PUMaC

2025 Algebra A

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. Find the sum of all real values of x satisfying

$$\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}} = x - \sqrt{x-\sqrt{x-\sqrt{x-\cdots}}}.$$

Proposed by Andrew Li

Solution: Let $y = \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}} = x - \sqrt{x-\sqrt{x-\sqrt{x-\cdots}}}$. Then we have

$$y = \sqrt{x+y} \quad \text{and} \quad x - y = \sqrt{y}$$

Turning each equation into a quadratic and solving for y gives

$$y = \frac{1 \pm \sqrt{1+4x}}{2} \quad \text{and} \quad y = \frac{2x+1 \pm \sqrt{1+4x}}{2}$$

Since x must be non-negative, we need only consider the cases

$$y = \frac{1+\sqrt{1+4x}}{2} = \frac{2x+1+\sqrt{1+4x}}{2}$$

and

$$y = \frac{1+\sqrt{1+4x}}{2} = \frac{2x+1-\sqrt{1+4x}}{2}$$

Solving for x gives the solutions $2-\sqrt{5}, 0, 2+\sqrt{5}$ but the solution $2-\sqrt{5}$ is extraneous. Thus, our final answer is $\boxed{2+\sqrt{5}}$.

2. Let $\star$ be an associative operation on the set of positive real numbers that satisfies

$$\underbrace{x \star x \star \cdots \star x}_{x \text{ appears } k \text{ times}} = \frac{x}{\sqrt{k}}$$

for any positive real number x and positive integer k. Let s_n denote the nth smallest positive integer with no prime factor greater than 5. Compute $s_1 \star s_2 \star s_3 \star \cdots$.

Proposed by David Ji

Solution: We have that

$$\sqrt{m}\star\sqrt{n} = (\underbrace{\sqrt{mn}\star\sqrt{mn}\star\cdots\star\sqrt{mn}}_{n \text{ times}})\star(\underbrace{\sqrt{mn}\star\sqrt{mn}\star\cdots\star\sqrt{mn}}_{m \text{ times}}) = \underbrace{\sqrt{mn}\star\sqrt{mn}\star\cdots\star\sqrt{mn}}_{m+n \text{ times}} = \frac{\sqrt{mn}}{\sqrt{m+n}}$$

which can be rearranged to

$$\left(\frac{1}{\sqrt{m}\star\sqrt{n}}\right)^2 = \left(\frac{1}{\sqrt{m}}\right)^2 + \left(\frac{1}{\sqrt{n}}\right)^2$$

Thus,

$$\left(\frac{1}{s_1 \star s_2 \star s_3 \star \cdots}\right)^2 = \left(\frac{1}{s_1}\right)^2 + \left(\frac{1}{s_2}\right)^2 + \left(\frac{1}{s_3}\right)^2 + \dots$$
$$= \left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \dots\right)\left(1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \dots\right)\left(1 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^4 + \dots\right)$$
$$= \frac{1}{1-\frac{1}{4}} \cdot \frac{1}{1-\frac{1}{9}} \cdot \frac{1}{1-\frac{1}{25}} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} = \frac{25}{16}$$

$$\implies s_1 \star s_2 \star s_3 \cdots = \boxed{\frac{4}{5}}$$

3. Find the largest possible value of
$$\frac{-2x+6y}{1+2x^2+3y^2}$$
over all real numbers x, y.

Proposed by Rohit Agarwal and David Ji

Solution: By Cauchy Schwarz, we have that

$$(-2x+6y)^2 \le \left((-\sqrt{2})^2 + (2\sqrt{3})^2\right)\left((\sqrt{2}\cdot x)^2 + (\sqrt{3}\cdot x)^2\right) = 14(2x^2+3y^2)$$

Let $z = 2x^2 + 3y^2$. Then

$$\frac{-2x+6y}{1+2x^2+3y^2} \le \frac{\sqrt{14z}}{1+z} = \frac{\sqrt{14}}{\sqrt{z}+\frac{1}{\sqrt{z}}} \le \boxed{\frac{\sqrt{14}}{2}}$$

by AM-GM. The equality conditions $(x, y) = (\frac{\sqrt{14}}{14}, -\frac{\sqrt{14}}{7})$ can be constructed by considering the equality cases of AM-GM and Cauchy-Schwarz.

4. Let
$$S = \sum_{j=1}^{2025} \binom{4050}{2025+j} \cos(3j).$$
What is the largest positive integer n such that 2^n divides $\lfloor S \rfloor$?

Proposed by David Ji

Solution: Let $\omega = \cos 1.5 + i \sin 1.5$. We have that

$$\begin{aligned}
\sum_{j=1}^{2025} \binom{4050}{2025+j} \cos(3j) &= \sum_{j=1}^{2025} \binom{4050}{2025+j} \frac{\omega^{2j} + \overline{\omega}^{2j}}{2} \\
&= \frac{1}{2} \sum_{k=0}^{2024} \binom{4050}{k} \omega^{2k-4050} + \frac{1}{2} \sum_{k=2026}^{4050} \binom{4050}{k} \omega^{2k-4050} \\
&= \frac{1}{2} \left[\sum_{k=0}^{4050} \binom{4050}{k} \omega^k \overline{\omega}^{4050-k} \right] - \frac{1}{2} \binom{4050}{2025} \\
&= \frac{1}{2} (\omega + \overline{\omega})^{4050} - \frac{1}{2} \binom{4050}{2025} \\
&= \frac{1}{2} (2\cos 1.5)^{4050} - \frac{1}{2} \binom{4050}{2025}
\end{aligned}$$

Thus, $\lfloor S \rfloor = -\frac{1}{2}\binom{4050}{2025}$ and so $n = \boxed{7}$ by Kummer's Theorem.

