Set 1

1. [**5**] Suppose real numbers x, y, z satisfy

$$x + y + 3z = 7$$
$$2x + 2y + 5z = 12$$

What is the value of z?

Proposed by: Mason Yu

Answer: $\boxed{2}$

Solution: Multiplying the first equation by 2, we have that $2x+2y+6z = 14$. We then subtract the second equation from the first, yielding that $2x + 2y + 6z - (2x + 2y + 5z) = 14 - 12$, or $z = \boxed{2}$.

2. [**5**] Tom has three straws of lengths 5, 6, 7, 8, and 9 inches, for a total of 15 straws. How many of his straws are shorter than at least half of all his other straws?

Proposed by: Mason Yu

Answer: $\boxed{9}$

Solution: Tom has $3 \times 5 = 15$ straws, so he can't have any straw shorter than $\frac{14}{2} = 7$ straws. Straws of length 5 and 6 inches are shorter than at least 3×3 straws, while straws of length and above 7 are shorter than at most $3 \times 2 = 6$ straws. Thus, only any straw of length 7 inches and above suffices, which is $3 \times 3 = \boxed{9}$.

3. [**5**] Kathy and Grace are taking a 3 hour train ride. Kathy is asleep for exactly 2 hours of the train ride, and Grace is asleep for exactly 1 hour. If there are exactly 30 minutes of the train ride where both are awake, for how many minutes is exactly one of them awake?

Proposed by: Mason Yu

Answer: $\boxed{2}$

Solution: Kathy is awake for 1 hour and Grace is awake for 2 hours. 30 minutes of these are shared between them. So there are 2 hours and 30 minutes total where at least one of them is awake. Subtract the 30 minutes shared between them and we have 2 hours, or $\boxed{120}$ minutes where exactly one of them is awake.

. . . .

Set 2

4. [**6**] An is putting stickers on her laptop. She has four different stickers and wants to put at least one, but not all, on her laptop. How many ways can she do this? The order in which she places the stickers does not matter.

Proposed by: Mason Yu

Answer: $\boxed{14}$

Solution: She can choose to put any sticker on her laptop or not, giving her 2 choices for each sticker. So our answer is $2^4 = 16$, but we have to subtract 2 to get $\boxed{14}$ since she can't put all the stickers or none of the stickers on.

5. [**6**] Angelina has 100 flowers, each of which is red, pink, or purple. She creates 10 bouquets containing 10 flowers each. Each bouquet must have at least 5 purple flowers, and more pink than red flowers. What is the greatest number of red flowers she can have in total?

   Proposed by: Mason Yu

   Answer: $\boxed{20}$

   Solution: Note that for each bouquet, we want to maximize the number of red flowers, meaning we minimize the number of pink and purple flowers. So if each bouquet has 5 purple flowers, then each bouquet must have at least 3 pink flowers and at most 2 red flowers. We pick this maximal number of red flowers for all 10 bouquets, yielding that she can have $\boxed{20}$ red flowers in total.

6. [**6**] Adeethyia wrote down a two-digit integer greater than 60, doubled it, subtracted 41, and got a two-digit positive integer with both digits identical. What is his number?

   Proposed by: Mason Yu

   Answer: $\boxed{70}$

   Solution: Let his integer be x. Then, $2x - 41$ must be a multiple of 11. Since we know that x is greater than 60, the expression must be greater than 79 and thus, must equal 99. Solving for x, we have that $2x - 41 = 99$, so $2x = 140$ and $x = \boxed{70}$.

. . . .

Set 3

7. [**7**] A fraction in simplified form, with value less than 1, has its numerator and denominator sum to 36. The numerator and denominator are both positive integers. How many possible values are there for the fraction?

   Proposed by: Mason Yu

   Answer: $\boxed{6}$

   Solution: Note that any integer less than 18 that is relatively prime to 36 will suffice. These can be enumerated as $1, 5, 7, 11, 13, 17$, of which there are $\boxed{6}$. (This is also just $\phi(18) = 6$).

8. [**7**] Amy has 40 mittens, 14 of which are brown, 16 are red, and 10 are gold. She selects at random in the dark. What is the smallest number of mittens Amy must select to guarantee 2 matching pairs of different colors?

   Proposed by: Amy Qiao

   Answer: $\boxed{19}$

   Solution: If Amy wants 2 pairs of 2 distinct colors, the worst case is that she exhausts all 16 red mittens before drawing a pair of a different color. After selecting all red mittens, Amy must draw either a brown or a gold mitten. The worst case is that she draws one of each of brown and gold before drawing another matching mitten. Therefore, she must draw $16 + 2 + 1 = \boxed{19}$ mittens to guarantee at least 2 matching pairs of 2 distinct colors.

9. [**7**] The three longest diagonals in a regular hexagon of side length 1 are rearranged to form the sides of a triangle. Then, the six remaining diagonals are rearranged to form the sides of a new regular hexagon. What is the ratio of the area of the triangle to the area of the new hexagon?

