share F full flasks, H half-full, and E empty, we learn that $E + H + F = n$ and $H/2 + F = n/2$, or $H + 2F = n$. Subtracting, $E - F = 0$ so $E = F$.

4. If $c > a + b$ then since $a + b + c = n$ we subtract to learn $2c > n$. But since the child receives as many empty as full flasks, and they receive c full flasks, they receive $2c$ flasks just counting the empty and full ones. In other words, they would receive more than n flasks, which we know is not allowed.

5. The two integer triangles with perimeter 7 are represented by $[1, 3, 3]$ and $[2, 2, 3]$. The five of perimeter 13 are $[1, 6, 6]$, $[2, 5, 6]$, $[3, 5, 5]$, $[3, 4, 6]$, and $[4, 4, 5]$.

6. We can find the possibilities by brute force, as we did for problem 2. In fact, due to the result of problem 7 we know the tables must be the same. Each triple from problem 2 should have 1 added to each entry to become a triangle triple for this problem.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$t(n)$	1	0	1	1	2	1	3	2	4	3	5	4	7

7. If $a+b \geq c$ then $(a+1)+(b+1) \geq (c+2) > (c+1)$. On the other hand, if $(a+1)+(b+1) > (c+1)$ then since these numbers are all integers we must have $(a+1)+(b+1) \geq (c+2)$ and the results follows subtracting 2 from each side of the inequality.

8. The correspondence $(a, b, c) \leftrightarrow [a+1, b+1, c+1]$ is the correspondence desired. A solution to the flask problem is a triple (a, b, c) with $0 \leq a \leq b \leq c$, $a+b \geq c$, and $a+b+c = n$ as shown by problems 3 and 4. Adding one to each term gives a triple $[a+1, b+1, c+1]$ with $1 \leq a+1 \leq b+1 \leq c+1$, satisfying the triangles inequality $(a+1)+(b+1) > (c+1)$ by problem 7, so adding one to each term leads to an integer triangle. Conversely, with an integer triangle, we can subtract one from each side length to obtain a triple that satisfies the flask problem. And, of course, $(a+1)+(b+1)+(c+1) = n+3$.

9. We need to write n as a sum of two positive integers, so let $0 < a \leq b$ with $a+b = n$. Since $a \leq b$ we have $2a \leq a+b = n$, so $2a \leq n$ or $a \leq n/2$. Since a must be a positive integer, there are $\lfloor n/2 \rfloor$ choices for a.

10. $[a, b, c]$ represents an integer triangle of perimeter $2n$ if and only if $1 \leq a \leq b \leq c$, $a+b > c$ and $a+b+c = 2n$. Then $n-a \geq n-b \geq n-c$ and $(n-a)+(n-b)+(n-c) = 3n-(a+b+c) = 3n-n = 2n$. We just need to show that $n-c$ is positive. But since $a+b > c$, we see that $a+b+c > 2c$ so $2n > 2c$ or $n > c$ and $n-c$ is positive.

11. Let $[[a, b, c]]$ be a triple of positive integers adding to $n+6$ with $n \geq 0$ and $a \geq b \geq c$. There are the following possibilities:

    Case 1: $c > 2$. Then $[[a-2, b-2, c-2]]$ is a triple of positive integers adding to n, of which there are $p_3(n)$.

    Case 2: $c = 2$. Then $c-2 = 0$ and $a+b = n+4$, so $(a-2)+(b-2) = n$. By problem 9 there are $\lfloor n/2 \rfloor$ such pairs if $b-2 > 0$, and there is one such if $b-2 = 0$, so altogether we have $\lfloor n/2 \rfloor + 1$ triples in this case.

    Case 3: $c = 1$. This time $a+b = n+5$ so $(a-2)+(b-2) = n+1$. If b is at least two then, as in the previous case, we get $\lfloor (n+1)/2 \rfloor + 1$ such pairs. But if b is one, we know $a = n+4$ and we have the triple $[[n+4, 1, 1]]$. So we get a total of $\lfloor (n+1)/2 \rfloor + 2$ triples when $c = 1$.

    Since c is positive, these three cases cover all possibilities, leading to a total of $p_3(n) + \lfloor n/2 \rfloor + \lfloor (n+1)/2 \rfloor + 3$ such triples. By checking the cases of n even or odd, we find $\lfloor n/2 \rfloor + \lfloor (n+1)/2 \rfloor = n$, obtaining the desired recurrence.

12. We use the recurrence proved in problem 11. First, for $n = 0, 1, 2, 3, 4, 5$ we can quickly calculate that $p_3(n) = 0, 0, 0, 1, 1, 2$ respectively, while $n^2/12$ has values 0, 1/12, 1/3, 3/4, 4/3, 25/12 which are closest to 0, 0, 0, 1, 1, 2 respectively. That is, both $p_3(n)$ and $\|n^2/12\|$ have the same first six starting values. But $\|(n+6)^2/12\| = \|(n^2+12n+36)/12\| = \|n^2/12+n+3\|$ and since $n+3$ is an integer, it won't change how $n^2/12$ is rounded. So $\|(n+6)^2/12\| = \|n^2/12\| + n + 3$ which is the same recurrence satisfied by $p_3(n)$. Since they have the same recurrence and the same starting values, they must be equal for all n.

13. The squares modulo 12 are 0, 1, 4, 9, 4, 1, 0, 1, 4, 9, 4, 1 (from 0^2 to 11^2), none of which is 6.

    Alternately, let q be the quotient and r the remainder on dividing n by 6. That is, let $n = 6q + r$. Then $n^2 = 36q^2 + 12qr + r^2$ and $n^2/12$ simplifies to $3q^2 + qr + r^2/12$. The fractional part of this is $r^2/12$ which has possible values of 0, 1/12, 1/3, 3/4, 4/3, and 25/12 for the possible values of r. None of these are half an integer.

14. Obviously the correspondence takes triples that add to n to triples that add to $n+3$ and is a one-to-one correspondence. All that needs to be shown is that $[a, b, c]$ represents a triangle if and only if $[a+1, b+1, c+1]$ does also. But $[a, b, c]$ is a triangle if and only if $0 < a$ and $a + b > c$. Then certainly $0 < a + 1$ and $(a+1) + (b+1) > c + 2 > c + 1$. On the other hand, if $a + b + c$ is odd, then $a + b$ and c have opposite parities. Then $(a+1) + (b+1)$ and $(c+1)$ have the same parity. So if $(a+1)+(b+1) > (c+1)$, the left side is at least *two* more than the right. Thus $a + b \geq (c+1)$ and so $a + b > c$. Finally, note that there are no even perimeter integer triangles with shortest side one, so the possibility that $a + 1 = 1$, leading to the problematic $a = 0$, is ruled out.

15. If $[a, b, c]$ is an integer triangle, we have $1 \leq a \leq b \leq c$ and $a + b > c$. If we let $\gamma = c - b$, $\alpha = b - a$, and $\beta = a + b - c - 1$ then α, β, and γ are all non-negative integers and satisfy the relationships with a, b, and c as given in the problem. They are unique because they are the only solution to the system of three linear equations in the three unknowns α, β, and γ given in the problem.

    Conversely, given nonnegative α, β, and γ, setting a, b, and c to the combinations given in the problem assures us that $b - a = \alpha \geq 0$, $c - b = \gamma \geq 0$, and $a \geq 1$ so $1 \leq a \leq b \leq c$. Furthermore, $a + b - c = \beta + 1 > 0$ so the numbers satisfy the triangle inequality and form an integer triangle.

16. Since each integer triangle of perimeter n has a unique α, β, and γ as given in the previous problem, we find that $a + b + c = 3 + 2\alpha + 3\beta + 4\gamma$ so $n - 3$ can be written as a sum of α 2's, β 3's, and γ 4's. Since the triangles and the choices of α, β, and γ are in one-to-one correspondence, so are the triangles and the ways of writing $n - 3$ as a sum of 2's, 3's, and 4's.

Rolling the Dice

The Background

This Power Problem is about calculating probabilities of rolls of dice. The dice will not always be your standard cube with one through six spots on the sides, though. In fact, our dice will be allowed to have any positive integer number of sides, and each of those sides could show any positive integer. To indicate a die with specific numbers on it sides, we will use the notation $[a\,b\,c \ldots d\,e\,f]$ to indicate that the sides have a, b, c, etc. dots. So a standard die would be notated as $[1\,2\,3\,4\,5\,6]$ while $[3\,8\,14]$ indicates a three sided die whose sides show the numbers 3, 8, and 14. For brevity, $[[n]]$ with a double bracket means a standard n-sided die, so $[[n]] = [1\,2\,3\,\ldots\,n]$. Dice may have the same number on multiple sides. All of our dice are fair, meaning that if the die has n sides, then each side will turn up with probability $1/n$.

A roll will consist of taking several dice (possibly not with the same numbers of sides) and rolling them all, and taking the sum of the numbers showing. We will say that one roll beats another if the sum of the numbers showing in that roll beats the sum showing in the other roll. If the sums are the same, we will call it a tie.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. Alice and Bob each roll a standard five-sided die. Compute the probability that Alice's roll will beat Bob's. [1 pt]

2. Alice rolls $[1\,3\,5\,7]$ and Bob rolls $[2\,4\,6\,8]$. Compute the probability that Alice wins. [1 pt]

3. Alice rolls two copies of $[2\,4\,6]$ while Bob rolls one $[1\,3\,5]$ and one $[3\,5\,7]$. Compute the probability that Alice beats Bob, the probability that Bob beats Alice, and the probability that they tie. [2 pts]

4. Alice rolls $[[m]]$ and Bob rolls $[[n]]$ where $m \geq n$. Compute, showing/explaining your work, the probability that they tie. [2 pts]

5. Let $p_1, p_2, \ldots, p_n$ be a sequence of positive rational numbers whose sum is one. Show that you can create a die whose probability of rolling a 1 is p_1, of rolling a 2 is p_2, and so on. [2 pts]

6. Construct two six-sided dice other than two copies of the standard $[[6]]$ whose probability of showing a total of each of $2, 3, 4, \ldots, 12$ is the same as the probability that two standard dice show that same total. [2 pts]

We will call two dice *interlaced* if each has at least one face that can both beat some face on the other and lose to some face on the other. For example, [1 2 5 6] and [3 3 4] are not interlaced, because each face on the first die either beats every face on the second or loses to every face on the second. On the other hand, [1 3 5] and [2 4] are interlaced because the 3 face on the first die can both beat and lose to the second die, while either face of the second die can both beat and lose to the first die.

If r is a rational number between zero and one inclusive, then it is easy to create two dice where the first has probability r of beating the second. Simply express r as a fraction p/q of nonnegative integers, and let the dice be [2] and [1 1 ... 1 3 3 ... 3] with p copies of 1 and q total sides. But notice that these dice are not interlaced. Of course, each of two interlaced dice has a positive probability of beating the other.

7. Given rational number r between zero and one *exclusive*, find two interlaced dice, A and B, where the probability of A beating B is r. Your dice may have any number of sides, and you are completely free to choose the numbers on those sides. [2 pts]

8. Given nonnegative rational numbers r and t, with $r + t \leq 1$, find two (not necessarily interlaced) dice A and B where the probability of A beating B is r and the probability of A and B tying is t. [1 pt]

9. Find two interlaced dice A and B where the probability that A and B tie is $4/5$. [1 pt]

If A and B are dice, we will say $A > B$ if the probability that A beats B is larger than the probability that B beats A. We will extend this notation to collections of dice, and say $A\,B\,\ldots > C\,D\,\ldots$ if the first collection of dice is more likely to beat the other than to lose to it.

The $>$ relation on sets of dice is not what you expect!

10. Let $A = [4\,4\,4\,4\,4\,4]$, $B = [2\,2\,2\,2\,6\,6]$ and $C = [1\,1\,5\,5\,5\,5]$. Show that $A > B$, $B > C$ and $C > A$. [3 pts]

So the $>$ relation on dice is not transitive. Normally, we would expect that if team A is likely to beat team B, and team B is likely to beat team C, then A should be the strongest team and should beat C. But not this time! These dice are more like the game of Rock-Paper-Scissors, where Rock beats Scissors and Scissors beats Paper, but Paper beats Rock.

11. Can we make longer cycles? Yes we can! For the four dice $A = [4\,4\,4\,4\,4\,4]$, $B = [3\,3\,3\,3\,7\,7]$, $C = [1\,1\,5\,5\,5\,5]$, and $D = [2\,2\,2\,6\,6\,6]$, determine the probability that each die beats each other die. [3 pts]

You may have noticed in the previous problem that while the dice are not transitive, they are not *completely* intransitive, in the sense that there are trios of the dice that are not intransitive. That is, $A \geq B$, $B \geq C$ and $A \geq C$. But the previous problem is, in fact, the closest you can come to complete intransitivity.

12. Consider a collection of four or more dice in which, for each two dice A and B, either $A > B$ or $B > A$ so that there are no pairs of dice that have equal probabilities of beating each other. Prove that there must be at least one transitive triangle. That is, among four or more dice, each of which is either $>$ or $<$ each other die, then there are three dice X, Y, and Z that satisfy $X > Y$, $Y > Z$, and $X > Z$. [3 pts]

Another strange property of $>$ among dice is that sometimes one die is more likely to beat another, but rolling two copies of each changes the dominant die—it is like rock beating scissors, but two scissors beats two rocks!

13. Let $A = [1\,1\,6\,6\,6\,6]$ and $B = [3\,3\,3\,3\,8\,8]$.
    (a) Determine whether $A > B$ or $B > A$, and with what probability. [1 pt]
    (b) Determine whether $AA > BB$ or $BB > AA$, and with what probability. [2 pts]
    (c) Determine whether $AAA > BBB$ or $BBB > AAA$, and with what probability. [2 pts]
    (d) Explain why, as the number of dice rolled grows, the probability that a large number of B dice beats the same number of A dice approaches one. [2 pts]

14. In all the previous examples of three nontransitive dice, the three win probabilities of one die over another were not equal. This can be remedied. Find three four-sided dice, A, B, and C, such that A beats B with probability $9/16$, B beats C with probability $9/16$ and C beats A with probability $9/16$. [2 pts]

15. Prove that there are no sets of nontransitive dice each of which has two sides. [2 pts]

16. Prove or find a counterexample: a cycle of n nontransitive dice must have at least one die with at least n sides. For instance, if you have a cycle of five dice $A > B > C > D > E > A$ then at least one of the dice must have at least five sides. [2 pts]

The dice $A = [3\,3\,3\,3\,3\,6]$, $B = [2\,2\,2\,5\,5\,5]$, and $C = [1\,4\,4\,4\,4\,4]$ can be easily checked to be nontransitive, with $A > B$, $B > C$, and $C > A$. And the nontransitivity reverses when you use two dice: $A\,A < B\,B$, $B\,B < C\,C$ and $C\,C < A\,A$. You could use this to play a game where you win most of the time. Let a friend choose one of the dice, then choose the one that beats it. Of course, after playing for a while your friend will catch on. So offer to instead choose first, and use two dice at a time. If you choose a pair of A's, your friend will probably choose to use a pair of C's, as C beat A more often when there was only one die per player. So you will probably still win!

Instead of moving to more dice, how about moving to more players? Using the dice below, you can let two friends each choose a die (different from each other), and then you will always be able to choose another die that will probably beat each of them!

$A = [2\,14\,17]$ $B = [7\,10\,16]$ $C = [5\,13\,15]$ $D = [3\,9\,21]$
$E = [1\,12\,20]$ $F = [6\,8\,19]$ $G = [4\,11\,18]$

17. For each of the 21 pairs of the above dice, determine which one of the remaining dice is $>$ each of them. For example, If your friends choose A and B, you should choose G, because $G > A$ and $G > B$. There—now you only have 20 pairs left to figure out! [4 pts]

The Solutions

1. We can count possibilities: If Bob rolls a 1 then Alice can roll a 2, 3, 4, or 5 and win. If Bob rolls a 2 then Alice has three winning rolls; she has two winning rolls if Bob rolls a 3, and only one if Bob rolls a 4. She can't win if Bob rolls 5. So altogether she has ten winning rolls, while there are $5 \times 5 = 25$ different combinations of rolls the two of them can have. So Alice's win probability is $10/25 = 2/5 = 0.4$.

   Alternately, if both roll the same number ($1/5$ probability) they tie. Otherwise, Alice wins half of the time and Bob the other half by symmetry. So Alice's win probability is $\frac{1-1/5}{2} = 2/5$.

2. We count possibilities for Alice winning as before. If Bob rolls a 2 then Alice wins with three different rolls. A 4 from Bob allows two different winning rolls from Alice, while a 6 only gives Alice one winning roll. If Bob rolls 8 Alice cannot win. There are $4 \times 4 = 16$ total roll combinations, so Alice wins $6/16 = 3/8 = 0.375$ of the time.

3. When Alice rolls her two dice, she can get a total of four in one way, six in two ways, eight in three ways, ten in two ways, and twelve one way. It is exactly the same as if she rolled a single nine-sided die with faces [4 6 6 8 8 8 10 10 12]. Looking at Bob's possible rolls, his possible combinations come out the same as rolling *exactly* the same nine-sided die! This symmetry means that Bob has exactly the same probability of beating Alice as she has of beating him. Thus, we just need to determine the probability of a tie, and the win probabilities can be determined as in the second solution to problem 1.

   There is one way they could tie with a total of four, $2 \times 2 = 4$ ways to tie with a total of six, $3 \cdot 3 = 9$ ways to tie with a total of eight, and four and one ways to tie with totals of ten and twelve, respectively. Thus, there are 19 possibilities of a tie. Since they are each rolling a nine-sided die, there are $9 \times 9 = 81$ possible combinations. So the probability of a tie is $19/81$.

   Then, the probability of a win by either is $\frac{1-19/81}{2} = 31/81$.

4. Since $m \geq n$, it is easiest to think of the probability of a tie as the probability that Alice matches Bob's roll. Since Alice gets each of her possible results with probability $1/m$ this must be the probability that her roll matches Bob's.

5. Since the p_i are all positive rational numbers, each can be expressed as a_i/b_i for positive integers a_i and b_i. We can certainly find a common multiple M of all of the b_i. Then we express a_i/b_i in the equivalent form a'_i/M. Since the p_i sum to one, the a'_i must sum to M.