5. Let $f(x) = 2x^2 - 1$. Find the smallest positive integer N such that the median of the real roots of the polynomial $\underbrace{(f \circ f \circ f \circ \ldots f \circ f)}_{f \text{ composed } N \text{ times}}(x) - x$ has absolute value at most 2^{-2025}.

Proposed by Rohit Agarwal, David Ji, and Atharva Pathak

Solution: Let $f^k(x)$ denote the function f applied k times. Note that all roots of the polynomial must fall within the interval $[-1, 1]$ because otherwise $|f(x)| > |x| \implies |f^N(x)| > |x|$. Then we can let $x = \cos\theta$ and we have that

$$f(\cos\theta) = \cos 2\theta \implies f^N(\cos\theta) = \cos(2^N\theta) \implies (2^N - 1)\theta = 2k\pi \quad \text{or} \quad (2^N + 1)\theta = 2k\pi \quad \text{where } k \in \mathbb{Z}$$

Thus, the 2^N roots of the polynomial $f^N(x) = 1$ are

$$1, \cos\left(\frac{2\pi}{2^N - 1}\right), \cos\left(\frac{4\pi}{2^N - 1}\right), \ldots, \cos\left(\frac{(2^N - 2)\pi}{2^N - 1}\right),$$
$$\cos\left(\frac{2\pi}{2^N + 1}\right), \cos\left(\frac{4\pi}{2^N + 1}\right), \ldots, \cos\left(\frac{2^N\pi}{2^N + 1}\right)$$

By symmetry, the two middle elements are going to be the ones which are closest to $\cos\frac{\pi}{2} = 0$. These happen to be

$$\cos\left(\frac{2^{N-1}\pi}{2^N - 1}\right) = -\sin\left(\frac{\pi}{2(2^N - 1)}\right) \quad \text{and} \quad \cos\left(\frac{2^{N-1}\pi}{2^N + 1}\right) = \sin\left(\frac{\pi}{2(2^N + 1)}\right)$$

Then the absolute value of the median is

$$\frac{1}{2}\left(\sin\left(\frac{\pi}{2(2^N - 1)}\right) - \sin\left(\frac{\pi}{2(2^N + 1)}\right)\right) \approx \frac{1}{2}\left(\frac{\pi}{2(2^N - 1)} - \frac{\pi}{2(2^N + 1)}\right) = \frac{\pi}{2(2^{2N} - 1)} \approx \frac{\pi}{2^{2N+1}}$$

Thus, we compute the minimal such N to be $\boxed{1013}$.

6. Let c be a real number, and let $f(t) = (t - 0.01) + (t^2 - c)i$ for every real number t. Suppose that $m(t)$ is a (not necessarily continuous) function from $\mathbb{R}$ to $\mathbb{C}$ such that

$$m(t)^2 + 1 + \frac{1}{m(t)^2} = (t-1)(t+1)$$

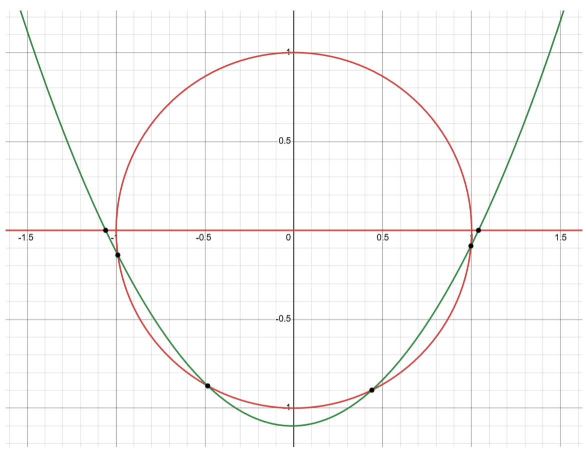
for every real number t. Over all such functions $f(t)$ and $m(t)$, what is the maximum possible number of distinct ordered pairs of real numbers (a, b) such that $f(a) = m(b)$?

Proposed by Austen Mazenko

Solution: Rearranging gives

$$\left(m(t) + \frac{1}{m(t)}\right)^2 = t^2 \implies m(t) + \frac{1}{m(t)} = t \quad \text{or} \quad m(t) + \frac{1}{m(t)} = -t \implies m(t) = \frac{\pm t \pm \sqrt{t^2-4}}{2}$$

Consider the branch $m(t) = \frac{t+\sqrt{t^2-4}}{2}$ first. If we think about $m(t)$ as a parametric equation in the complex plane, for $t \leq -2$ we have a line at $\Im(z) = 0$ from $z = 0$ to $z = -1$, for $-2 < t < 2$ we have a semicircular arc above the imaginary axis from $z = -1$ to $z = 1$, and for $t \geq 2$ we have a line at $\Im(z) = 0$ from $z = 1$ to $z = \infty$. The other four possible branches are reflections across the real and imaginary axes. Graphing $f(t)$ and $m(t)$ as parametric equations on the complex plane yields the following graph, where each black point can be achieved two times by two different branches under different parameter values.



