Algebra A

1. Let
$$a_k = 0.\overbrace{0\ldots0}^{k-1\ 0's}1\overbrace{0\ldots0}^{k-1\ 0's}1$$
The value of $\sum_{k=1}^{\infty} a_k$ can be expressed as a rational number $\frac{p}{q}$ in simplest form. Find $p+q$.

2. If $a_1, a_2, \ldots$ is a sequence of real numbers such that for all n,
$$\sum_{k=1}^{n} a_k \left(\frac{k}{n}\right)^2 = 1,$$
find the smallest n such that $a_n < \frac{1}{2018}$.

3. Let $x_0, x_1, \ldots$ be a sequence of real numbers such that $x_n = \frac{1+x_{n-1}}{x_{n-2}}$ for $n \geq 2$. Find the number of ordered pairs of positive integers (x_0, x_1) such that the sequence gives $x_{2018} = \frac{1}{1000}$.

4. Suppose real numbers a, b, c, d satisfy $a+b+c+d = 17$ and $ab+bc+cd+da = 46$. If the minimum possible value of $a^2+b^2+c^2+d^2$ can be expressed as a rational number $\frac{p}{q}$ in simplest form, find $p+q$.

5. For $k \in \{0, 1, \ldots, 9\}$, let $\epsilon_k \in \{-1, 1\}$. If the minimum possible value of $\sum_{i=1}^{9} \sum_{j=0}^{i-1} \epsilon_i \epsilon_j 2^{i+j}$ is m, find $|m|$.

6. Let a, b, c be non-zero real numbers that satisfy $\frac{1}{abc} + \frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. The expression $\frac{4}{a^2+1} + \frac{4}{b^2+1} + \frac{7}{c^2+1}$ has a maximum value M. Find the sum of the numerator and denominator of the reduced form of M.

7. Let the sequence $\{a_n\}_{n=-2}^{\infty}$ satisfy $a_{-1} = a_{-2} = 0$, $a_0 = 1$, and for all non-negative integers n,
$$n^2 = \sum_{k=0}^{n} a_{n-k}a_{k-1} + \sum_{k=0}^{n} a_{n-k}a_{k-2}.$$
Given a_{2018} is rational, find the maximum integer m such that 2^m divides the denominator of the reduced form of a_{2018}.

8. $$\frac{p}{q} = \sum_{n=1}^{\infty} \frac{1}{2^{n+6}} \frac{(10 - 4\cos^2(\frac{\pi n}{24}))(1-(-1)^n) - 3\cos(\frac{\pi n}{24})(1+(-1)^n)}{25 - 16\cos^2(\frac{\pi n}{24})}$$
where p and q are relatively prime positive integers. Find $p+q$. where p and q are relatively prime positive integers. Find $p+q$.