   Now we make the die [1 1 ... 1 2 2 ...] where there are a'_1 copies of 1, a'_2 copies of 2, and so on. This die has $a'_1 + a'_2 + \cdots = M$ sides, so its probability of showing 1 is $a'_1/M = p_1$, and so forth, so we have succeeded in creating the desired die.

6. Since there must be exactly one way to roll a total of two, there must be a 1 on each die. Now we need two ways to roll a total of three. Instead of putting a 2 on each die, why don't we try putting *two* 2's on one die and none on the other?

   Then we need to have three ways to roll a total of four. We could try putting three 3's on one die, or two on one and one on the other. Trying the various possibilities, and continuing to try various combinations of numbers on the dice, we quickly find some ways of continuing fail to allow us to reach our goal. In fact, there is only one pair of dice that work, called *Sicherman dice* and they are the dice [1 2 2 3 3 4] and [1 3 4 5 6 8].

7. We will try to modify the simple solution for non-interlaced dice. Start by expressing $r = p/q$ as a fraction of positive integers. To give ourselves some room to maneuver, we will start with the dice [4 4 ... 4] and [1 1 ... 1 7 7 ... 7] where we will feel free to use as many or as few copes of 4 as we need, and we have p copies of 1 and $q - p$ copies of 7.

   Let's start interlacing the dice by moving one of the 4's up to a 6, and one of the 7's down to a 5. This should slightly increase the probability that the first die beats the second. But can we cancel this increase by changing another of the 4's to a 2 and one of the 1's to a 3? Let's try this in the simplest case—using only two copies of 4 in the first die. So after the change we have the dice [2 6] and [1 1 ... 1 3 5 7 7 ... 7] where there are now $p-1$ copies of 1 and $q-p-1$ copies of 7. It should now be obvious that if the first die rolls a 2, it beats $p-1$ rolls on the second die, while if it rolls a 6 it will beat $p+1$ rolls on the second die. Thus, altogether, the first die wins on $p - 1 + p + 1 = 2p$ rolls. Since there are $2 \times q = 2q$ possible combinations of rolls, the probability that the first die wins is $2p/2q = p/q$ as desired.

   There are obviously many other solutions to this problem. For example, since there are now no 4's on either of the dice, the numbers 3 and 5 on the second die are equivalent and could both be 3's, 4's, or 5's. Different sets of numbers could be used on the faces. All correct answers will be accepted for full credit.

8. If we start with non-interlaced dice, the idea most similar to the original plan in the text prior to question 7 would be to express r as p/q and t as s/q where p, s, and q are nonnegative integers. We know that we can find a common denominator, so we are not making special assumptions when we use the same denominator q for both r and t. Then we could take the first die to be [2], and the second die to be 1 1 ... 1 2 2 ... 2 3 3 ... 3] with p copies of 1, s copies of 2, and $q - p - s$ copies of 3. This may be quickly checked to give the desired win and tie probabilities.

9. Interlaced dice with ties are much harder to produce. Especially if the tie probability is large, most of the faces of both dice must be tied with each other. For example, in the present case, at least eighty percent of the sides on one die must tie with a side on the other. In fact, going with only eighty percent doesn't even come close. If we used $A =$ [2 3 3 3 3 3 3 3 3 4] (ten sides, with eight 3's) and $B =$ [1 3 3 3 3 3 3 3 3 5] (ten sides, with eight 3's) we would only obtain a tie $8/10 \times 8/10 = 64\%$ of the time.

   In fact, we need to make the *product* of the probabilities that each die rolls the tie number to be 4/5, while still interlacing them. Two convenient numbers that multiply to 4/5 are $8/9 \times 9/10$. So we will make 8/9 of the numbers on one die equal to 9/10 of the numbers on the other. One example is [1 3 3 ... 3 5] with 18 copies of 3, and [2 3 3 ... 3 4] with 16 copies of 3. It is now simple to check that there are 18×16 possibilities of getting a tie of 20×18 possible roll combinations, yielding a 4/5 probability of rolling a tie.

10. Since die A always rolls a 4, it is simple to see that it beats die B 2/3 of the time and loses to die C equally often, so $A > B$ and $C > A$. Die B will beat C if B rolls 2 and C rolls 1,

which happens $2/3 \times 1/3 = 2/9$ of the time, or anytime die B rolls a 6, which happens $1/3$ of the time. So die B wins $2/9 + 1/3 = 5/9$ of the time, and thus $B > C$.

11. $A > B$ since A beats B $2/3$ of the time. $C > A$ because C beats A $2/3$ of the time. A and D each win exactly half of the time compared to each other. We have $B > C$ since B beats C $5/9$ of the time. B beats D when B rolls 3 and D rolls 2 ($2/3 \times 1/2 = 1/3$ of the time) or any time B rolls a 7 (another $1/3$ of the time) for a total win probability of $2/3$. Finally, $D > C$ because D can roll a 2 and C a 1 ($1/2 \times 1/3 = 1/6$ of the time) or D can roll a 6 ($1/2$ of the time), so D's win probability over C is $1/6 + 1/2 = 2/3$. So here we have $A > B > D > C > A$, $A = D$, and $B > C$.

12. Among any set of four dice, there are $\binom{4}{2} = 6$ pairs of dice. Since in any pair one beats the other, there must be six cases of one die winning out over another. So by the pigeonhole principle, at least one of the dice beats at least two of the others. Call that die X. Then among those dice it beats, pick any pair Y and Z where $Y > Z$. Now we have our trio X, Y, and Z which are transitively related.

13. (a) The only way for A to beat B is if A rolls a 6 and B a 3. This happens $2/3 \times 2/3 = 4/9$ of the time. So $B > A$ with probability $5/9$.

    (b) Two copies of A roll a total of 2 with probability $1/9$, a 7 with probability $4/9$ and 12 with probability $4/9$. Two copies of B roll a total of 6 with probability $4/9$, 11 with probability $4/9$, and 16 with probability $1/9$. So $A\,A$ beats $B\,B$ when the totals are 7 versus 6, 12 versus 6, or 12 versus 11, each of which happens $16/81$ of the time. Thus $A\,A > B\,B$ with a win probability of $48/81 = 16/27$.

    (c) Finally, we compute that $A\,A\,A$ can total 3 ($1/27$ of the time), 8 ($6/27$), 13 ($12/27$) or 18 ($8/27$), while $B\,B\,B$ can total 9, 14, 19, or 24, with probabilities $8/27, 12/27, 6/27$, and $1/27$ respectively. Going through all the possibilities and adding the probabilities, we find $B\,B\,B > A\,A\,A$ with win probability $473/729$.

    (d) When we roll a large number of dice, the average of all the dice rolled tends to get closer and closer to the average of the numbers that the die could show. The *Law of Large Numbers* specifies the exact details of this, but we needn't get that technical here. Suffice it to say that the A averages about $26/6$ per die thrown, while B averages $28/6$ per die. Since we're throwing the same number of dice of each variety, the average—and hence the total—of the B's should be higher, and B should win, and the more dice thrown, the more likely this is to happen.

If this seems difficult to believe, consider the case where one die has 100 sides, all having a 2 on them, and a second die has 99 sides with a 1, and one side with 1 million. The first die is 99% likely to beat the second. But throwing more and more dice, it is more and more likely that the 1 million will show up, and then the die with all 2's will be way behind and stay there. The present case isn't quite as extreme, but it is exactly the same principle.

14. The dice $[1\,4\,7\,7], [2\,6\,6\,6]$, and $[3\,5\,5\,8]$ are one solution. They were discovered by Daniel Tiggerman.

15. Let A, B, and C be two-sided dice with $A > B > C > A$. Let the sides of A be a_1 and a_2 with $a_1 \geq a_2$ and similarly for B and C.

    First, we must have $a_1 \geq b_1$, for if $b_1 > a_1$ then B beats A at least half of the time, and we will not have $A > B$. But, similarly, $b_1 \geq c_1$ and $c_1 \geq a_1$. So $a_1 = b_1 = c_1$. But now for $A > B$, we must have $a_2 > b_2$. But, similarly, $b_2 > c_2$ and $c_2 > a_2$. Since we are comparing integers, $>$ really is transitive in this case, so these last three inequalities imply that $a_2 > a_2$ which is impossible. This clearly extends to cycles of as many dice as needed.

16. While the previous problem shows that a cycle of three or more nontransitive dice must have at least one die with at least three sides, the statement is not true in general. In fact, $A = [1\,5\,5], B = [4\,4\,4], C = [3\,3\,3], D = [2\,2\,6]$ has $A > B > C > D > A$ (with win probabilities $2/3, 1, 2/3, 5/9$ respectively). So we have a nontransitive cycle of four dice none of which have more than three sides. It should be clear that we can make a nontransitive cycle of any length by inserting more constant-sided dice between B and C.

    If you read ahead, the dice in problem 17 are a cycle of seven nontransitive dice, all of which have three sides, so those dice also provide a counterexample!

17. The table below lists the one die that beats both the die at the top of its column and the beginning of its row.

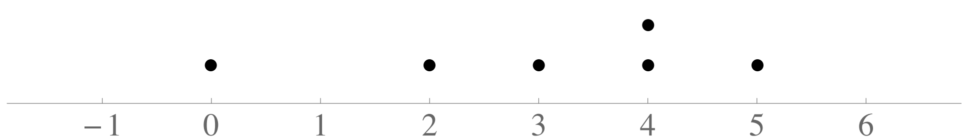
	A	B	C	D	E	F	G
A		G	F	G	D	D	F
B	G		A	G	A	E	E
C	F	A		B	A	B	F
D	G	G	B		C	B	C
E	D	A	A	C		D	C
F	D	E	B	B	D		E
G	F	E	F	C	C	E	

Chip-Firing Games

The Background

Consider an ordinary number line (only the integers will be of concern) where a number of counters or "chips" have been placed on some of the points. On the line below, two chips have been placed on the 1-point and four chips on the 4-point.

This configuration will be described by saying that the "1-pile has two chips, and the 4-pile has four chips." The *chip-firing game* proceeds by sending chips from one point to another according to the following simple rule: if a pile has two or more chips in it, that pile may "fire" one chip to each of its neighbors. In the example above, each of the two piles may be fired, and the result of firing both is the arrangement below.



At this point, the 4-pile could be fired again, leaving it empty but putting a second chip on the 3-pile and the 5-pile, and the game could continue from there. If all the piles in a configuration have one chip (or no chips, allowing empty piles) the configuration will be called *stable*. The game will end if it reaches a stable configuration, because there is nothing else to do. All starting configurations eventually reach a stable configuration. It turns out that the ending configuration doesn't depend on the order in which the piles are fired. That is, if two or more piles can be fired, they can be fired in any order, including firing other piles that weren't fireable but became so during the process, and the final stable configuration of chips will always be the same (we will prove this later). For example, in the original configuration, the game could have begun by firing the 4-pile twice, leaving two chips in each of the 1-, 3-, and 5-piles. Then the 3-pile could be fired, even before the 1-pile which was originally fireable, even though the 3-pile wasn't. All orders of firing lead to the same final stable configuration.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. Determine the final configuration resulting from the example, and the total number of firings needed to obtain it. (By total it is meant to count each and every firing, counting multiple times if a pile is fired more than once.) [3 pts]

2. Let k be a positive integer, and place one chip in each pile from the 1-pile up to the k-pile. Then place a second chip in the 1-pile. Determine the final stable configuration resulting from this starting configuration, and the total number of firings that occur. (You do not need to prove your result, just describe it. The necessary induction proof for this problem will be considered "obvious.") [3 pts]

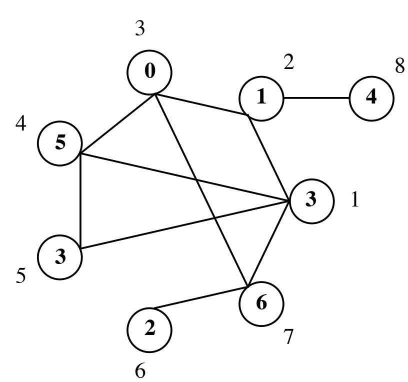
3. For n from 1 to 6, determine the stable configuration resulting from starting with n chips in the 0-pile and no other chips in any other pile. [3 pts]

4. Determine the number of firings that occur in each case of the previous problem. [2 pts]

5. State and prove a conjecture describing the stable configuration resulting from starting with an arbitrary number, n, of chips in the 0-pile and no other chips. [4 pts]

6. State and prove a formula (explicit or recursive) involving n that will compute the number of firings that occur in the previous problem. [2 pts]

Instead of just the integers on a number line, the chip-firing game can be played on a more general game board. A game board consists of a number of piles, $P_1, P_2, \ldots P_n$, and a list of connections. A connection tells when two piles should be considered "next to" each other. That is, if P_3P_7 is one of the connections, then if pile P_3 fires, one of its chips will go to pile P_7, and vice versa. We might draw a game board as seen below. Piles are represented as circles, and the connections as lines or curves from one circle to another. The boldfaced numbers inside the circles will represent the number of chips in that pile, while the number outside the pile simply represents the pile number. Piles P_1 and P_5 each have three chips in them in the diagram, while pile P_3 has no chips at present.



The game board will be required to be connected in the sense that there is a sequence of connections from any one pile to any other. That is, it must be possible for a chip starting in any pile to end up in any other pile if it is fired enough times from one pile to a connected pile. The game also uses only a finite number of chips.

The number of connections to any given pile is called its *valence*. In the example, P_1 has valence 4 while P_8 has valence 1. The firing rule in the game played on a general board is that any pile which currently has at least as many chips as its valence may fire one chip to each of its neighbors. Thus, P_1, P_2, and P_3 in the diagram may not fire, while all the rest of the piles can. If P_5 fires, the result is that P_1 will now have four chips (and now be able to fire), P_4 will have six, and P_5 will drop to one, while no other pile is changed.

7. Draw the diagram that would result from the example if every pile that can fire initially does so, while no pile that could not fire initially (such as P_1) does so, even if it becomes able to. It may be assumed that firing order does not matter. [2 pts]

8. Prove that, from any configuration of chips on any game board, if it is possible to fire both piles P_x and P_y then the result of firing P_x first and then firing P_y is the same as firing P_y first then P_x. (This, along with some induction, allows you to rearrange any firing order into any other that fires the same piles, justifying the assumption in the previous problem.) [2 pts]

For convenience, let P be the total number of piles, C the total number of connections, and N the total number of chips in the game. In the example game board, $P = 8, C = 10$, and $N = 24$.

9. (a) Show that if the total number of chips on the game board is more than $2C - P$ than the game will never reach a stable configuration. That is, show that there must always be some pile that contains more chips than its valence. [2 pts]

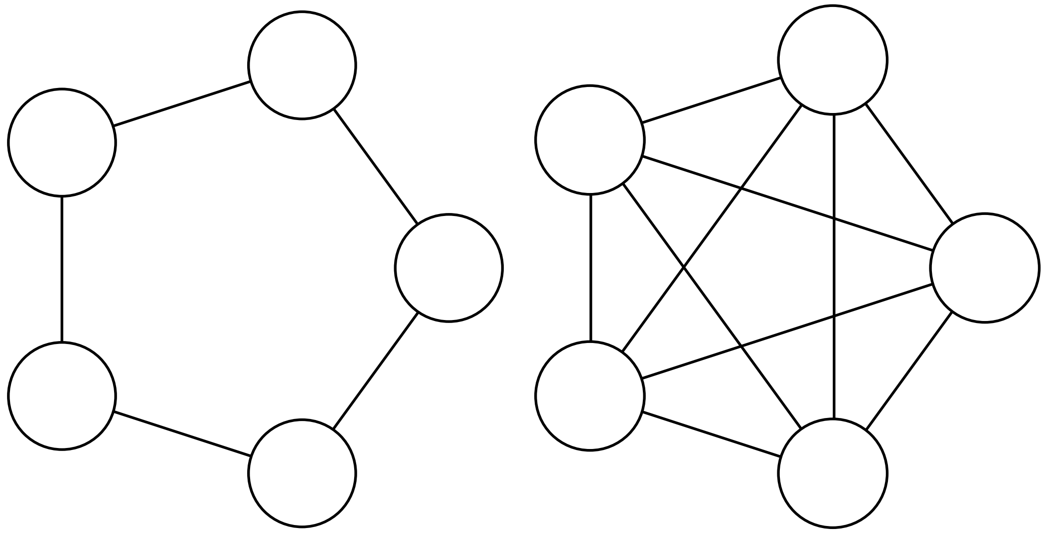
   (b) Show that this bound is tight. That is, show that for any game board there is an initial configuration with exactly $2C - P$ total chips for which the game is in a stable configuration. [2 pts]

10. Prove that if a game never reaches a stable configuration then *every* pile must fire infinitely often. [3 pts]

11. Prove that if a game does reach a stable configuration then there is at least one pile that *never* fires. [3 pts]

Problem 9 showed that if there are more than $2C - P$ chips on the board the game must go on forever without reaching a stable configuration. There is a similar theorem, stating that if there are fewer than C chips on the board then the game will always reach a stable configuration, and this bound is tight in that if there are at least C chips in the game then there are some configurations that never become stable. This will not be proved (but see the article referenced at the end of this contest packet).

12. The game board below left has $C = 5$ connections. Find, with justification, an initial configuration of five chips which never leads to a stable configuration. [2 pts]



13. The game board above right has $C = 10$ connections. Find, with justification, an initial configuration of ten chips which never leads to a stable configuration. [2 pts]

A game board where every pile is connected to every other pile is called *complete*. The board in problem 13 is a complete board with five piles.