   Proposed by: Jerry Sun

   Answer: $\boxed{\dfrac{2}{9}}$

   Solution: Let s be the area of the original hexagon. By doubling a side of the original hexagon, we can construct the triangle and see that it has area $\frac{4}{6}s = \frac{2}{3}s$. The new hexagon has side length $\sqrt{3}$ by special right triangles, and so has area $3s$. Hence, the desired ratio is $\frac{\frac{2}{3}}{3} = \boxed{\dfrac{2}{9}}$.

Set 4

10. [**8**] Let m be the nearest integer multiple of 101 to 10,000,000,000 and n be the nearest integer multiple of 99 to 10,000,000,000. Find the difference $|m - n|$.

    Proposed by: Sebastian Weinberger

    Answer: $\boxed{2}$

    Solution: By inspection, $m = 10{,}100{,}000{,}000 - 101{,}000{,}000 + 1{,}010{,}000 - 10{,}100 + 101 = 10{,}000{,}000{,}001$ and $n = 9{,}999{,}999{,}999$ such that their difference is $\boxed{2}$.

11. [**8**] Define the symbol

    $$a \oplus b = (a \bmod b) + (b \bmod a),$$

    where $a \bmod b$ is the remainder of a when divided by b. What is the value of $2 \oplus 3 \oplus \cdots \oplus 2026$, evaluated from left to right?

    Proposed by: An Cao

    Answer: $\boxed{2026}$

    Solution: Note that $a \oplus (a+1) = (a \bmod (a+1)) + ((a+1) \bmod a) = a + 1$. Then by induction, $1 \oplus 2 \oplus \cdots \oplus n = n$. So the answer is $\boxed{2026}$.

12. [**8**] Two regular tetrahedra (triangular pyramids) of equal size are placed adjacent edge-to-edge on a flat surface. If their edges have length 1, what is the distance between their top vertices?

    Proposed by: Ethan Brady

    Answer: $\boxed{\dfrac{1}{\sqrt{3}}}$

    Solution: Project the top vertices to the base of the tetrahedra. The top vertices are the same distance away as these projected points, and these projected points are exactly the centers of the base equilateral triangles. Now working in 2D, we use the properties of 30-60-90 triangles to find the distance from a center to an edge is $\frac{1}{2\sqrt{3}}$, so the distance between the centers is $\boxed{\dfrac{1}{\sqrt{3}}}$.

Set 5

13. [**9**] A positive integer's odd divisors (including 1) add up to an odd positive integer n. What is the sum of the four smallest possible values of n?

Proposed by: Mason Yu

Answer: $\boxed{102}$

Solution: The positive integers must have an odd number of odd factors, so their greatest odd factor must be a perfect square. The sums of the factors of the four smallest odd perfect squares are $1+1+3+9+1+5+25+1+7+49=\boxed{102}$.

14. [**9**] You have a row of 20 balls: two labeled 1, two labeled 2, up to two labeled 10, arranged in ascending order (1, 1, 2, 2, ..., 10, 10). Every turn, you can swap two adjacent balls. What is the minimum number of turns you need to take to rearrange the 20 balls in decreasing order?

Proposed by: Jonathan Xu

Answer: $\boxed{180}$

Solution: A inversion is a pair of balls a and b such that a is labeled by a smaller number than b, but a is placed before b. In the initial state, the first ball labeled 1 is in an inversion with every ball after it except the second ball labeled 1. The second ball is in an inversion with every ball after it. Similarly for the balls labeled 2, and so on, so that the total number of inversions is

$$18+18+16+16+14+14+\cdots+2+2+0+0=4(9+8+7+\cdots+1+0)=4\cdot 45=180.$$

In the final state, since the balls are in decreasing order, the number of inversions is 0.

We claim that the minimal number of turns is 180. To see that the number of turns is at least 180, observe that in each turn, by swapping a pair of balls a and b, only the relative ordering of a and b is affected. Therefore, only the status of whether a and b are in an inversion can change, so the number of inversions can decrease by at most 1. Since we start with 180 inversions and end with 0, we need at least 180 turns.

To see that 180 is achievable, consider swapping the second 1 to the last position in 18 steps, then swapping the first 1 to the second to last position in 18 steps, then swapping the second 2 to the third to last position in 16 steps, then the first 2 to the fourth to last position in 16 steps, and so on. This process sends every ball to its position in the final ordering with a total of

$$18+18+16+16+\cdots+2+2+0+0=\boxed{180}$$

steps as desired.

15. [**9**] Cayley wants to travel to a point A which is 100 meters away from her current location. Unfortunately, Cayley is a crab, so she can only travel in semicircular steps of arc length 1. What is the least total integer distance Cayley can travel, in meters, to get to point A?

Proposed by: Mason Yu

Answer: $\boxed{158}$

Solution: Notice that the distance that Cayley can travel after 1 step is $\frac{2}{\pi}$, so the maximum distance that she can travel after n steps is $\frac{2n}{\pi}$. So $\frac{2n}{\pi}\geq 100$ therefore $n>50\pi$ so $n=168$ is the least possible number of steps she can take. So the least distance she can travel is $\boxed{158}$ meters.