Thus, the greatest possible number of distinct ordered pairs (a, b) is $2 \times 6 = \boxed{12}$

7. Let

$$\omega = e^{i\pi/2025} = \cos\left(\frac{\pi}{2025}\right) + i \sin\left(\frac{\pi}{2025}\right).$$

David writes the complex numbers $2, \omega, \omega^2, \ldots, \omega^{4049}$ in a line going from left to right on a blackboard. Then, each minute, if the two leftmost complex numbers on the blackboard are z_1 and z_2, David erases the two numbers and replaces them with

$$\frac{1}{z_1 + z_2} - \frac{1}{\frac{1}{z_1} + \frac{1}{z_2}}.$$

After 4049 minutes, there is only one number remaining. What is the last remaining number?

Proposed by David Ji

Solution: We have that

$$\frac{1}{z_1+z_2}-\frac{1}{\frac{1}{z_1}+\frac{1}{z_2}}=\frac{1-z_1z_2}{z_1+z_2}$$

which is the reciprocal of the tangent angle addition formula. Let $z_1 * z_2$ denote the replacement of z_1 and z_2. Note that for any complex numbers z and z', we have that

$$-\frac{i-z_1*z_2}{i+z_1*z_2}=\frac{1}{\frac{i-z_1}{i+z_1}\cdot\frac{i-z_2}{i+z_2}}$$

This invariant can be discovered by considering the extension of the arctangent function to the complex numbers, $\arctan z=-\frac{i}{2}\ln\left(\frac{i-z}{i+z}\right)$. Thus, if $Z=2*\omega*\omega^2*\cdots*\omega^{4049}$ we have that

$$\begin{aligned}
-\frac{i+Z}{i-Z}&=\left(\frac{i-2}{i+1}\right)\left(\frac{i-\omega}{i+\omega}\right)\left(\frac{i+\omega^2}{i-\omega^2}\right)\left(\frac{i-\omega^3}{i+\omega^3}\right)\left(\frac{i+\omega^4}{i-\omega^4}\right)\left(\frac{i-\omega^5}{i+\omega^5}\right)\cdots\\
&=\frac{\left(\frac{i-2}{i+2}\right)\left(\frac{i-1}{i+1}\right)\prod_{k=0}^{4049}\frac{i-w^k}{i+w^k}}{\left(\prod_{k=0}^{2024}\frac{i-w^{2k}}{i+w^{2k}}\right)^2}=\frac{\left(\frac{i-2}{i+2}\right)\left(\frac{i-1}{i+1}\right)\left(\frac{i^{4050}-1}{(-i)^{4050}-1}\right)}{\left(\frac{i^{2025}-1}{(-i)^{2025}-1}\right)^2}\\
&=\frac{(i-2)(i+1)}{(i+2)(i-1)}=\frac{-3-i}{-3+i}\\
\implies Z&=\boxed{3}
\end{aligned}$$

8. Let S be the set of ordered triples of positive integers (x,y,z) such that $y(x+y+z)=zx$ and

$$x^2+y^2+z^2+xy+yz+zx=2\frac{(x+y)(y+z)(z+x)}{x+y+z}+145.$$

Find the sum of $100x+10y+z$ over all $(x,y,z)\in S$.

Proposed by David Ji

Solution: The first equation is equivalent to

$$(x+y)^2+(y+z)^2=(x+z)^2$$

If we let $a=x+y$, $b=y+z$, and $c=z+x$ be the side lengths of a right triangle, we have that

$$x^2+y^2+z^2+xy+yz+zx=\frac{1}{2}a^2+\frac{1}{2}b^2+\frac{1}{2}c^2=c^2=4R^2$$

where R is the circumradius of the triangle. Then

$$\frac{(x+y)(y+z)(z+x)}{x+y+z}=\frac{abc}{s}=4\cdot\frac{abc}{4K}\cdot\frac{K}{s}=4Rr$$

where s is the semiperimeter, K is the area, R is the circumradius, and r is the inradius of the triangle.
In this geometric interpretation, the second equation translates to

$$4R^2=8Rr+145 \implies 4d^2=145$$

where d is the distance between the incenter and circumcenter of the triangle. Coordinate geometry now yields the equation

$$(x-y)^2 + (z-y)^2 = 145$$

which results in the cases $1^2 + 12^2 = 145$ and $8^2 + 9^2 = 145$.
The case $1^2 + 12^2 = 145$ does not give integer solutions, but the case $8^2 + 9^2 = 145$ gives the triples $(14, 6, 15)$ and $(15, 6, 14)$. Thus, our final answer is $1400+60+15+1500+60+14 = \boxed{3049}$.