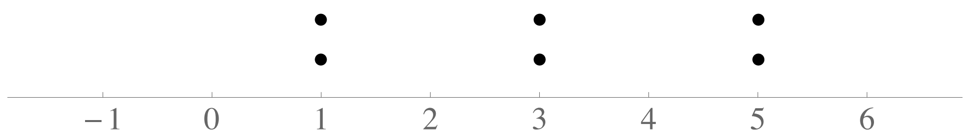
14. Consider the chip-firing game played on a complete board with n piles. Consider the initial configuration where one pile has no chips, a second pile has one chip, a third pile has two chips, and so forth until the last pile is given $n-1$ chips. (Since every pile is connected to every other, it is irrelevant which pile is which.) Prove that this is a "Garden of Eden" configuration. That is, the only way to arrive at this distribution of chips by firing a pile from another configuration is for that other configuration to also have one pile of zero chips, one pile of one chip, …, one pile of $n-1$ chips. [2 pts]

15. Prove that the configuration in problem 14 is the *only* configuration of $0+1+\cdots+(n-1) = \frac{n(n-1)}{2} = C$ chips on a complete board that never becomes stable (up to rearrangement of the piles, which, as pointed out in the previous problem, is irrelevant by symmetry). [3 pts]

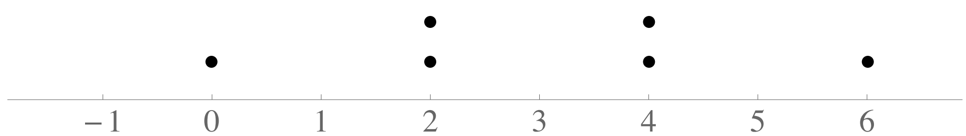
Note: Chip-firing games are a relatively new mathematical development, with investigations starting in the mid-1980's. Physicists study the *sandpile model*, which is essentially the same concept, as a way to learn about "self-organized criticality," which is the ability of many physical systems to find their own stable states. There are many nice references online to get a deeper look at this topic. A good print article is: A. Björner, L. Lovász and P. W. Shor, *Chip-firing game on graphs*, Europ. J. Combin., 12 (1991) 283–291.

The Solutions

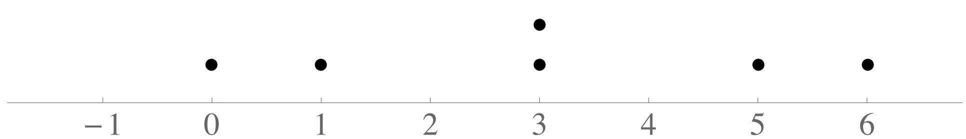
1. Perhaps the easiest way to work this out is to proceed as follows. First, fire the 4-pile twice, leaving two chips in each of the 1-, 3-, and 5-piles.



Now, fire each of these piles, leaving the arrangement below (total firings: 5).

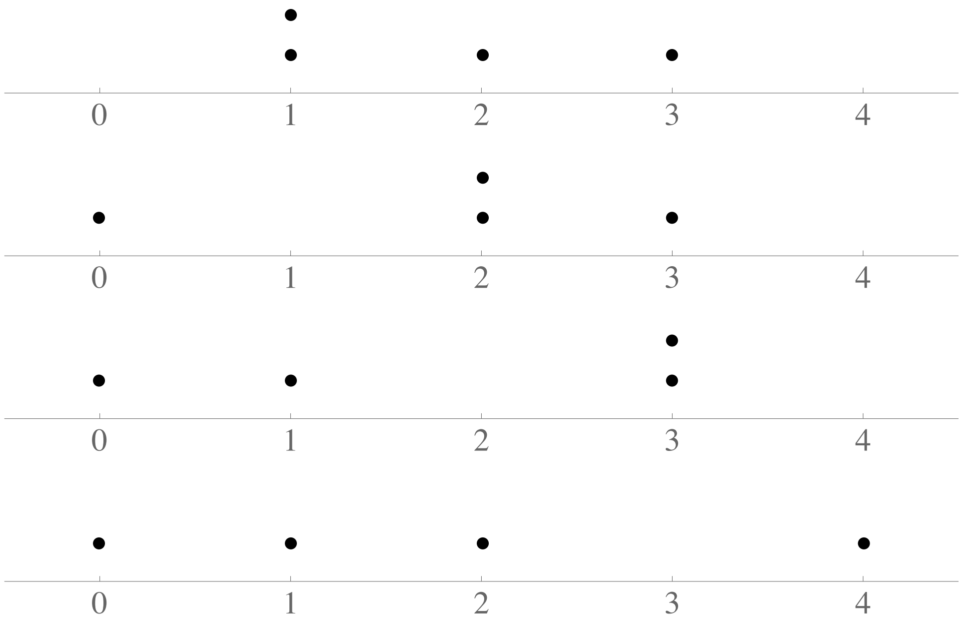


Now fire the 2-pile and the 4-pile (total firings: 7) to obtain the configuration below.



Finally, fire the 3-pile. Thus, after $\boxed{8}$ firings a configuration is reached in which each pile has at most one chip and no more firings are possible. The final configuration is shown below.

2. The only pile that can be fired to start is the 1-pile, so fire it. This puts a chip on the 0-pile, and an additional chip on the 2-pile. The game is complete if $k = 1$, but if $k > 1$ there are now two chips on the 2-pile. So fire it, putting a chip on the 1-pile and an additional chip on the 3-pile. And so on. After piles 1 through a, with $a < k$, have been fired, there will be single chips on piles 0 through $a - 1$, no chips on a, two chips on $a + 1$, and one chip on piles $a + 2$ through k. Eventually there is one chip on piles 0 through $k - 2$ and two chips on pile k, with nothing beyond that. Finally firing the k-pile puts a single chip on piles 0 through $k - 1$ and on pile $k + 1$ with no other non-empty piles. The sequence is shown below for $k = 3$.



Note that each pile from 1 to k has been fired exactly once, so the procedure requires $\boxed{k}$ firings. Also note that the pattern described works when $k = 1$ as well, for then the starting configuration is just two chips in the 1-pile, which is fired to obtain the stable configuration with one chip in each of the 0-pile and the 2-pile.

3. If there is just one chip, then the game is already in a stable configuration so for $n = 1$ the final configuration is one chip in the 0-pile.

   For $n = 2$, fire the 0-pile, leaving one chip each on the 1-pile and the -1-pile. This configuration is stable.

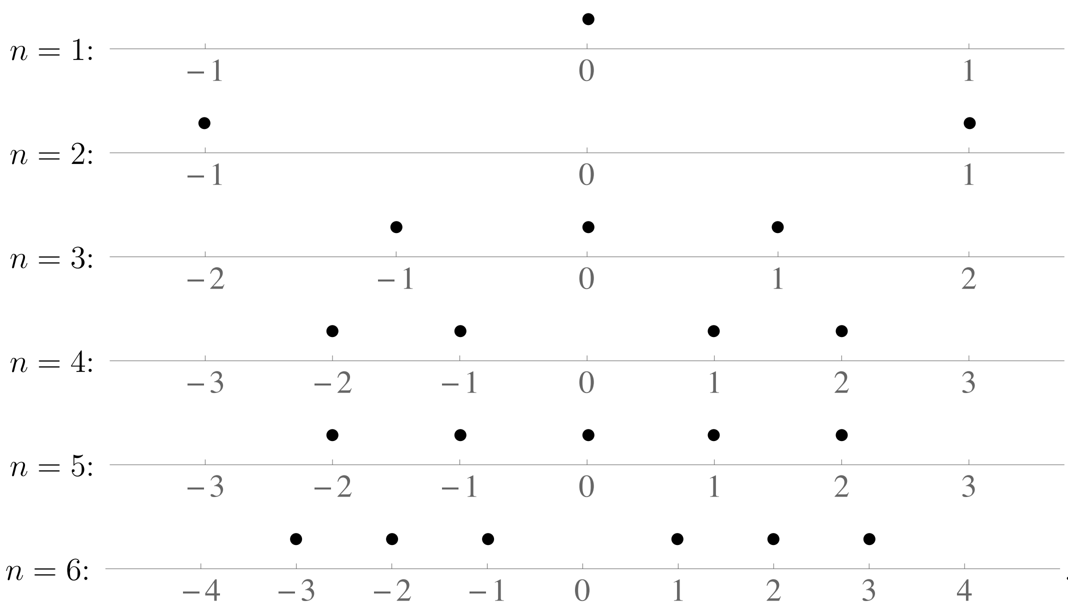
   For $n = 3$, fire the 0-pile, leaving one chip each in the 1-, 0-, and -1-piles, which is a stable configuration.

   For $n = 4$, start by firing the 0-pile twice, putting two chips each in the 1-pile and the -1-pile. Firing these puts one chip in the 2-pile and one in the -2-pile, and two chips in the 0-pile. The 0-pile is now fired resulting in a stable configuration with one chip each on the -2-, -1-, 1-, and 2-piles.

   The $n = 5$ case plays out just like the $n = 4$ case except the bottom chip on the 0-pile never moves, resulting in final configuration of one chip in each pile from -2 to 2.

   Finally, for $n = 6$ note that we can play this out by spreading out the top four chips in the 0-pile as in the $n = 4$ case. This puts a single chip on piles -2, -1, 1 and 2 and leaves two chips in the 0-pile. Now fire the 0-pile. This leaves a situation like problem 2 with $k = 2$ on the positively numbered piles and a mirror image on the negatively numbered piles. Play each of these out, which will leave one chip on piles -3, -1, 1, and 3, and two chips on the 0-pile. Again fire the 0-pile, and then the 1- and -1-piles, and finally the 0-pile again. We find that the final configuration is a single chip on each pile from -3 to 3 except that the 0-pile is empty.

Graphically, the answers are as shown below:



4. The previous solution more-or-less counted the number of firings as it went. There were no firings for $n = 1$ and only one for $n = 2$ and $n = 3$. For $n = 4$ and $n = 5$ we fired the 0-pile twice, then each of the 1- and -1-piles once, then the 0-pile again, for a total of five firings.

   For $n = 6$ it takes five firings to spread out the top four chips. Then one firing of the 0-pile to obtain the situation of problem 2 and its mirror image. Each of these requires two firings (total so far: 10) to reach the configuration with one chip on the -3-, -1-, 1-, and 3-piles and two in the 0-pile. We fire the 0-pile again (note that this puts two chips in each of the 1- and -1-piles, which is again the position of problem 2, this time with $k = 1$—recursion, anyone?). We now fire piles 1, -1, and finally 0 again. Total: 14 firings.

   To summarize, the number of firings required for $n = 1$ through 6 are $\boxed{0, 1, 1, 5, 5, \text{ and } 14}$.

5. If n is even, the stable configuration resulting from starting with n chips in the 0-pile is a single chip in each pile from $-n/2$ to $n/2$ except the 0-pile, which is empty. If n is odd, the stable configuration is one chip in each pile from $-(n-1)/2$ to $(n-1)/2$.

   It will be simpler to handle the even and odd cases of n separately, even cases first. So let $n = 2r$. Note that if $r = 1$ ($n = 2$) the stable configuration is a single chip in piles -1 and 1. This will be the base case of an induction.

   Now assume that the stable configuration starting with $2(r-1)$ chips in the 0-pile is a single chip in each pile from $-(r-1)$ to $r-1$ except the 0-pile.

   Starting with $2r$ chips in the 0-pile, we may think of them as 2 chips and another $2(r-1)$ chips. Leaving the bottom two chips alone, we inductively spread the top $2(r-1)$ chips (note that any firing available in the $2(r-1)$-chip configuration is still available playing out the $2r$-chip game, because all piles are the same except the 0-pile which has two extra chips in it). So we can move to a position where there is a single chip in each pile from $-(r-1)$ to $(r-1)$ and *two* chips in the 0-pile. Now fire the 0-pile.

The result of this is to put two chips in the 1- and -1-piles, and one chip in each other pile from $-(r-1)$ to $(r-1)$ except that the 0-pile is empty. Thus, piles 1 through $(r-1)$ are in the situation of problem 2, and the mirror image of this situation obtains in the negatively-numbered piles.

According to problem 2, the result of playing this out is to end with a single chip in piles 0 through $r-2$, and one more chip in pile r. Of course, the negatively-numbered piles play out the same say, putting a single chip in piles 0 through $-(r-2)$ and one in pile $-r$. Since both the positive and negative put chips back into the 0-pile, there are again two chips in that pile. That is, the configuration at this point is a single chip in piles $-r, -(r-2), -(r-3), \ldots, -2, -1, 1, 2, \ldots, r-3, r-2, r$ and two chips in the 0-pile. Fire the 0-pile again.

The result of this is to put us once again in the position of problem 2, only this time with $k = r-2$. So we again play out that scenario, this time finishing with a chip in pile $r-1$ but no chip in pile $r-2$ (and similarly for piles $-(r-1)$ and $-(r-2)$, and two chips in the 0-pile. Firing the 0-pile again will put us in the position of problem 2, with $k = r-3$ (and its mirror image in the negative numbers, of course).

So we may continue using the result of problem 2 to fill in each pile from r down to 2 (technically, requiring another induction!). The final firing of the 0-pile puts chips on the 1- and -1-piles, completing the proof in the even case.

Now that we have completed the proof in the case of $n = 2r$ even, assume $n = 2r+1$. Leaving the bottom chip alone for the time being, the top $2r$ chips play out as in the even case to spread the chips to one chip in each pile from $-r$ to r except the 0-pile. But in actuality there is the bottom chip left in the 0-pile, so the proof is complete—there is a single chip in each pile from $-r$ to r.

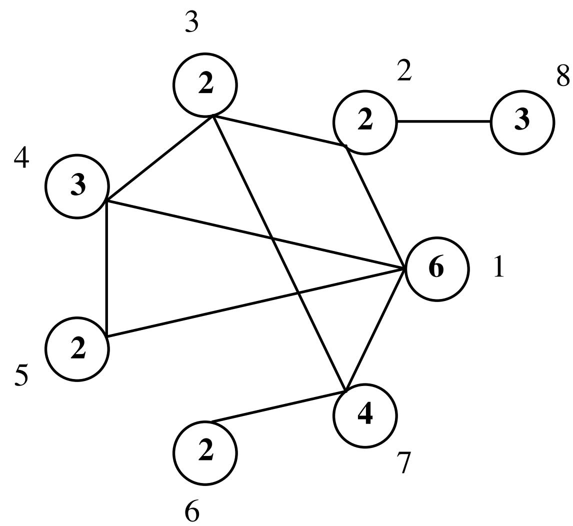
6. The inductive proof above shows how to count the firings needed to arrive at the stable configuration. First, note that from the last paragraph of the previous proof, the number of firings for the $2r+1$ case is exactly the same as for the $2r$ case, so let $r = \lfloor n/2 \rfloor$. The total number of firings necessary is $\boxed{1^2 + 2^2 + \cdots + r^2 = r(r+1)(2r+1)/6}$.

   This figure can be arrived at as follows. Let $F(r)$ be the number of firings needed when $n = 2r$. Then, as in the induction above, after $F(r-1)$ moves there are two chips in the 0-pile and one in each other pile from $-(r-1)$ to $r-1$.

   At this point, we fired the 0-pile, leaving the position of problem 2 with $k = r-1$ in both the positive and negative sides. Each of these requires $r-1$ firings, so we have performed $1 + 2(r-1) = 2r-1$ firings. And now the 0-pile fires again, returning to the scenario of problem 2 but with $k = r-2$. And so forth. So to complete the entire induction step requires $(2r-1) + (2(r-1)-1) + \cdots + 3 + 1$ firings. It is well known that the sum of the odd number up to $2r-1$ is r^2. Thus, we see that the difference $F(r) - F(r-1) = r^2$. Since when $r = 1$ it require $1^2 = 1$ firing, the given formula holds for all r.

7. The result of firing a vertex is that it loses a number of chips equal to its valence, while

each connected vertex gains one chip. So when all the initially fireable vertices P_4 through P_8 are fired the result is that P_1 gains three chips since it can't fire but has three neighbors that can, so it finishes with six chips. On the other hand, P_4 loses three but only gains one, finishing with three chips. The other vertices can be handled similarly. Of course, none of the connections between the piles will change, so the final result will be as shown in the diagram.



8. First, firing P_x can only add chips to P_y, not take any chips away. So P_y can be fired after P_x as well as before it, and vice versa. Second, each pile other than P_x or P_y gains two chips if it is connected to both P_x and P_y, regardless of the order in which they are fired. Similarly, a pile that is connected to only one of P_x or P_y will gain one chip regardless of order, while a pile that is connected to neither will not change. Under both firing orders, both P_x and P_y lose a number of chips equal to their valence, both gaining one back if they are connected to each other. So all piles end up with the same number of chips regardless of the order in which the firings are performed.

9. (a) If a pile has at least as many chips as its valence, then it can fire. So the only way the position can be stable is if each pile has at most $v_P - 1$ chips, where v_P means the valence of pile P. Adding this over all the piles gives $\sum(v_P - 1) = \sum v_P - P$. But note that the sum of all the valences of all the piles must equal $2C$ because each connection adds one to the valence of two different piles. Thus, a stable configuration has at most $2C - P$ chips, and more than this number of chips will always have at least one pile with at least as many chips as its valence by the pigeonhole principle.

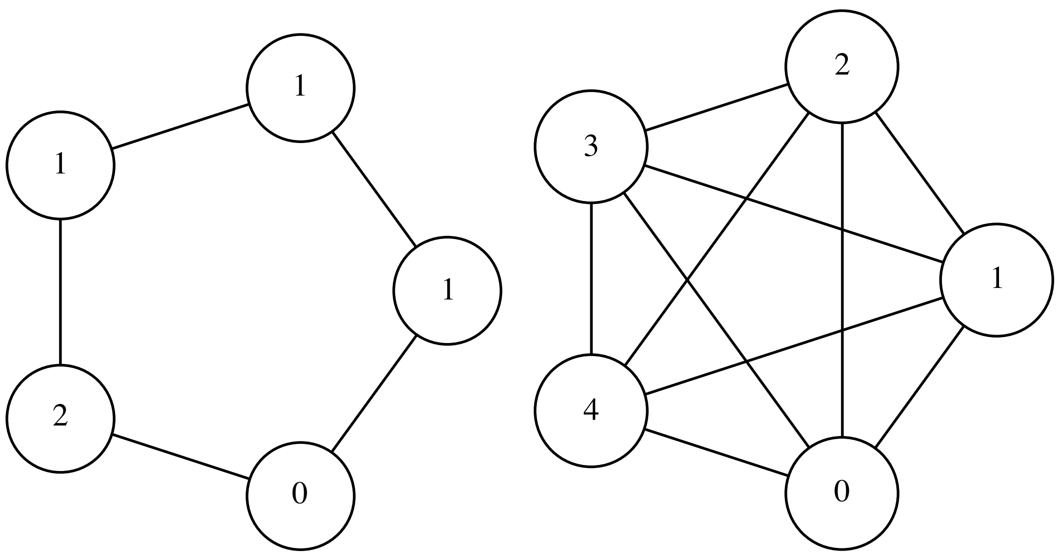
   (b) If each pile has exactly $v_P - 1$ chips, no pile can fire and the configuration is stable. As noted above, this gives a total of $2C - P$ chips.

10. By the pigeonhole principle, there must be at least one pile P_a that fires infinitely often. Choose any pile to which it is connected, P_b. Every time P_a fires, P_b gains a chip. Of course, P_b can never have more than N chips, as that is the total number of chips in the game. So P_b must fire infinitely often to avoid acquiring more than N chips. So any neighbor of a pile that fires infinitely often must itself fire infinitely often.

Since the game board is connected, every pile is either connected to P_a, or connected to a pile that is connected to P_a, or .... Every pile is connected to a pile that fires infinitely often, and therefore fires infinitely often itself.

11. Assume that the game reaches a stable configuration but that every pile has fired at least once. Consider the pile P_r that has fired least recently. Since all piles have fired, certainly every pile to which P_r is connected has fired since the last time P_r fired. But then P_r has acquired at least one chip from each pile to which it is connected, and currently must contain at least as many chips as its valence, so it can fire and the game has not reached a stable configuration. This contradiction shows that if each pile fires at least once a stable configuration can never be reached, proving the original statement by contradiction.

12. The arrangement below left is one initial configuration that will never lead to a stable configuration. The justification is that firing the only pile that can be fired—the pile with two chips in it—leads to a configuration that is identical except "rotated" one position clockwise. So the new configuration also has a pile that is ready to fire, and the game will continue forever.



13. The arrangement above right is one initial configuration that will never lead to a stable configuration. The only pile that can fire is the one with four chips in it, and firing leads to an identical configuration "rotated" one position clockwise as in the previous solution.

14. Consider the pile which currently has no chips in it. If the current distribution was arrived at by firing a pile in another configuration, the no-chip pile must have been the pile that fired. This is because if any other pile fired, since each pile is connected to each other pile the zero-chip pile would have received a chip from that other pile, and couldn't have zero chips in it now! So the zero-chip pile was the pile that fired to achieve the current distribution. Since it now has zero chips and is connected to $n-1$ other piles, it had $n-1$ chips in it. Each other pile received one chip from it. That is, the piles that currently have one, two, ..., $n-1$ chips used to have zero, one, ..., $n-2$ chips. So the previous configuration also had one pile with zero chips, one with one, ..., one with $n-1$ chips as required.

15. Since there are only a finite number of chips and finite number of piles, there are only a finite number of configurations. Since the game never becomes stable some configuration must repeat. Now between two repetitions of the same configuration *every* pile must fire at least once. Otherwise, we could repeat the sequence of firings that leads from the repeated configuration to itself over and over, giving an infinite game that never fires some particular pile, contradicting the result of problem 10.

    Now consider the firings between one pair of repeats of a configuration. For each pile, there is a last time that it fires before the repeated configuration shows up again. For a pile p, let f_p be the number of (other) piles that fire after this last firing but before the configuration repeats. For example, if from a certain configuration the piles fire in order 5, 3, 4, 5, 2, 1, 3, 2, 3 before the configuration is restored, then $f_1 = 3, f_2 = 1, f_3 = 0, f_4 = 6$, and $f_5 = 5$. Note that f_p is defined for each pile because each pile must fire between repetitions. Also note that for different piles p and q it must be the case that $f_p \neq f_q$; in fact if the last firing of p is before the last firing of q then $f_p > f_q$. Finally, note that when the configuration repeats, each pile has at least f_p chips in it because it has acquired one chip from each other pile that fired after it, and may have had chips remaining in it after it fired for the last time.

    Combining these facts about the f_p it becomes clear that at the time of the repeat, there must be a pile with at least zero chips, a different pile with at least one chip, a different pile with at least two chips, and so on until one pile that has at least $n-1$ chips. Adding these, there are at least $0 + 1 + 2 + \cdots + (n-1) = n(n-1)/2$ total chips in the game. But since the total number of chips is $n(n-1)/2$, no pile may have any extra chips in it at the time of the repeat. That is, at the time the configuration first repeats, there is a pile of zero chips, a pile of one chip, . . . , a pile of $n-1$ chips. As shown in problem 14, this distribution is a "Garden of Eden" and could not have come from any other distribution. Thus, the only distribution of $n(n-1)/2$ chips on a complete game board of n piles that never becomes stable is this one.

Parking Problems

The Background

Imagine a one-way street with n parking spaces numbered $1, 2, \ldots, n$. A car turns on to the street and may park at any available space, but once the car drives past a space it cannot return and park there later.

Now imagine that each driver has a preferred space. He or she turns onto the street and proceeds to the preferred space, ignoring whether the lower-numbered spaces are empty or not. If the preferred space is available, the driver parks. If not, the driver pulls ahead and parks in the first available empty space with a higher number than their preferred space. If there are no spaces left, they cannot park their car.

Example: For a street with three parking spaces, three drivers come along with preferences 2, 3, and 2 respectively. The first driver parks in space 2, the second in space 3, but the third driver passes by space 1, then finds spaces 2 and 3 full, so they cannot park. On the other hand, even if all three drivers prefer space 1, they can all park because the first driver gets space 1, the second finds it full and proceeds to park in space 2. The third driver must go on to space 3, but at least they get to park!

A *preference sequence* of length n is an ordered list $(p_1, p_2, \ldots, p_n)$ of n positive integers between 1 and n inclusive. A preference sequence is a *parking sequence* if driver 1 has parking space preference p_1, driver 2 has preference p_2, and so forth, and every driver can park. Thus $(1, 1, 1)$ is a parking sequence, while $(2, 3, 2)$ is not.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. (a) List all preference sequences of length 2. [1 pt]

   (b) Which of the preferences sequences of length 2 are parking sequences? [1 pt]

2. For each preference sequence below, list the spaces that will be taken by each driver, in order. If the driver can't park, use an "X." For example, the sequence $(2, 3, 2)$ should have answer 23X, while $(1, 1, 1)$ will be answered 123. A more complicated example: $(2, 4, 4, 2)$ leads to 24X3.

   (a) $(4, 5, 6, 3, 3, 3)$ [1 pt]

   (b) $(2, 4, 2, 4, 2, 4)$ [1 pt]

   (c) $(3, 3, 2, 2, 1, 1)$ [1 pt]

3. Compute, with explanation, the number of preference sequences of length n. [2 pts]

4. There are 16 parking sequences of length 3. List them all. [2 pts]

Our ultimate goal will be to compute the number of parking sequences of length n.

5. Prove that if any driver allows the person behind them in line to go in front of them, it does not affect whether or not the drivers can all park. That is, show that

$$(p_1, \ldots, p_{j-1}, p_j, p_{j+1}, p_{j+2}, \ldots, p_n)$$

is a parking sequence if and only if

$$(p_1, \ldots, p_{j-1}, p_{j+1}, p_j, p_{j+2}, \ldots, p_n)$$

is a parking sequence. [3 pts]

Applying the result of the previous problem repeatedly shows that the order of drivers actually makes no difference. That is, given any preference sequence $(p_1, p_2, \ldots, p_n)$, we can rearrange it at will without changing whether or not it is a parking sequence. For instance, to make parking more orderly we might *sort* a sequence into non-decreasing order: $(2, 4, 2, 4, 2, 4)$ would become $(2, 2, 2, 4, 4, 4)$ for example, while $(3, 3, 2, 2, 1, 1)$ would become $(1, 1, 2, 2, 3, 3)$.

6. Prove that if preference sequence $S = (p_1, p_2, \ldots, p_n)$ is a parking sequence, then if some driver changes their preference to a *lower*-numbered space, the resulting preference sequence is also a parking sequence. That is, if any p_j is replaced by a smaller number $p_j - a$ where $1 \leq a \leq p_j - 1$, then the resulting sequence is again a parking sequence. [2 pts]

7. Prove that a preference sequence is a parking sequence if and only if, when sorted, the j^{th} entry is less than or equal to j. For example, when we sort $(2, 4, 2, 4, 2, 4)$ into $(2, 2, 2, 4, 4, 4)$ the very first entry is larger than 1, so neither of these two preference sequences is a parking sequence. [2 pts]

The Circular Parking Garage

Instead of our one-way street, let's suppose that n cars enter a parking garage that has $n+1$ spaces in it. The car enters at space 1, and proceeds toward higher-numbered spaces until it gets to its preferred space. It then takes the first available space after that. Fortunately, this garage is circular, so that if all the spaces from its preferred space up to space $n+1$ are all already taken, the car gets to start over at space 1. For example, if $n = 3$ and the preference sequence is $(3, 3, 3)$, then the first car parks in space 3, the second in space 4, and the third goes around the circle and parks in space 1.

8. For the $n = 6$ preference list $(2, 4, 2, 4, 2, 4)$, parking in a garage with seven parking spaces, list the order in which the spaces are filled. (For example, in the case of $(3, 3, 3)$ above, the spaces were filled in the order 3, 4, 1.) [1 pt]

9. In the example of $(3,3,3)$, space 2 remained empty; because there are n cars and $n+1$ spaces, there will always be exactly one empty space. Let $n=5$ cars park in a garage with six spaces. For each parking space k from $k=1$ to $k=4$, give a preference sequence for which that space remains vacant. (Your answer should be four preference sequences, one for each k from 1 to 4.) [2 pts]

10. Explain why, with n cars and $n+1$ spaces in the garage, space n will always be filled. (This explains why $k=5$ was not asked for in the previous problem!) [2 pts]

Since there are now $n+1$ spaces available, we might as well allow drivers to prefer to park in that last space. A *garage preference sequence* of length n is an ordered list $(p_1, p_2, \ldots, p_n)$ of n positive integers between 1 and $n+1$ inclusive. Since the garage is circular, every driver can park, and there will always be one empty space.

11. If space k remains empty, explain why at least one driver's preferred space is $k+1$. (Since the garage is circular with $n+1$ spaces, we must also take our numbers modulo $n+1$, so that if space $k=n+1$ is the empty space, then $k+1$ will refer to space 1.) [2 pts]

12. Let garage parking sequence $S=(p_1, p_2, \ldots, p_n)$ leave space k empty. Show that if every driver increases their preference by the same amount, the empty space is affected the same way. That is, $(p_1+a, p_2+a, \ldots, p_n+a)$ will lead to space $k+a$ remaining empty. (Again, numbers are to be taken modulo $n+1$.) [2 pts]

13. (a) If a garage preference sequence leaves space $n+1$ empty, explain why it must be a (regular) preference sequence. [1 pt]

    (b) If a garage preference sequence leaves space $n+1$ empty, explain why it must, in fact, be a parking sequence. [1 pt]

14. Prove that there are exactly $(n+1)^{n-1}$ parking sequences of length n. [3 pts]

15. Use the result of the previous problem to prove that $(n+1)^{n-1} > n!$ for all integers $n \geq 2$. [2 pts]

Parallel Parking

Often when parking a car along the side of a street, the driver pulls just past the parking space and then backs up into the space. We will try something similar. Given a preference sequence, each driver drives to their preferred spot. If they can park there, they do so. If that spot is already taken, but the previous spot is not, they back into that space (we are no longer in the circular garage, so there is no space previous to space 1 this time). Otherwise, they drive ahead to the first empty space if there is one. For example, given the preference sequence $(2,4,2,4,2,4)$ the first driver parks in space 2 and the second in space 4. The third driver can't park in their preferred space, so they back into space 1, and the fourth driver backs into space 3. The fifth driver can't park in space 2 or back into space 1, so they proceed to the first empty space, which is space 5. The last driver will get space 6. We will call a preference sequence a *parallel parking sequence* if all n cars can find parking spaces using the parallel parking rule. Note that $(2,4,2,4,2,4)$ is a parallel parking sequence even though it is not a parking sequence.

16. Of the $3^3 = 27$ preference sequences of length 3, we know there are $4^2 = 16$ parking sequences. It is a fact that there are 24 parallel parking sequences of length 3. Find the three preference sequences of length 3 that are not parallel parking sequences. (For the record, all four preference sequences of length 2 are parallel parking sequences.) [1 pt]

17. Show that, as opposed to parking sequences, the order of drivers *does* matter with parallel parking. That is, show that there is a preference sequence that can be reordered into a sequence that is a parallel parking sequence, and also into an order which is not a parallel parking sequence. [1 pt]

18. Show, however, that if $S = (p_1, p_2, \ldots, p_n)$ is a parallel parking sequence and $p_j \leq p_{j+1}$, then the sequence obtained from S by swapping p_j and p_{j+1} is also a parallel parking sequence. [2 pts]

19. Show that every parking sequence is a parallel parking sequence. [2 pts]

20. Compute, with explanation, the number of parallel parking sequences of length 4. Please try to be logical about this—a complete listing will not earn full credit. [2 pts]

Author's Note: The author does not know of a formula (either explicit or recursive) for the number of parallel parking sequences of length n. The sequence is not known to the *Online Encyclopedia of Integer Sequences*, so any information on this is probably new mathematical ground.

The Solutions

1. (a) Each of the two drivers can prefer space 1 or space 2, so there are four possibilities: $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$.

   (b) In each of the first three cases, both drivers can park. For $(2,2)$ both drivers bypass space 1, but can't park in the same space, so this is not a parking sequence.

2. (a) The first four drivers get their choice of space, but the last two cannot park, so the spaces are filled in order $\boxed{\text{4563XX}}$.

   (b) The spaces fill up in the order $2,4,3,5,6$ but the last driver can't park. The result: $\boxed{\text{24356X}}$.

   (c) This time, all drivers can park. The order the spaces fill is $\boxed{342516}$.

3. Since each of n drivers can prefer any of the n spaces, the multiplication principle tells us that the number of preference sequences is n^n.

4. Tediously, we list out all the possibilities.

$$\begin{array}{cccccccc} (1,1,1) & (1,1,2) & (1,2,1) & (2,1,1) & (1,1,3) & (1,3,1) & (3,1,1) & (1,2,2) \\ (2,1,2) & (2,2,1) & (1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1) \end{array}$$

5. We will use brute force to consider all the possibilities. Let $p_j = x$ and $p_{j+1} = y$ be the places where the drivers prefer to park. So we need to consider the effect of changing preference sequence $(\ldots, x, y, \ldots)$ to $(\ldots, y, x, \ldots)$. If $x = y$ so that the two drivers who swapped positions wanted the same space, it is clear the preference sequence does not change, so whether or not it is a parking sequence can't change either.

   Assume the original sequence is a parking sequence and driver j ends up parking in space s while $j+1$ ends up in space t. Note $x \le s$ and $y \le t$.

   Case 1: $\boldsymbol{x \le s < y}$. Then, after the change in driver order, driver $j+1$ has exactly the same choices as before. Because even though they now drive in before driver j, the space, s, that is now empty is before y and thus $j+1$ won't park there. So $j+1$ parks in space t as before. This leaves spaces s still available to driver j, and no changes result, so the new order is still a parking sequence.

   Case 2: $\boldsymbol{x < y \le s}$. Note that $t > s$ because otherwise j would have parked in space t in the original driver order. So now the earliest space available to driver $j+1$ is space s. Now space s isn't available to driver j—just like it wasn't available to $j+1$ in the original order since j had just parked there. But now j can proceed to park in space t, since that is now the earliest available space. The rest of the cars now park as in the original order, and once again the result is a parking sequence.

   Case 3: $\boldsymbol{y \le t < x}$. Once again, all drivers park in the same spots as originally; driver j sacrifices nothing by allowing $j+1$ go first, because the space that driver $j+1$ parks in is lower-numbered than j wants.

Case 4: $y < x \leq t$. Note that, as in case 2 above, $s < t$ or driver j would have originally parked in t and not s. So we actually have $y < x \leq s < t$. In this case, the first space available to $j + 1$ now that they are going first is space s. But, again as in case 2, j will end up parking in t and no other driver's spaces is affected, and the resulting sequence is again a parking sequence.

These are all the possibilities when the original sequence was a parking sequence, and in each case the resulting sequence was again a parking sequence. So a swap of two drivers cannot change a parking sequence into a non-parking sequence. Nor can it change a non-parking sequence into a parking sequence, or swapping the drivers back to their original positions would mean that the original sequence was also a parking sequence. So a swap of driver positions does not affect whether or not a sequence is a parking sequence.

6. We will prove this in the case $a = 1$ and then apply the result recursively for $a > 1$. That is, we will prove the result in case one driver decreases their preference by one.

   First, if space $p_j - 1$ is already occupied, the change in preference has no effect on the outcome. Driver j must drive past the spot and park in p_j or later, just as if their preference had not changed.

   On the other hand, if $p_j - 1$ is available, that must mean in the original preference list there is some driver k with $k > j$ who parked in that space. For every driver k with $k > j$ who, in the original preference list, parked in a space earlier than p_j, use the previous problem to let that driver come before driver j. This will cause all the spaces from 1 to $p_j - 1$ to become full before driver j gets to park, and so space $p_j - 1$ isn't available. The previous paragraph then assures us we still have a parking sequence.

7. First, note that the preference sequence $(1, 2, \ldots, n)$ is a parking sequence. By the previous problem, any of these numbers can be reduced and leave a parking sequence. So if a sequence can be sorted into one where each term is less than or equal to the corresponding term of $(1, 2, \ldots, n)$ it must be a parking sequence.

   On the other hand, if (when sorted) any $p_j > j$ then there are at least $p - (j - 1) = p - j + 1$ cars that must park in the $p - j$ spots numbered $p - j + 1$ and higher. By the pigeonhole principle some car will not be able to park.

   An alternate proof of the "if" portion can be obtained by induction. For $k < n$ assume that the first k cars have parked in spaces 1 through k. Driver $k + 1$ has preference for a space numbered at most $k + 1$. Spaces 1 through k are already full, so driver $k + 1$ wlll park in space $k + 1$. (This is a strong induction that doesn't actually need a base case (the statement is vacuously true), though the case $k = 1$ is clearly true if the vacuous base case makes you nervous.) So the n cars will park in spaces 1 through n, and the sequence is a parking sequence.

8. The spaces will fill in order 2, 4, 3, 5, 6, 7.

9. There is more than one possible answer for each k; these are samples.
   - For $k = 1$: (2, 3, 4, 5, 5).
   - For $k = 2$: (1, 3, 4, 5, 5).
   - For $k = 3$: (1, 2, 4, 5, 5).
   - For $k = 4$: (1, 2, 3, 5, 5).

10. The only way a car could park in space $n + 1$ is that it had an earlier preference, say p, but spaces p through n were already full. This implies that to have car in space $n + 1$, space n must be full. But since there are n cars and $n + 1$ spaces, it is not possible for both spaces n and $n + 1$ to be empty by the pigeonhole principle.

11. The argument is exactly the same as the previous problem. Since space k is empty, space $k+1$ must be occupied. There are only two ways for this to happen: a driver prefers that space, or a driver had to bypass space k because it was full. The latter is a contradiction, so it must be the former.

12. For preference list $(p_1, p_2, \dots, p_n)$ let $[s_1, s_2, \dots, s_n]$ be the list of the spaces in the order in which a car parks in them. We will show by induction that $(p_1 + a, p_2 + a, \dots, p_n + a)$ leads to $[t_1, t_2, \dots, t_n]$ with $t_j = s_j + a$. Clearly, since there are no cars parked yet when the first driver comes, they can park in their preferred spot. So $s_1 = p_1$ and $t_1 = p_1 + a = s_1 + a$, so the base case is established.

    Now assume that for some $r < n$ we have $t_1 = s_1 + a$, $t_2 = s_2 + a$, $\dots$, $t_r = s_r + a$. Then either spot p_{r+1} is unoccupied in the original garage—in which case $p_{r+1} + a$ is unoccupied in the shifted garage, and $t_{r+1} = p_{r+1} + a = s_{r+1} + a$. Otherwise, by induction, the occupied spaces in the shifted garage are the shifts of the occupied spaces in the original garage, and the first open space available for driver $r + 1$ will be the shift by a of the first open space in the original garage. That is, $t_{r+1} = s_{r+1} + a$ in this case also, completing the induction.

    Since the occupied spaces are all shifted by a, the unoccupied space must also be shifted by a, as required by the problem.

13. (a) Since space $n + 1$ is empty, no driver can prefer that space, so this garage preference sequence only contains numbers from 1 to n, and is thus a standard preference sequence.

    (b) If the sequence is not a parking sequence, some driver would not be able to park in any of the spaces 1 through n. Since their preference is at most n, the first available space would be $n+1$. So if the sequence is not a parking sequence, space $n+1$ will be occupied.

14. Assemble the previous questions into a complete explanation. Given any garage preference sequence $(p_1, p_2, \dots, p_n)$, each shift $(p_1 + a, \dots, p_n + a)$ leaves a different space empty (problem 12). So exactly one of the $n+1$ possible shifts (including a shift by zero) leaves space $n+1$ unoccupied. By problem 13 this shift must be a parking sequence. Now there are $(n + 1)^n$ possible garage preference sequences (each of the n drivers has $n + 1$ spaces they can select as their preference). Exactly $1/(n+1)$ of these is a parking sequence (and every parking sequence is a garage preference sequence, so every parking sequences arises in this way), leaving a total of $(n + 1)^{n-1}$ parking sequences.

15. The sequence $(1, 2, \ldots, n)$ is a parking sequence, and by problem 5 any of the $n!$ reorderings of this sequence is also a parking sequence. But for $n \geq 2$, there are other parking sequences, such as $(1, 1, \ldots, 1)$. So there are more parking sequences than arrangements of $(1, 2, \ldots, n)$, hence $(n+1)^{n-1} > n!$.

16. The three non-parallel parking sequences are $(3, 3, 3)$, $(2, 3, 3)$, and $(3, 2, 3)$.

17. From the previous problem we know that $(2, 3, 3)$ is not a parallel parking sequence, but its rearrangement $(3, 3, 2)$ is: the first driver parks in space 3, the second tries to park there but has to back into space 2, and then the last driver tries to park in space 2 but has to back into space 1.

18. If driver j initially parked in a space numbered $p_{j+1} - 2$ or less, then driver $j+1$ cannot park in that space even if they enter the lot before driver j and end up backing into a space, so exactly the same parking spaces will be occupied by the same cars regardless of the order of drivers j and $j+1$. On the other hand, if driver j initially ends up in space $p_{j+1} - 1$ or higher, then driver $j+1$ will get that space first, and then driver j will get the space that would have originally gone to $j+1$. (Exception: if spaces p_{j+1} and $p_{j+1} - 1$ are both available, driver $j+1$ will get space p_{j+1} and driver j will get $p_{j+1} - 1$ regardless of the order in which they enter the street.) In any case, all drivers will be able to park, so the sequence remains a parallel parking sequence.

19. Consider a preference sequence that is not a parallel parking sequence. Let driver i be the first driver who is unable to park. Then spaces $p_i - 1$ through n (a total of $n - p_i + 2$) are all already taken. Since a driver can back up at most one space when parking, if the ability to back up were taken away from the drivers, each car in one of these spaces would have to have parked in one of the spaces p_i through n under the original parking rules. But there are $n - p_i + 1$ such spaces for $n - p_i + 2$ cars, so by the pigeonhole principle someone doesn't get to park under the original parking rules. Thus any non-parallel parking sequence is not a parking sequence, and conversely each parking sequence must be a parallel parking sequence.

20. There are $\boxed{203}$ such sequences. This can be arrived at as follows.

   - Any sequence of length 4 having at least two numbers less than or equal to 2 is a parallel parking sequence. For the first two such drivers will get spaces 1 and 2, and then the other two drivers can fit into spaces 3 and 4. This is clear from the fact that all four sequences of length 2 are parallel parking sequences.
     - There are $2^4 = 16$ sequences where we choose each of the four entries to be either 1 or 2.
     - There are 32 sequences with a single 3 (four places to put the 3 and $2^3 = 8$ choices to make the other three numbers either 1 or 2).
     - There are another 32 sequences with a single 4.
     - For sequences with two 3's or 4's, there are $\binom{4}{2} = 6$ choices of where to put them, times $2^2 = 4$ choices of whether each preference is a 3 or a 4, and times $2^2 = 4$ choices of the two small numbers being 1 or 2. That leads to $6 \cdot 4 \cdot 4 = 96$ possibilities.

So altogether this gives 176 sequences.

- $(3, 3, 3, 1)$, $(3, 3, 1, 3)$, $(3, 1, 3, 3)$, $(1, 3, 3, 3)$, and $(3, 3, 3, 2)$, for five more.
- Each sequence above can have any one of the 3's replaced by a 4. Also, both of the first two 3's can be replaced by 4's. That adds another 20 sequences.
- The two remaining parallel parking sequences are $(3, 3, 2, 3)$ and $(3, 3, 2, 4)$.

Note: There are several combinatorial problems whose answer is $(n + 1)^{n-1}$. A couple interesting ones are:

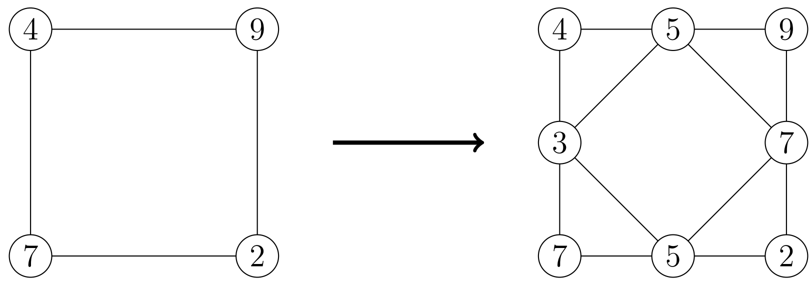
- The number of spanning trees of a complete graph on n distinct points.
- If you take items numbered from 1 to n and place them randomly in a circle, this is the number of ways you could make exactly $n - 1$ swaps of positions of two objects in the circle, and end up with the circle sorted from 1 to n in a clockwise order.
- If you take a sequence of n integers from 1 to n, this is the number of such sequences where the first few terms add up to n. That is, of the 27 sequences of three numbers from 1 to 3 (i.e., "preference sequences" in the terminology of this problem), there are 16 of them where the first few terms add to 3: $(1, 1, 1), (1, 2, x), (2, 1, x)$ and all nine sequences of the form $(3, x, y)$.

Incredibly, the parking sequences are also closely related to "critical arrangements" of the chip firing game from the fall contest! These are configurations that are stable—no pile can fire—but with the addition of a single chip the configuration is able to fire and never becomes stable after any finite number of firings.

Diffygons

The Background

When you were learning subtraction, you may have practiced with an exercise called a "diffy box." This is simply a box with numbers in its four corners, and your assignment was to compute the difference of the two numbers at the ends of each side and write the difference in the middle of the side, as shown below.



Of course, there is no need to draw the box. The numbers can just be listed in an ordered quadruple, starting in one corner and going around the box. For instance, the box above might be given by $(4, 9, 2, 7)$. It might also be listed as $(9, 4, 7, 2)$ because it doesn't matter in which corner the listing starts or in which direction around the box it goes. To compute the diffy box the differences between the terms in the list should be listed, starting in the same corner and going around the box in the same direction, ending with the difference between the last and first numbers in the list. Thus $(4, 9, 2, 7) \to (5, 7, 5, 3)$, while $(9, 4, 7, 2) \to (5, 3, 5, 7)$.

Notice that it is not important to have exactly four numbers—the "diffy operation" can be applied to any polygon with numbers at its vertices or, more practically, with lists of numbers of any length. Hence our title, "diffygons." Formally, let $A = (a_1, a_2, \ldots, a_{n-1}, a_n)$ be an ordered n-tuple. Its *differences* will be the ordered n-tuple

$$\Delta A = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|, |a_n - a_1|).$$

Please note: even though it doesn't matter where we start or in which direction we travel around our polygons, if you are working with a specific n-tuple and find its differences they must be listed in the order given in the definition. For example, $\Delta(2, 6, 1) = (4, 5, 1)$ and not $(5, 1, 4)$ or $(4, 1, 5)$ even though those are just rotated or reflected versions of the same diffy triangle. While these answers are not wrong, the contest graders will be looking for specific answers, and putting your numbers in a different order might cause the grader to mark an answer incorrect, even though it is correct, because it doesn't match the answer key. *Since you have now been warned, you will not be allowed to protest if a correct answer is marked incorrect because the numbers are out of order!*

As the original diagram suggests, once the differences have been found, they form a new box. The differences of this new box can then be found, and then the differences of that box, and so on. Again, we will actually just work with the lists of numbers. So we will apply the operation Δ recursively. This will be indicated by a superscript. Thus $\Delta(4,9,2,7) = (5,7,5,3)$, $\Delta^2(4,9,2,7) = \Delta(\Delta(4,9,2,7)) = \Delta(5,7,5,3) = (2,2,2,2)$, and $\Delta^3(4,9,2,7) = (0,0,0,0)$. Since further applications of Δ obviously have no effect, define the *longevity* of $(4,9,2,7)$ to be 3, because that many applications of Δ brings it to this unchanging state of all zeroes. In precise terms, given that $\Delta^k A = (0,0,\ldots,0)$ and $\Delta^{k-1}A$ is not all zeroes, the longevity of A is defined to be k.

To make sure you understand the concepts and notation, try the quick problems below. Answers are upside-down at the bottom of the page.

1. Let $A = (2,6,-3,1,4,8)$. Compute:
   (a) ΔA.
   (b) $\Delta^3 A$.
2. Determine the longevity of $(1,5,8,14)$.
3. Explain why the longevity of $(1,2,3)$ does not exist.

1. (a) (4, 9, 4, 3, 4, 6), (b) (0, 4, 0, 1, 0, 3)
2. Repeatedly applying Δ leads to $(4,3,6,13)$, $(1,3,7,9)$, $(2,4,2,8)$, $(2,2,6,6)$, $(0,4,0,4)$, $(4,4,4,4)$,$(0,0,0,0)$. Since the seventh application produces all zeroes but the sixth does not, the longevity is 7.
3. Repeatedly applying Δ leads to $(1,1,2)$, $(0,1,1)$,$(1,0,1)$,$(1,1,0)$,$(0,1,1)$,$\ldots$. The same three non-zero triples repeat forever. So since no number of applications of Δ can result in all zeroes, the longevity does not exist.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

The notation used throughout this set of problems will be as it was in the background: If $A = (a_1, a_2, \ldots, a_{n-1}, a_n)$ is an ordered n-tuple, then

$$\Delta A = (|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|, |a_n - a_1|)$$

is its differences, and repeated applications of Δ are indicated by superscripts. Please make sure to keep lists of differences in their proper order! A diffy *box* will always be a diffygon with exactly four sides (and we will, of course, just use ordered lists of four numbers); diffy*gons* can be lists of any length n, representing polygons with numbers at each of their n vertices.

1. The diffy box $A = (9, 17, 31, 57)$ has one of the largest longevities among all diffy boxes with four positive integers less than 100.

   (a) Compute $\Delta^4 A$. [1 pt]

   (b) Compute the longevity of A. [1 pt]

Given that $A = (a_1, a_2, \ldots, a_n)$ is an ordered n-tuple, and that c and k are real numbers, the notation $A + c$ will mean to add c to each term A and $k \cdot A$ will mean to multiply each term of A by k. That is, $A + c = (a_1 + c, a_2 + c, \ldots, a_n + c)$ and $k \cdot A = (ka_1, ka_2, \ldots, ka_n)$.

3. Let $A = (4, 2, 8, 3, 6)$.

   (a) Compute ΔA. [1 pt]

   (b) Compute $3 \cdot A + 7$. [1 pt]

   (c) Compute $\Delta(3 \cdot A + 7)$. [1 pt]

You might have noticed that the result of part (c) of the previous problem is simply 3 times the result of part (a). This is no accident.

4. Let $A = (a_1, a_2, \ldots, a_n)$ be an arbitrary n-tuple of real numbers.

   (a) Prove that $\Delta(A + c) = \Delta A$ for any real number c. [2 pts]

   (b) Prove that $\Delta(k \cdot A) = |k| \cdot \Delta A$ for any real number k. [2 pts]

5. Let $A = (a_1, a_2, \ldots, a_n)$ be an arbitrary n-tuple of real numbers. Show that there is an n-tuple B whose minimum element is 0 and for which $\Delta B = \Delta A$. [2 pts]

If A is an n-tuple, define its *range*, notated $r(A)$ to be the difference between its maximum and minimum elements, whether or not that difference actually appears in its list of differences. Thus, the range of $(2, 6, 0, -3, 4)$ is $r(2, 6, 0, -3, 4) = 6 - (-3) = 9$.

6. (a) Prove that $r(A + c) = r(A)$. [2 pts]
   (b) Prove that $r(k \cdot A) = |k| \cdot r(A)$. [2 pts]
   (c) Prove that $r(\Delta A) \leq r(A)$. [2 pts]

Amazingly, even with arbitrary, seemingly unrelated real numbers as their entries, diffy boxes tend to end up at all zeroes after a short amount of time.

7. Begin with $A = (\pi, e, 4, \sqrt{2})$. Compute the results of repeatedly applying Δ to A until it becomes all zeroes. (Make sure to show the results of each step; if you don't reach all zeroes after a few applications of Δ you have made a computational error.) [2 pts]

If you have played around with diffy boxes, you may have noticed that they *always* seem to go to zero eventually. This is *almost* true, even (as you just saw) if the entries aren't integers! This doesn't happen for all numbers of sides of diffygons (it happens when the number of sides is a power of two, to be precise) and the proof when the entries are not all integers is difficult. But for integer entries and diffy boxes with four sides, you will only need one more piece:

8. Given that $A = (a_1, a_2, a_3, a_4)$ is a list of four integers, show that all entries of $\Delta^4 A$ are even integers. [2 pts]

9. Prove that the longevity of any diffy box of four integers is finite. [4 pts]

Define the *tribonacci* numbers as follows: $t_0 = 0$, $t_1 = 0$, $t_2 = 1$, and for $n > 2$, $t_n = t_{n-1} + t_{n-2} + t_{n-3}$. The sequence is similar to the Fibonacci sequence except that *three* consecutive numbers are added instead of just two. Thus the sequence starts $0, 0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots$. Let T_n be the diffy box $(t_n, t_{n+1}, t_{n+2}, t_{n+3})$.

10. Given that $n \geq 2$, prove that $\Delta^3 T_n = 2 \cdot T_{n-2}$. [2 pts]

11. Compute an exact formula in terms of n for the longevity of T_n. [2 pts]

The result of the last problem shows that there are diffy boxes that can last arbitrarily long before becoming all zeroes. Allowing *real* entries, there are diffy boxes that can last forever.

12. Let t be the unique real root of the cubic polynomial $t^3 - t^2 - t - 1 = 0$. (The value of t is approximately 1.839.) Show that $\Delta(1, t, t^2, t^3) = (t - 1) \cdot (1, t, t^2, t^3)$. [2 pts]

Thus, the diffy box $(1, t, t^2, t^3)$ lasts forever, with every application of Δ simply multiplying the previous result by $t - 1$. Of course, since $0 < t - 1 < 1$, repeatedly multiplying by $t - 1$ causes the result to have a limit of zero, even though it never actually reaches a sequence of all zeroes. It can be shown that a diffy box of four real numbers has a finite longevity if and only if it is not a rotation or reflection of $k \cdot (1, t, t^2, t^3) + c$ for some real numbers k and c.

The situation is very different for diffy triangles and other shapes!

13. Let $A = (a_1, a_2, \ldots, a_n)$ be an ordered n-tuple of real numbers representing a diffygon with n sides.

    (a) Show that the longevity of A is 1 if and only if A is constant and not already all zeroes (that is, $a_1 = a_2 = \cdots = a_n \neq 0$). [1 pt]

    (b) Show that if n is odd then there are *no* diffygons with n entries with finite longevity ≥ 2. [3 pts]

For n-sided diffygons, if n is a power of 2 then the behavior seen for diffy boxes ($n = 4 = 2^2$) dominates. That is, almost all such diffygons have finite longevity. The proof for integer diffygons is basically identical to the proof contained in this Power Contest, although it takes more steps to guarantee all even terms. For arbitrary real numbers, almost all diffygons have finite longevity, with exceptions similar to the one demonstrated above for $n = 4$.

When n has an odd factor, however, the behavior of the previous problem takes over, and unless the diffygon can be split into identical chunks that are a power of 2 in length (for example, $(1, 4, 6, 3, 1, 4, 6, 3, 1, 4, 6, 3)$ which repeats the chunk $(1, 4, 6, 3)$ of length 4), its repeated differences will fall into a repeating pattern (as happened in the final question of the background sheet).

Problems 10 and 11 showed that there are diffy boxes that have arbitrarily large or even infinite longevity. This can only happen if there is one corner of the box and a direction to go around the box starting in that corner for which the entries appear in increasing order. Otherwise, the longevity can never be more than six!

14. Prove this last statement, by classifying all the non-increasing diffy boxes according to their longevity. That is, we have already shown that a diffy box has longevity 1 if and only if its entries are all the same non-zero constant. It is easy to see that (a, b, a, b) has longevity 2 for any real a and b; what other boxes will have longevity 2? Longevity 3? By problem 4 you can shift the box so that its minimum entry is zero, if convenient. By the end of your classification, you should have included all boxes $(0, a, b, c)$ that do not satisfy $0 < a < b < c$ and shown that all such boxes have longevity six or less. [5 pts]

The Solutions

1. Simply apply the difference operation and compute the results:

   (a) $\Delta(9, 17, 31, 57) = (8, 14, 26, 48)$, $\Delta^2(9, 17, 31, 57) = (6, 12, 22, 40)$, $\Delta^3(9, 17, 31, 57) = (6, 10, 18, 34)$, and finally, $\Delta^4(9, 17, 31, 57) = \boxed{(4, 8, 16, 28)}$.

   (b) Continuing, $\Delta^5(9, 17, 31, 57) = (4, 8, 12, 24)$, $\Delta^6(9, 17, 31, 57) = (4, 4, 12, 20)$, $\Delta^7(9, 17, 31, 57) = (0, 8, 8, 16)$, $\Delta^8(9, 17, 31, 57) = (8, 0, 8, 16)$, $\Delta^9(9, 17, 31, 57) = (8, 8, 8, 8)$, and finally, $\Delta^{10}(9, 17, 31, 57) = (0, 0, 0, 0)$ so the longevity is $\boxed{10}$.

2. (a) $\Delta(4, 2, 8, 3, 6) = \boxed{(2, 6, 5, 3, 2)}$.

   (b) $3 \cdot A + 7 = (12, 6, 24, 9, 18) + 7 = \boxed{(19, 13, 31, 16, 25)}$.

   (c) $\Delta(19, 13, 31, 16, 25) = \boxed{(6, 18, 15, 9, 6)}$.

3. These follow easily from properties of arithmetic and absolute value.

   (a) Given any two real numbers a and b, note that $(a + c) - (b + c) = a + c + b - c = a - b$ and likewise the absolute values of both sides are equal. Apply this to each slot of ΔA.

   (b) Given any two real numbers a and b, note that $ka - kb = k(a - b)$ and taking absolute values yields $|ka - kb| = |k(a - b)| = |k||a - b|$. Thus, each slot of $\Delta(k \cdot A)$ is simply $|k|$ times the corresponding entry of ΔA, which is the definition of $|k| \cdot \Delta A$.

4. Let $c = \min(a_1, a_2, \ldots, a_n)$ and let $B = A + (-c)$. Since $c \leq a_k$ for any k, $a_k - c \geq 0$ and since $c = a_{\min}$ for whichever subscript(s) correspond to the smallest a, there is at least one zero among the entries of B. So B is an n-tuple whose minimum element is 0, and by problem 3(a), $\Delta B = \Delta A$.

5. (a) Note that adding a constant to all numbers in a list increases or decreases both the minimum and the maximum number in the list by the same amount. Thus, the argument from 3(a) applies to the range.

   (b) Similarly, the argument from 3(b) applies here, since if k is positive both the minimum and maximum are multiplied by the same amount (which then factors out of the difference). If k is negative, the role of the maximum and minimum are interchanged, but then $-k = |k|$ is factored out of the difference. If $k = 0$ then all products become zero as does the range.

   (c) Use part (a) and problem 4 to obtain from A another n-tuple B whose minimum element is zero and whose range is the same as that of A. The maximum element of B must be $r(A)$. But since all the elements of B are nonnegative, the minimum elements of ΔB is nonnegative and the maximum element of ΔB is at most the maximum element of B. Thus, $r(\Delta A) = r(\Delta B) \leq r(A)$.

6. To keep track of the relative sizes of the numbers, approximate $\pi \approx 3.1$, $e \approx 2.7$, and $\sqrt{2} \approx 1.4$. So the results of recursively applying Δ to $(\pi, e, 4, \sqrt{2})$ are:

$$\begin{aligned}
\Delta(\pi, e, 4, \sqrt{2}) &= (\pi - e, 4 - e, 4 - \sqrt{2}, \pi - \sqrt{2}),\\
\Delta^2(\pi, e, 4, \sqrt{2}) &= (4 - \pi, e - \sqrt{2}, 4 - \pi, e - \sqrt{2}),\\
\Delta^3(\pi, e, 4, \sqrt{2}) &= (a, a, a, a), \text{where } a = e + \pi - \sqrt{2} - 4,\\
\Delta^4(\pi, e, 4, \sqrt{2}) &= (0, 0, 0, 0).
\end{aligned}$$

7. The table below lists every possible combination of even (e) and odd (o) entries a diffy box could have, and what combination it will have after Δ is applied. Naturally, $o - e$ and $e - o$ both lead to o while $e - e$ and $o - o$ lead to e when computing Δ. Of course, the table is quite redundant (for example, (e, o, o, o) and (o, e, o, o) are just rotations of each other.

A	ΔA	$\Delta^2 A$	$\Delta^3 A$	$\Delta^4 A$
(o,o,o,o)	(e,e,e,e)	(e,e,e,e)	(e,e,e,e)	(e,e,e,e)
(o,o,o,e)	(e,e,o,o)	(e,o,e,o)	(o,o,o,o)	(e,e,e,e)
(o,o,e,o)	(e,o,o,e)	(o,e,o,e)	(o,o,o,o)	(e,e,e,e)
(o,o,e,e)	(e,o,e,o)	(o,o,o,o)	(e,e,e,e)	(e,e,e,e)
(o,e,o,o)	(o,o,e,e)	(e,o,e,o)	(o,o,o,o)	(e,e,e,e)
(o,e,o,e)	(o,o,o,o)	(e,e,e,e)	(e,e,e,e)	(e,e,e,e)
(o,e,e,o)	(o,e,o,e)	(o,o,o,o)	(e,e,e,e)	(e,e,e,e)
(o,e,e,e)	(o,e,e,o)	(o,e,o,e)	(o,o,o,o)	(e,e,e,e)
(e,o,o,o)	(o,e,e,o)	(o,e,o,e)	(o,o,o,o)	(e,e,e,e)
(e,o,o,e)	(o,e,o,e)	(o,o,o,o)	(e,e,e,e)	(e,e,e,e)
(e,o,e,o)	(o,o,o,o)	(e,e,e,e)	(e,e,e,e)	(e,e,e,e)
(e,o,e,e)	(o,o,e,e)	(e,o,e,o)	(o,o,o,o)	(e,e,e,e)
(e,e,o,o)	(e,o,e,o)	(o,o,o,o)	(e,e,e,e)	(e,e,e,e)
(e,e,o,e)	(e,o,o,e)	(o,e,o,e)	(o,o,o,o)	(e,e,e,e)
(e,e,e,o)	(e,e,o,o)	(e,o,e,o)	(o,o,o,o)	(e,e,e,e)
(e,e,e,e)	(e,e,e,e)	(e,e,e,e)	(e,e,e,e)	(e,e,e,e)

Since the entry in the Δ^4 column is always (e, e, e, e) all diffy boxes containing only integers yield all even integers after four applications of Δ.

8. The idea of the proof is that applying Δ four times, as in the previous problem, leads to all even entries, so 2 can be factored out of the diffy box, *and also out of the range*. But the range stays the same or shrinks when applying Δ. Since the range is finite, factors of two cannot continue to be removed forever!

   Specifically, let n be an integer so that $2^n > r(A)$. Define $A_0 = A$. Since $\Delta^4 A_0$ consists of all even terms, define A_1 so that $\Delta^4 A_0 = 2 \cdot A_1$. Recursively define A_k to be such that $\Delta^4 A_{k-1} = 2 \cdot A_k$. Inductively, $\Delta^{4k} A_0 = 2^k \cdot A_k$.

   Now consider the range: $2^n > r(A) = r(A_0) \geq r(\Delta^{4n} A_0) = r(2^n A_n) = 2^n r(A_n)$, the last equality courtesy of problem 5(b). Thus, $r(A_n) < 1$ but since all numbers involved are integers the only way this can happen is if $r(A_n) = 0$, meaning all entries of A_n are equal. Then $\Delta A_n = (0,0,0,0)$. Conclusion: $\Delta^{4n+1}A = (0,0,0,0)$.

9. This will be a brute force computation. Note that the tribonacci numbers are increasing for indices larger than 1. Then

$$\Delta(t_n, t_{n+1}, t_{n+2}, t_{n+3}) = (t_{n+1} - t_n, t_{n+2} - t_{n+1}, t_{n+3} - t_{n+2}, t_{n+3} - t_n).$$

Applying the recursive definition $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ to these combinations yields $(t_{n-1} + t_{n-2}, t_n + t_{n-1}, t_{n+1} + t_n, t_{n+2} + t_{n+1})$. Note that since indices are as small as $n-2$ the assumption that $n \geq 2$ was necessary.

Now apply Δ again: $\Delta^2 T_n = (t_n - t_{n-2}, t_{n+1} - t_{n-1}, t_{n+2} - t_n, t_{n+2} + t_{n+1} - t_{n-1} - t_{n-2})$. The last entry may be simplified, by adding and subtracting t_n:

$$\begin{aligned} t_{n+2} + t_{n+1} - t_{n-1} - t_{n-2} &= t_{n+2} + t_{n+1} + t_n - t_n - t_{n-1} - t_{n-2} \\ &= t_{n+2} + t_n \end{aligned}$$

and noting that $t_{n+1} - t_n - t_{n-1} - t_{n-2} = 0$.

Thus $\Delta^2 T_n = (t_n - t_{n-2}, t_{n+1} - t_{n-1}, t_{n+2} - t_n, t_{n+2} + t_n)$.

It will simplify the next step to use the recurrence relation to replace $t_{n+1} - t_{n-1}$ with its equivalent $t_n + t_{n-2}$ and $t_{n+2} - t_n$ with $t_{n+1} + t_{n-1}$. Thus

$$\Delta^2 T_n = (t_n - t_{n-2}, t_n + t_{n-2}, t_{n+1} + t_{n-1}, t_{n+2} + t_n).$$

Apply Δ one more time to produce

$$\Delta^3 T_n = (2t_{n-2}, t_{n+1} - t_n - t_{n-2} + t_{n-1}, t_{n+2} - t_{n+1} - t_{n-1} + t_n, t_{n+2} + t_{n-2}).$$

The second and third entries can be simplified by replacing the highest index term using the recurrence relation and simplifying, yielding $2t_{n-1}$ and $2t_n$ respectively. The final entry is simplified as $t_{n+2} + t_{n-2} = t_{n+1} + t_n + t_{n-1} + t_{n-2} = t_{n+1} + t_{n+1} = 2t_{n+1}$, replacing the top index term and then grouping the three lowest-index terms together in the result. The final result is $\Delta^3 T_n = (2t_{n-2}, 2t_{n-1}, 2t_n, 2t_{n+1}) = 2 \cdot T_{n-2}$, as desired.

10. Compute the longevity of T_0 and T_1 (each arrow represents applying Δ): $T_0 = (0,0,1,1) \longrightarrow (0,1,0,1) \longrightarrow (1,1,1,1) \longrightarrow (0,0,0,0)$ and $T_1 = (0,1,1,2) \longrightarrow (1,0,1,2), \longrightarrow (1,1,1,1) \longrightarrow (0,0,0,0)$. So both of these boxes have longevity 3.

Now if n is even, say $n = 2k$, then applying the results of problems 3 and 9 shows that $\Delta^{3k} T_n = 2^k \cdot T_0$ and then three more applications of Δ yields all zeroes. So the longevity is $3k + 3$.

In case n is odd, say $n = 2k + 1$, the result is the $\Delta^{3k} T_n = 2^k T_1$ and, again, three more applications yields all zeroes.

Combining these yields the formula $3n/2 + 3$ for n even and $3(n-1)/2 + 3$ for n odd. These can be combined in a formula such as $3\lfloor \frac{n}{2} \rfloor + 3$.

11. Once again, simply compute

$$\begin{aligned}\Delta(1,t,t^2,t^3) &= (t-1,t^2-t,t^3-t^2,t^3-1)\\ &= (t-1,t(t-1),t^2(t-1),(t^2+t+1)(t-1))\\ &= (t-1)\cdot(1,t,t^2,t^2+t+1).\end{aligned}$$

Of course, since $t^3-t^2-t-1=0$, the last term in the diffy box can be replaced by t^3 which is the desired result.

12. Of course, an n-tuple that is all zeroes has a longevity of zero, but no other n-tuple has longevity zero.

    (a) The longevity will be one if and only if applying Δ yields all zeroes, which occurs if and only if all the numbers in the n-tuple are the same.

    (b) If the longevity of $A=(a_1,a_2,\ldots,a_n)$ is 2, then ΔA has longevity 1, so by part (a) must have the form $(c,c,\ldots,c)$ for some non-zero c. Thus $|a_2-a_1|=c$ or $a_2-a_1=\pm c$. Similarly, $a_3-a_2=\pm c$, and so forth until $a_n-a_{n-1}=\pm c$ and $a_1-a_n=\pm c$.

    Add all n of these equations. On the left-hand side, each a_i occurs once with a plus and once with a minus, so the left-hand side is zero. On the right-hand side there will be k $+$'s and $n-k$ $-$'s. So the result of adding the $\pm c$'s on the right-hand side is $(2k-n)c$. But c is not zero, and since n is odd $2k-n$ cannot be zero either. Thus the right-hand side cannot equal zero, forcing a contradiction. So no odd-length n-tuple can have longevity equal to 2.

    Of course, since the longevity of ΔA is one less than the longevity of A, no odd-length n-tuple can have longevity longer than 2 either.

13. To make the following simpler, first consider any diffy box where two non-adjacent elements are equal, which can be rotated to look like (a,b,a,c). Start applying Δ: $\Delta(a,b,a,c)=(x,x,y,y)$, where $x=|b-a|$ and $y=|c-a|$. Then $\Delta^2(a,b,a,c)=(0,z,0,z)$ where $z=|x-y|$. Of course, $\Delta^3(a,b,a,c)=(z,z,z,z)$ and $\Delta^4(a,b,a,c)=(0,0,0,0)$. So if two non-adjacent elements are equal, the longevity is at most four.

    From earlier work, only $(0,0,0,0)$ has longevity zero, and only (c,c,c,c) with $c\neq 0$ has longevity 1. All other diffy boxes must have larger longevity. Thus problems 3 and 4 apply without changing the longevity. The diffy box can be rotated so that its minimum element comes first. The results of problems 3 and 4 can then be applied so that this first entry can be assumed to be zero and the first non-zero entry can be assumed to be 1.

    The possibilities for diffy boxes are thus scaled, translated, and/or rotated versions of one of the following cases:

    $(0,0,0,1)$: This category includes all boxes where the smallest number occurs three times. Applying Δ yields $(0,0,0,1)\longrightarrow(0,0,1,1)\longrightarrow(0,1,0,1)\longrightarrow(1,1,1,1)\longrightarrow(0,0,0,0)$ so if three of the entries are equal the longevity is four.

    $(0,0,1,1)$: From the previous case, diffy boxes with this pattern are seen to have longevity 3.

$(0,0,1,a)$**:** If $a>1$ then this case yields $(0,0,1,a)\longrightarrow(0,1,a-1,a)\longrightarrow(1,|(2-a)|,1,a)$. If $a<1$ then this case yields $(0,0,1,a)\longrightarrow(0,1,1-a,a)\longrightarrow(1,a,|(2a-1)|,a)$. For both possibilities, two applications of Δ led to a box with equal non-adjacent terms, which has already been shown to take at most four more steps to reach all zeroes.

$(0,1,0,a)$**:** For any a the longevity is at most four because of the two non-adjacent equal entries.

$(0,1,1,0)$**:** Has longevity 3 from the second case.

$(0,1,1,1)$**:** Applying Δ yields $(1,0,0,1)$ which is again the second case, so diffy boxes where the largest number occurs three times have longevity four.

$(0,1,1,a)$**:** The evolution is $(0,1,1,a)\longrightarrow(1,0,|a-1|,a)\longrightarrow(1,|a-1|,x,|a-1|)$ where the value of x doesn't matter because two non-adjacent entries are equal, so this case has longevity at most six.

$(0,1,a,0)$**:** This is rotated from $(0,0,1,a)$ above.

$(0,1,a,1)$**:** This has two non-adjacent equal entries.

$(0,1,a,a)$**:** This evolves as $(0,1,a,a)\longrightarrow(1,|a-1|,0,a)\longrightarrow(x,|a-1|,a,|a-1|)$ so this case has longevity at most six, regardless of the value of x.

$(0,1,a,b)$**:** Since the terms aren't allowed to increase, either $a<1$ or $b<a$. *But not both!* Otherwise the terms would strictly increase if we went around the box in the other direction.

In the case $a<1$, the evolution will be $(0,1,a,b)\longrightarrow(1,1-a,b-a,b)\longrightarrow(a,|b-1|,a,|b-1|)$ which lasts at most two more steps. If $a>1$ and $a>b$ then $(0,1,a,b)\longrightarrow(1,a-1,a-b,b)\longrightarrow(|a-2|,|b-1|,|a-2b|,|b-1|)$ which lasts at most four more steps.

Since all possibilities have now been accounted for, the conclusion that any box where the terms do not strictly increase must have longevity at most six is proven.

Whose Seat Am I In?

The Background

Note: in this problem there are two kinds of people, "absent-minded" and "clear-headed." To make it easier to keep them separated, absent-minded people will be referred to using masculine pronouns and clear-headed people by feminine pronouns.

There are n people, numbered 1 through n by the order in which they walk into a room. They are to sit in n seats, also numbered 1 through n. Person number 1 is supposed to sit in seat number 1, person 2 in seat 2, and so on. That is, each person is assigned to sit in his or her own particular seat. Of course, some of these people are absent-minded, and forget where they are supposed to sit. These people instead sit in an empty seat at random (which might be their own assigned seat), with equal probability for sitting in each empty seat. A person who is not absent-minded will sit in her own seat unless someone else is already sitting there, in which case she will choose a random unoccupied empty seat, with equal probability for each seat.

One absent-minded person

To begin, explore the situation where there are n people, and only person 1 is absent-minded.

If there are just $n = 2$ people, person 1 sits in his own seat half the time, while he sits in seat 2 the other half of the time. There are two possibilities, and person n gets to sit in her own seat with probability exactly 1/2.

Next, examine the possibilities when there are three people, and only person 1 is absent-minded. The notation (a, b, c) will be used to mean that person a is sitting in seat 1, person b in seat 2, and person c in seat 3. This will be called a *seating arrangement* or simply *arrangement* for short, and similar notation will be used for more people sitting in more seats. Note that the arrangements $(1, 3, 2)$ and $(2, 3, 1)$ are not possible with just one absent-minded person, because person 1 did not sit in seat 2, so it was available for person 2 to sit in when she walked into the room. Not being absent-minded, she would have sat in her own seat 2 so the middle number in the triple would have been 2, not 1 or 3. The remaining possibilities are:

Person 1 sits in seat	with probability	Person 2 then sits in seat	with probability	Person 3 must sit in seat	Notation	Probability
1	1/3	2	1	3	$(1, 2, 3)$	1/3
2	1/3	1	1/2	3	$(2, 1, 3)$	1/6
		3	1/2	1	$(3, 1, 2)$	1/6
3	1/3	2	1	1	$(3, 2, 1)$	1/3

Notice that there are four possible seating arrangements, and person 3 gets to sit in her own seat with probability $1/3 + 1/6 = 1/2$. Also note that she can only sit in seats 1 or 3.

The Problems

There are 40 points available on this contest. The numbers at the end of each problem are the number of points that problem is worth.

1. Make a table like the one above for four people, with only the first person being absent-minded. Hint: there should be eight possible seating arrangements, and amazingly person 4 still has probability exactly 1/2 of sitting in her own seat. Feel free to turn the paper sideways ("landscape mode") so your table isn't cramped on the page. [4 pts]
2. In the four-person scenario, in which seats can person 4 sit? [1 pt]

When there were two people, there were 2 seating arrangements; with three people there were 4 arrangements, and with four people there were 8 possible seating arrangements. Person n ended up sitting in either her own seat or in seat 1, with probability 1/2 for each. While it can be dangerous to generalize from just a few examples, the natural generalization here turns out to be correct.

3. For any $n \geq 2$ and k with $2 \leq k \leq n$, explain why person k can only end up sitting in seat 1 or seat j with $j \geq k$. In other words, person k cannot end up sitting in seats 2 through $k-1$. (In particular, person n can only end up sitting in seat 1 or her own seat n.) [2 pts]
4. For any $n \geq 2$ and any k, $2 \leq k \leq n$, choose any subset consisting of k of the n people that includes person 1. Explain why there is exactly one possible seating arrangement where each person in this set is in the wrong seat and each person not in this set is in the correct seat. [2 pts]
5. Prove that there are exactly 2^{n-1} possible seating arrangements. [2 pts]
6. Prove that, for any $n \geq 2$, the probability that person n sits in her own seat n is 1/2. [2 pts]
7. (a) When $n = 3$, determine the probability that person 2 is in her correct seat. [1 pt]
   (b) When $n = 4$, determine the probability that person 3 in in her correct seat. [1 pt]
   (c) When $n = 4$, determine the probability that person 2 is in her correct seat. [1 pt]

The observations in this last problem also generalize:

8. When there are n people, and only person 1 is absent-minded, show that the probability that person k ($k > 1$) will end up sitting in her correct seat is $\dfrac{n-k+1}{n-k+2}$. [3 pts]
9. (a) With $n = 3$, compute the probability that there are exactly s people sitting in their own seats, where s is each integer from 0 up to 3. [1 pt]
   (b) Compute the expected value of the number of people sitting in their own seats when $n = 3$. [1 pt]
10. (a) With $n = 4$, compute the probability that there are exactly s people sitting in their own seats, where s is each integer from 0 up to 4. [1 pt]

(b) Compute the expected value of the number of people sitting in their own seats when $n = 4$. [1 pt]

11. Determine and prove a formula depending on n that computes the expected value of the number of people sitting in their own seats for each given n. [3 pts]

Multiple absent-minded people

To continue, explore the situation where there are n people, and the first a of them are all absent-minded, where a can be any integer between 1 and n. To receive credit for the next three problems, work or explanations must be shown (you can check your answers with the formulas in the problems that follow them).

12. Let $n = 3$ and $a = 2$.

    (a) Compute the number of possible seating arrangements. [1 pt]

    (b) Compute the probability that person 3 sits in her own seat. [1 pt]

13. Let $n = 4$ and $a = 2$.

    (a) Compute the number of possible seating arrangements. [1 pt]

    (b) Compute the probability that person 3 sits in her own seat. [1 pt]

    (c) Compute the probability that person 4 sits in her own seat. [1 pt]

14. (a) Compute the number of possible seating arrangements for $n = 5$ and $a = 2$. [1 pt]

    (b) Compute the number of possible seating arrangements for $n = 5$ and $a = 3$. [1 pt]

These patterns persist as well:

15. Given that $k > a$, show that person k can only sit in seats with numbers either at least k or at most a. [1 pt]

16. (a) Prove that person n will sit in her own seat with probability $\dfrac{1}{a+1}$. [1 pt]

    (b) Prove that person k (with $k > a$) will sit in her own seat with probability $\dfrac{n-k+1}{n-k+a+1}$. [2 pts]

17. Prove that with n people, the first a of whom are absent-minded, there are $a!(a+1)^{n-a}$ possible seating arrangements. [3 pts]

The Solutions

1. Here are the eight possible seating arrangements from four people, with just the first being absent-minded.

Person 1 sits in seat	with probability	Person 2 then sits in seat	with probability	Person 3 then sits in seat	with probability	Person 4 must sit in seat	Notation	Overall probability
1	1/4	2	1	3	1	4	$(1, 2, 3, 4)$	1/4
2	1/4	1	1/3	3	1	4	$(2, 1, 3, 4)$	1/12
		3	1/3	1	1/2	4	$(3, 1, 2, 4)$	1/24
		3	1/3	4	1/2	1	$(4, 1, 2, 3)$	1/24
		4	1/3	3	1	1	$(4, 1, 3, 2)$	1/12
3	1/4	2	1	1	1/2	4	$(3, 2, 1, 4)$	1/8
		2	1	4	1/2	1	$(4, 2, 1, 3)$	1/8
4	1/4	2	1	3	1	1	$(4, 2, 3, 1)$	1/4

Just to double-check: the sum of all the probabilities in the last column does add to one, and the probabilities where person 4 is in seat 4 (rows 1, 2, 3, and 6 in the table) do add to 1/2 as advertised.

2. From the table above, note that the 4 is in either the first entry or the last entry in each quadruple, so person 4 must be in either seat 1 or seat 4.

3. Consider person k and seat j with $1 \neq j < k$. Person k cannot sit in seat j because if person 1 didn't accidentally sit in seat j, then person j came into the room before person k and her seat was available, so she sat in it. In either case, seat j is now unavailable for person k to sit in.

4. Let the subset be $\{1, p_2, p_3, \ldots, p_k\}$ where $1 < p_2 < p_3 < \cdots < p_k$. The only way in which these people, and no others can sit in the wrong seats is if person 1 sits in seat p_2, person p_2 sits in seat p_3, and so on, with the last person in the subset, person p_k, sitting in seat 1. Clearly for only these people to be in the wrong seats and everyone else to be in their assigned seats, none of these people can sit in a seat other than the seats with numbers in this subset, because that would displace someone whose number is not in the subset. And by the result of the previous problem everyone in this subset must either be in his or her own seat, a higher-numbered seat, or seat 1. None can be in their own seat if they are all to be in a wrong seat. So person p_k must be in seat 1 because there is no wrong-numbered seat with number higher than p_k. Working backward, since seat 1 is now filled by person p_k, the only wrong seat available for person p_{k-1} is seat p_k. Then since person p_{k-2} can't be in seat 1 or seat p_k he or she must be in seat p_{k-1}, and so on until person 1 must be found in seat p_2. A full proof would use induction, but the problem only asked for an explanation rather than a complete proof.

5. Two approaches suggest themselves.

   The first is to build on the previous problem. Clearly there is one arrangement where everyone is in the correct seat, and no arrangement where exactly one person is in the wrong seat. Now if person 1 sits in his correct seat 1, so will everyone else, so if there is anyone in a wrong seat, person 1 must be among them. We know from the previous problem that all sets of at least two people that include person 1 give rise to exactly one seating arrangement where that set is the set of people in wrong seats. Now a subset of two or more people that includes person 1 can be realized as any non-empty subset of $\{2, 3, \ldots, n\}$ with person 1 added to it. Since there are $n - 1$ elements in $\{2, 3, \ldots, n\}$, it is well-known that there are 2^{n-1} subsets, and hence $2^{n-1} - 1$ non-empty subsets. Together with the one arrangement where everyone is seated correctly, the total is 2^{n-1}.

   A second approach is to condition on whether or not the last person is sitting in her correct seat. Let the number of seating arrangements with n people be denoted A_n. If person n is in the correct seat, then each arrangement of $n - 1$ people becomes an arrangement of n people by placing person n in her correct seat. So there are A_{n-1} arrangements of n-people where person n is in the correct seat. Each arrangement of $n - 1$ people where some people are in the wrong seat also becomes an arrangement of n people with some in wrong seats by putting person n in seat 1 and the person who had sat in seat 1 into seat n (this is certainly an arrangement with some people in wrong seats, and by the previous problem there is only one such arrangement possible for any particular subset of people). This adds $A_{n-1} - 1$ more arrangements (because some of the first $n - 1$ people had to be in incorrect seats, so the arrangement of $n-1$ people where everyone is in his or her correct seat has been discounted). Finally, there is one more arrangement where person 1 and person n swap seats, so the total is $A_n = A_{n-1} + (A_{n-1} - 1) + 1 = 2A_{n-1}$. Together with the obvious initial values $A_1 = 1$ and $A_2 = 2$, and the values $A_3 = 4$ and $A_4 = 8$ as have been discovered in earlier work, this recursion clearly results in $A_n = 2^{n-1}$.

6. It is tempting to observe that, in the second solution to the previous problem, half of the arrangements had person n in her correct seat and half did not and conclude that the probability is $1/2$. But that conclusion is not justified. Not all arrangements have the same probability, and this must be taken into account. A nice recursive argument is as follows. It has already been observed that for $n = 2$ people the probability of person n sitting in her correct seat is $1/2$.

   Using the principle of strong induction, assume it has already been shown that the desired probability is $1/2$ when there are between 2 and n people, inclusive. Consider the case of $n+1$ people. There is a $1/(n+1)$ probability that person 1 will sit in seat 1, in which case everyone will sit in his or her own seat, including person $n+1$. There is also a $1/(n+1)$ probability that person 1 will sit in seat $n+1$, so that person $n+1$ will now definitely not be in her own seat. That leaves a probability of $\dfrac{n-1}{n+1}$ that person 1 will sit in seat k, with $2 \le k \le n$. Now all people with numbers between 2 and $k - 1$ inclusive, if any, will sit in their own seats. So seats 2 through k are filled by people 1 through $k - 1$. Person k now must sit in a seat at random, and the situation is as if there were $(n + 1) - (k - 1) = n + 2 - k$ people waiting for seats, with person k taking the role of the absent-minded person 1. Since $2 \le k \le n$, $n+2-k$

is between 2 and n, so the probability that the last person sits in her own seat has inductively been assumed to be $1/2$. Thus, the total probability that person $n+1$ is in her own seat is

$$\frac{1}{n+1}\cdot 1+\frac{1}{n+1}\cdot 0+\frac{n-1}{n+1}\cdot\frac{1}{2}=\frac{1}{2}$$

as desired.

7. These probabilities can be computed from the tables for $n=3$ and $n=4$ that have already been created.

   (a) Revisiting the table in the beginning of the question, person 2 is in her correct seat unless person 1 sat there first, which has probability $1/3$, so the probability that person 2 is in her correct seat is $\boxed{2/3}$.

   (b) Examining the table from problem 1 shows person 3 to be sitting in her correct seat in rows 1, 2, 5, and 8, whose probabilities sum to $1/4+1/12+1/12+1/4=\boxed{2/3}$.

   (c) Again referring back to the table from problem 1, person 2 is in her correct seat unless person 1 already sat there (which happens $1/4$ of the time) so her probability of being in the correct seat is $\boxed{3/4}$.

8. The problem is similar to problem 6. In that problem, it was shown that (if there are at least two people) than the last person sits in her own seat with probability $1/2$. This problem extends that to say that if there are at least three people, then person $n-1$ sits in her own seat with probability $2/3$; if there are at least four people than person $n-2$ sits in her own seat with probability $3/4$ and so forth. The probabilities are determined by how far the person is from the *end* of the line.

   Let $j=n-k$ denote how far the person is from the end of the line. Thus, $j=0$ is the last person in line (whom we already know has probability $1/2$ to sit in her own seat), $j=1$ represents the second-to-last person, and so on. The goal is to prove that person $n-j$ sits in her own seat with probability $\dfrac{j+1}{j+2}$.

   For a fixed j, the base case occurs when $k=n-j=2$. Person 2 always sits in her own seat with probability $\dfrac{n-1}{n}$ because there is only a $\dfrac{1}{n}$ probability that person 1 sits in seat 2, and otherwise person 2 gets to sit in her own seat. So for all choices of j, with $k=2$, the probability is $\dfrac{n-1}{n}$. Substituting $n=k+j$ and $k=2$ leaves a probability of $\dfrac{j+1}{j+2}$ as desired.

   Now using strong induction, assume that the probability that person $n-j$ is in her own seat is $\dfrac{j+1}{j+2}$ for all values of k from 2 up to $n-j$. Now let $k=n+1-j$. There are four possibilities:

   - Person 1 sits in one of the seats numbered at least $k+1$. Then person k will sit in her own seat. There are j of these seats, so this happens with probability $\dfrac{j}{n+1}$.

- Person 1 sits in seat k. Then person k will not get to sit in her own seat. This happens with probability $\dfrac{1}{n+1}$.
- Person 1 sits in seat 1, and everyone gets to sit in his or her own seat. This happens with probability $\dfrac{1}{n+1}$.
- Person 1 site in a seat s where s is numbered from 2 to $k-1$. Just like problem 6, if person s is renamed person 1 and people 1 through $s-1$ and seats 2 through s are ignored, by the inductive assumption person k now gets to sit in her seat with probability $\dfrac{j+1}{j+2}$. Since there are $k-2$ seats in this range, this scenario happens with probability $\dfrac{k-2}{n+1}$.

Putting these pieces together, the probability that person k is sitting in her own seat is

$$\begin{aligned}\frac{j}{n+1}\cdot 1+\frac{1}{n+1}\cdot 0+\frac{1}{n+1}\cdot 1+\frac{k-2}{n+1}\frac{j+1}{j+2}\\ =\frac{1}{n+1}\left(j+1+\frac{(k-2)(j+1)}{j+2}\right)\\ =\frac{1}{n+1}\frac{(j+1)(j+2+k-2)}{j+2}\end{aligned}$$

Noting that $k=n+1-j$, this simplifies to $\dfrac{j+1}{j+2}$ as desired. Since the value of j was arbitrary and induction has now shown that the result holds for all values of k (and hence $n=j+k$) the proof is complete.

Alternate Solution: The following is a composite of several student solutions that are a lot more elegant than the above computation. From problem 3 person k cannot sit in seats 2 through $k-1$. So there are $k-2$ seats in which she never sits, leaving $n-k+2$ seats in which person k is allowed to sit. She will sit in her correct seat *unless* the first person ahead of her that chose a seat numbered either 1 or k through n specifically chose seat k. Since this choice is random and equally likely to choose any seat, the chance that seat k is chosen is $1/(n-k+2)$ leaving a probability of $\dfrac{n-k+1}{n-k+2}$ that seat k is available.

9. (a) Running through all possible values of s:

$s=0$: Consulting the table shows one case where no one sits in the correct seat, with probability 1/6.

$s=1$: There are two arrangements in the table with just one person in the correct seat; their total probability is 1/2.

$s=2$: This is impossible.

$s=3$: There is only one arrangement where everyone sits in the correct seat, and it has probability 1/3.

(b) The expected value of the number is the sum of the number of people in correct seats times the probability that this number of people are in their correct seats. In this case,

$$\sum_{s=0}^{3} s \cdot p(s) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + 2 \cdot 0 + 3 \cdot 13 = \boxed{\frac{3}{2}}.$$

10. (a) Repeating the above calculations using the table in problem 1:

$s = 0$**:** One arrangement with probability 1/24.
$s = 1$**:** Three arrangements with total probability 1/4.
$s = 2$**:** Three arrangements with total probability 11/24.
$s = 3$**:** Not possible.
$s = 4$**:** One arrangement with probability 1/4.

(b) This time the total is

$$0 \cdot \frac{1}{24} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{11}{24} + 3 \cdot 0 + 4 \cdot \frac{1}{4} = \boxed{\frac{13}{6}}.$$

11. If the expected value of the number of people in correct seats is denoted E_n, then either the explicit formula $\boxed{n - \sum_{k=1}^{n-1} \frac{1}{k} = n - H_{n-1}}$ (where H_m is the m^{th} *harmonic number*, $H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$) or the recursive formula $\boxed{E_n = \begin{cases} 1 & n = 1 \\ E_{n-1} + \frac{n-1}{n} & n > 1 \end{cases}}$ (or equivalents) are acceptable.

These formulas can be arrived at as follows. Clearly $E_1 = 1$ and it is fairly straightforward to compute $E_2 = 1$. The previous problems computed $E_3 = 3/2$ and $E_4 = 13/6$. Checking the differences between these, they grow by 0, then 1/2, then 2/3 so the recursive formula might be conjectured based on these samples. If the recursive formula is correct, it is straightforward to obtain the explicit formula since $E_n = E_{n-1} + \frac{n-2}{n-1} = E_{n-1} + 1 - \frac{1}{n-1} = E_{n-2} + 2 - \frac{1}{n-1} - \frac{1}{n-2} = \cdots$ until $E_1 = 1$ is reached.

Now to justify the recursion! Claim:

$$E_{n+1} = \frac{1}{n+1}\left(n + 1 + \sum_{k=0}^{n-1}\left(k + E_{n-k} - \frac{1}{n-k}\right)\right)$$

This follows by examining the possible seats where person 1 might sit. Each seat has probability $\frac{1}{n+1}$, explaining the fraction in front of the expression on the right-hand side. If person 1 sits in seat 1, then all $n+1$ people will sit in their correct seats, which is the $n+1$ term outside the sum on the right-hand side.

What is the expected number of people in correct seats if person 1 sits in one of the seats from 2 up to $n+1$? Assume he sits in seat $k+2$ (so that k ranges from 0 to $n-1$). Then persons 2 through $k+1$ (a total of k people) sit in their correct seats before person $k+2$ walks into the room and notices her seat is taken. That accounts for the k in the sum.

When person $k+2$ walks into the room, she sees seat 1 empty, seats 2 through $k+1$ (correctly) filled, and her seat $k+2$ incorrectly filled. The situation is just like the regular game played with $n-k$ people *except* that she can't sit in her own seat—she would have to sit in seat 1 instead. Normally if she sits in her own seat so will everyone else, while if she sits in someone else's seat, a chain of people will be displaced until someone else randomly sits in her seat. But in this case the chain will end when someone sits in seat 1. Either in the regular game or in this case, everyone in the chain is in the wrong seat and everyone not in the chain is in the right seat, so this doesn't affect the expected value. But in the case where person $k+2$ was supposed to sit in her own seat and instead had to sit in seat 1, only the remaining $n-k-1$ people sit in their own seats, not $n-k$ people because person $k+2$ is in seat 1 instead of her own seat. Thus, in computing the expected value for this situation, the number of people in correct seats is reduced by one in the case where everyone is supposed to sit in their own seat, which happens with probability $\dfrac{1}{n-k}$. So instead of E_{n-k} the expected number of additional people sitting in correct seats once person $k+2$ walks into the room is $E_{n-k}-\dfrac{1}{n-k}$. This accounts for the remainder of the terms in the sum.

Finally, the sum is manipulated into the recursion formula:

$$\begin{aligned}
E_{n+1} &= \frac{1}{n+1}\left(n+1+\sum_{k=0}^{n-1}\left(k+E_{n-k}-\frac{1}{n-k}\right)\right)\\
&= \frac{1}{n+1}\left(n+1+E_n-\frac{1}{n}+\sum_{k=1}^{n-1}\left(k+E_{n-k}-\frac{1}{n-k}\right)\right)\\
&= \frac{1}{n+1}\left(E_n+\frac{n-1}{n}+n+\sum_{k=0}^{n-2}\left(k+1+E_{n-1-k}-\frac{1}{n-1-k}\right)\right)\\
&= \frac{1}{n+1}\left(E_n+\frac{n-1}{n}+n+n-1+\sum_{k=0}^{n-2}\left(k+E_{n-1-k}-\frac{1}{n-1-k}\right)\right)\\
&= \frac{1}{n+1}\left(E_n+\frac{n-1}{n}+\frac{n(n-1)}{n}+\left[n+\sum_{k=0}^{n-2}\left(k+E_{n-1-k}-\frac{1}{n-1-k}\right)\right]\right)\\
&= \frac{1}{n+1}\left(E_n+\frac{n^2-1}{n}+nE_n\right)\\
&= \frac{1}{n+1}\left((n+1)E_n+\frac{(n+1)(n-1)}{n}\right)\\
&= E_n+\frac{n-1}{n}.
\end{aligned}$$

Alternate Solution: Many student solutions noted a much simpler approach. The probability that person 1 is in his own seat is $1/n$ while by problem 8 the probability for person k to be in her own seat, $k>1$, is $\dfrac{n-k+1}{n-k+2}$. The result now follows from linearity of expectation.

12. (a) Since person 1 has three choices and person 2 has two choices of where to sit, and then person 3 sits in whatever seat is still available, there are $3 \cdot 2 \cdot 1 = \boxed{6}$ possible seating arrangements.

(b) Of these six arrangements, each of which has equal probability of occurring, person 3 is in her seat for two of them, $(1, 2, 3)$ and $(2, 1, 3)$, leading to a probability of $2/6 = \boxed{1/3}$.

13. (a) If person 1 sits in seat 1 the remaining three people are in the $n = 3$, $a = 1$ case where there are four arrangements. Similarly if person 1 sits in seat 2 there are four arrangements possible. If person 1 sits in seat 3 then wherever person 2 sits (three choices), person 3 will have two choices, making six more arrangements. If person 1 sits in seat 4, there are four possibilities: $(2, 4, 3, 1)$, $(4, 2, 3, 1)$, $(3, 4, 2, 1)$, and $(4, 3, 2, 1)$. There are a total of $\boxed{18}$ possibilities.

(b) Person 3 gets to sit in her own seat unless person 1 or person 2 sat there. The probability is $3/4$ that person 1 did not sit there, and then a probability of $2/3$ that person 2 didn't sit there either, making the probability $\frac{3}{4} \cdot \frac{2}{3} = \frac{2}{4} = \frac{1}{2}$ that neither sat in seat 3, so a complementary probability of $\boxed{\frac{1}{2}}$ that person 3 can sit there.

(c) The arrangements where person 4 is in seat 4 are $(1, 2, 3, 4)$, $(2, 1, 3, 4)$, $(1, 3, 2, 4)$, $(3, 1, 2, 4)$, $(2, 3, 1, 4)$, and $(3, 2, 1, 4)$. These occur with probabilities $1/12$, $1/12$, $1/24$, $1/24$, $1/24$, and $1/24$ respectively, which add to $\boxed{\frac{1}{3}}$.

14. (a) There are $\binom{5}{2} = 10$ possibilities where persons 1 and 2 could sit. Count based on these possibilities, then double the result (because persons 1 and 2 could swap places). If persons 1 and 2 are in seats:

- 1 and 2: everyone else sits in her correct seat—**1** arrangement.
- 1 and 3: the remaining people are in the $n = 3$, $a = 1$ case (reassigning person 3 to sit in seat 2), which has **4** arrangements.
- 1 and 4: person 3 sits in her correct seat, leaving **2** choices for the others.
- 1 and 5: since persons 3 and 4 get their correct seat, this leads to only **1** arrangement.
- 2 and 3: like the case of 1 and 3, there are **4** arrangements.
- 2 and 4: like the case of 1 and 4, there are **2** arrangements.
- 2 and 5: like the case of 1 and 5, there is **1** arrangement.
- 3 and 4: the remaining people all sit randomly, so **6** arrangements.
- 3 and 5: if person 3 sits in seat 1 or seat 2, persons 4 and 5 will have no choices in their seats. But if person 3 sits in seat 4, person 4 will then choose between two seats. Altogether this gives **4** arrangements.
- 4 and 5: person 3 will sit in her own seat, and then person 4 will choose between the **2** remaining seats.

Adding the numbers in bold leads to 27 arrangements, which is doubled (see above) to yield a final total of $\boxed{54}$ seating arrangements.

(b) This is actually a little easier to count! Since the first three people choose randomly where they will sit, the only thing that matters is whether or not one of them sits in seat 4. If one does, then person 4 will choose one of the remaining two seats randomly. But if seat 4 is open, person 4 sits there and person 5 takes whatever seat is left. Of the $\binom{5}{3} = 10$ choices of where the first three people could sit, four of them miss seat 4 and six do not. So there are $6 \cdot 2 + 4 = 16$ possibilities for where persons 4 and 5 are sitting. For each of these, the first three people sit in their three seats in a random order, so six permutations. Thus, there are a total of $6 \cdot 16 = \boxed{96}$ possible seating arrangements.

15. Once person k enters the room, all lower-numbered people have already entered. Each person with a number lower than k and higher than a will sit in her own seat unless it is already taken. Thus all seats with numbers lower than k and higher than a are already taken by the time person k enters, meaning she must sit in a seat with a number either at least k or at most a.

16. Since part (a) is the special case of part (b) where $k = n$, only part (b) will be proven.

The strategy is similar to problem 8, where what matters is the person's position relative to the end of the line, not the beginning. So again let $j = n - k$ be the person's distance from the end of the line. Since $k = n - j > a$, we have $n > a + j$. So for any given j, the minimum possible number of people waiting to get into the room is $a + j + 1$. Let $c = n - a$, the number of clear-headed (i.e., not absent-minded) people. Note that c must be at least one since $c + a = n \geq k > a$.

Denote by $P(a, c, j)$ the probability that, given a absent-minded people followed by c clear-headed people, that person $k = n - j = a + c - j$ will sit in her own seat. The goal will be to prove that $P(a, c, j) = \dfrac{j+1}{j+a+1}$. Substituting $j = n - k$ then gives the desired result.

It should be clear that when $a = 0$, $P(0, c, j) = 1 = \dfrac{j+1}{j+a+1}$ since if there are no absent-minded people then everyone will sit in their assigned seat. It should also be clear that $j < c$ for $j = n - k = a + c - k$ and $k > a$. In fact, when $j = c - 1$ then $k - a = 1$ and person k is going to choose a seat immediately after all the absent-minded people choose their seats. In this case, a seats are taken at random, so the probability that person k gets her own seat is $\dfrac{n-a}{n}$. Thus, $P(a, c, c-1) = \dfrac{n-a}{n}$. Note that this equals $\dfrac{a+c-a}{a+c} = \dfrac{c}{a+c} = \dfrac{j+1}{j+a+1}$ as desired.

To compute $P(a, c, j)$, let person 1 randomly choose to sit in seat s.

With probability $\dfrac{a}{n}$, s will be less than or equal to a. Swap the names on seats 1 and s. So now person 1 is in the correct seat, there are $a - 1$ more absent-minded people to sit in $n - 1$ seats, so the probability that person k now gets to sit in her own seat is $P(a - 1, c, j)$.

With probability $\dfrac{k-1-a}{n}$, s will be larger than a but less than k. In this case, someone who wasn't absent-minded will become displaced from their seat, and that person will choose a random seat before person k gets to look for a seat. So essentially one seat has been taken away, but there are the same number of absent-minded people ahead of person k. The probability that person k will get her own seat in this case is $P(a, c-1, j)$.

With probability $\frac{1}{n}$, s will equal k. Then person k cannot sit in her own seat, and this case contributes zero to the overall probability.

Finally, with probability $\frac{j}{n}$, s will be greater than k. Person 1 and seat s are now eliminated leaving $n-1$ people of whom $a-1$ are absent-minded. There are still c clear-headed people, and person k has moved up one position in line, so k should be replaced by $k-1$. But since both n and k decrease by 1, j remains fixed. So the probability that (original) person k sits in the right seat will be $P(a-1, c, j)$ in this case.

Combining these yields $P(a, c, j) = \frac{a+j}{n}P(a-1, c, j) + \frac{k-1-a}{n}P(a, c-1, j)$.

Now perform a strong double-induction on a and c. In the base cases where $a = 0$ or $c = j+1$ the probability that person k was in the correct seat was $\frac{j+1}{j+a+1}$. So assume that $P(a, c, j) = \frac{j+1}{j+a+1}$ when either $a < A$ or $c < C$ for some A and C. The result is that

$$\begin{aligned} P(A, C, j) &= \frac{A+j}{n}P(A-1, C, j) + \frac{k-1-A}{n}P(A, C-1, j) \\ &= \frac{A+j}{n}\frac{j+1}{j+A-1+1} + \frac{k-1-A}{n}\frac{j+1}{j+A+1} \\ &= \frac{j+1}{n}\left(1 + \frac{k-1-A}{j+A+1}\right) \\ &= \frac{j+1}{n}\frac{j+k}{j+A+1} \end{aligned}$$

and noting that $n = j+k$ leaves $\frac{j+1}{j+A+1}$ as desired.

17. Let $c = n-a$. The proof will proceed by induction on c. Clearly when $c = 0$ there are $n = a$ people, all of whom are absent-minded, so all sit in seats chosen uniformly at random. Obviously there are $a!$ arrangements.

Claim: If $A = (p_1, p_2, \ldots, p_n)$ is a possible arrangement of $n = a+c$ where the first a are absent-minded, there are exactly $a+1$ possible arrangements of $n+1$ people that can be obtained from it by adding one more person, p_{n+1}. These arrangements can be obtained from A by appending p_{n+1} to the end of A, or by placing p_{n+1} in one of the first a positions and moving the person currently in that position to the end of the arrangement. The $a+1$ arrangements so obtained are all distinct from each other, and are distinct from any arrangement obtained by applying the same operation to any other possible arrangement of n people, and all possible arrangements are obtainable in this way.

The claim says that any possible arrangement can be built up one person at a time from exactly one possible arrangement of fewer people, and the building process can be done in exactly one way. According to the claim there are $a+1$ arrangements of $a+c+1$ people obtainable from arrangements of $a+c$ people. So the number of possible arrangements is multiplied by $a+1$ each time a person is added. So using the $a!$ arrangements of the first a people, $c=0$, as a base case, simple induction on c shows that the formula for the number of possible arrangements is correct if there are $n=a+c$ people in total.

For example, if $a=3$, one possible arrangement of seven people would be $(7,5,2,4,3,1,6)$. This happened because persons 1, 2, and 3 came in and sat randomly in seats 6, 3, and 5 respectively. When person 4 came in, her seat was available so she took it. Then person 5's seat was unavailable, so she randomly sat in seat 2. Person 6 then randomly sat in seat 7, and person 7 had to take the only available seat, seat 1. According to the claim, the following arrangements are all possible for $a=3$ with eight people: $(7,5,2,4,3,1,6,8)$, $(8,5,2,4,3,1,6,7)$, $(7,8,2,4,3,1,6,5)$, and $(7,5,8,4,3,1,6,2)$. Checking that these are possible arrangements is routine; clearly they are all distinct from each other. It is not quite as obvious that none of these could be obtained by starting with a different possible arrangement of seven people, nor is it obvious that *every* possible arrangement of eight people can be obtained by adding an eighth person to some possible arrangement of seven people in this way. Those are the contents of the claim.

There are several parts to the claim. Going through them methodically:

Let $A=(p_1,p_2,\ldots,p_a,p_{a+1},\ldots,p_{a+c})$ be a possible seating arrangement of $n=a+c$ people. Then if there are $n+1$ people, the first n can certainly sit in the seats shown by arrangement A as long as none of them randomly choose to sit in seat $n+1$. In this case, person $n+1$ sits in seat $n+1$ and that is the first of the arrangements indicated in the claim. On the other hand, anyone sitting in seat s where $s\le a$ chose to do so at random, because those seats belong to the absent minded-people, and the only time someone does not have a choice is if she is not absent-minded and her seat is available. So if the person sitting in seat s had instead chosen to sit in seat $n+1$ and everyone between that person and person $n+1$ makes the same choice as they did in forming arrangement A, then seat s will be the only seat available to person $n+1$, so they will perforce sit there, obtaining the other arrangements indicated by the claim. So each arrangement of n people leads to $a+1$ arrangements of $n+1$ people.

These $a+1$ arrangements are clearly distinct from each other, because person $n+1$ is in a different seat in each one.

If A' is an arrangement of $n+1$ people obtained from arrangement A, and B' is obtained from arrangement B, then A' and B' can only be equal if A and B were the same to start with. This is because for A' and B' to be equal, person $n+1$ must be in the same seat in both. If this is the last seat in both, then A and B were originally just $A'=B'$ without the last entry. If, instead, this is seat s with $1\le s\le a$ then A' and B' can only be equal if they have the same person sitting in the last chair. But then A is just A' with this person moved from chair $n+1$ to chair s and then the last chair being taken away. The same is true of B. All other people in these arrangements must be in the same chairs in both A and B, so A and B must be equal. Thus, arrangements coming from different arrangements of n people lead to different arrangements of $n+1$ people.

Finally, it must be shown that all possible arrangements of $n+1$ people can be obtained from some arrangement of n people by the method of the claim. So let A' be a possible arrangement of $n+1$ people. By problem 15, person $n+1$ can only be in seat s where $1 \le s \le a$ or $s = n+1$. Clearly if $s = n+1$ then all of the first n people chose or were required to sit in the first n seats, so A obtained from A' by deleting the last entry (representing that person $s+1$ is sitting in seat $s+1$) is a possible arrangement of n people, and one that can be expanded as in the claim to obtain A'.

On the other hand, if $1 \le s \le a$ let the person who is sitting in seat $n+1$ be person p. It must be shown that the arrangement A obtained from A' by substituting p instead of p_{n+1} in seat s and eliminating seat $n+1$ is a possible seating arrangement of n people. But this is certainly the case, because seat s was available to person $n+1$, so it would certainly have been available earlier to person p when they chose to sit in seat $n+1$. If they had chosen to sit in seat s, and everyone else makes the same choices as they made to form A', then just before person $n+1$ comes into the room the first n people are sitting in arrangement A.

Author's Note: The author does not know of any practical applications of this problem. He just thought it was a curious result. A recent article on this problem in The American Mathematical Monthly uses statistical methods and random variables to get at some of these questions in more depth. The article is: Norbert Henze and Günter Last, *Absent-Minded Passengers*, The American Mathematical Monthly, 126.10, December 2019. The article has references to other sources on the problem and its extensions.

One possible extension is: what if the absent-minded people are not grouped together at the beginning of the group? It should not be too hard to show that for any given clear-headed person, all that matters is the number of absent-minded people who preceded her into the room, not whether they were all grouped together or not.

And in case you were wondering if there was any relation between the counting ideas related to this problem, where an $(a+1)^{n-a}$ appears, and the parking sequences problem from last year, where an $(a+1)^{a-1}$ appears, well, I think you already know the answer to that ....

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