Set 6

16. [**10**] Jenny has 4 different colored crayons and a 16×16 unit grid. On the grid, she colors each edge of length 1 with a random color. Estimate the number of (not necessarily 1×1) squares, on average, with edges in the grid that have all four sides the same color. Express your answer as a base 10 decimal.

    If you are within 5% of the right answer, you get 10 points. Every additional 2% error deducts one point (minimum zero points).

    Proposed by: Mason Yu

    Answer: $\boxed{4.0137}$

    Solution: Use linearity of expectation and break up cases into square side lengths.

    For 1×1 squares, the expected number is $16^2 \cdot \frac{1}{4^3} = 4$ since for any individual square the probability that the sides are all the same color is $\frac{1}{4^3}$.

    Analogously for 2×2 squares this is $15^2 \cdot \frac{1}{4^7} = \frac{225}{16384} \approx 0.0137$.

    All terms grow exponentially smaller and can be ignored. So our answer is approximately 4.0137.

17. [**10**] Richard and Yifan take turns picking balls out of a bag without replacement, with Richard drawing first. Richard wins if he picks a green ball before Yifan picks a purple ball. Suppose there are 2 green, 1 purple, and 999 yellow balls in the box. Estimate the probability that Richard wins. Express your answer as a base 10 decimal.

    If you are within 5% of the right answer, you get 10 points. Every additional 2% error deducts one point (minimum zero points).

    Proposed by: Mason Yu

    Answer: $\boxed{0.5838}$

    Solution: Note that because the number of yellow balls is large, we can estimate the first draw of a non-yellow-ball as randomly selected between Richard and Yifan.

    Suppose it's selected by Richard. Then, the probability that it's green is $\frac{2}{3}$, in which case Richard wins. In the other $\frac{1}{3}$ case, Yifan has to select both green balls, which has probability approximately $\frac{1}{4}$. So in this case, Richard has probability $\frac{11}{12}$ of winning.

    Now, suppose the first draw of a non-yellow ball is selected by Yifan. Then, the probability that it's purple is $\frac{1}{3}$, in which case Yifan wins. Otherwise, if Yifan first selects a green ball, there is one of each ball left. So once this case happens there is $\frac{1}{4}$ probability that Yifan and Richard both select each other's balls, in which there's no winner, and thus since we're estimating the probability that they win as equal, there's $\frac{3}{8}$ probability that that Richard wins. So then the probability that Richard wins if Yifan draws the first ball is $\frac{2}{3} \cdot \frac{3}{8} = \frac{1}{4}$.

    Since we estimated the probabilities of drawing the first ball as equal, the estimated probability is $\frac{11}{12} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{7}{12}$ which is sufficient for the points.

    In fact, the exact solution can be derived via a recurrence with the explicit formula $\frac{7y+20}{12(y+2)}$ which holds for odd y.

18. [**10**] Determine whether each of the following statements are true or false.

1. $6 \times 7 > 67$
2. $67 \times 69 > 68^2$
3. $6767 > 67 \times 67 + 6 \times 7 \times 6 \times 7$
4. $6^7 > 7^6$
5. $67^{76} > 76^{67}$
6. $10^{225} > 225^{100}$
7. $10^{250} > 250^{100}$
8. $101^{102^{103}} > 103^{102^{101}}$
9. $67! \cdot 68! \cdot 69! > 180!$
10. $100^{250!} > 250^{100^{250}}$

Submit a 10-tuple of letters corresponding to the answers above (where T denotes that the inequality is true, F denotes that it is false, and X denotes that you wish to leave the question blank).

Your score is the number of correct answers minus the number of incorrect answers (minimum zero points).

Proposed by: Mason Yu

Answer: $\boxed{FFTTTFTTFF}$

Solution: 1. $6 \times 7 = 42 < 67$ so the statement is false.

2. $67 \times 69 = (68-1)(68+1) = 68^2 - 1 < 68^2$ so the statement is false.

3. $67 \times 67 + 6 \times 7 \times 6 \times 7 = 6253$ so the statement is true.

4&5. For positive integers $x, y > e$, note $\frac{\ln x}{x} < \frac{\ln(y)}{y}$ if $x > y$. Therefore $x > y$ implies $x^y < y^x$. So $6^7 > 7^6$ and $67^{76} > 76^{67}$.

6. Take 25th roots of both sides. Then, we just need to compare 10^9 and 225^4. Dividing both sides by 10^8, we need to compare 10 and 2.25^4. The right hand side is clearly greater, so this statement is false.

7. Take 50th roots of both sides. Then, we just need to compare 10^5 and 250^2. Now, $\frac{250^2}{10^5} = \frac{25^2}{10^3} = \frac{625}{1000} < 1$ so this statement is true.

8. Since logarithms are an increasing function, we can take the logarithm of $101^{102^{103}}$ and $103^{102^{101}}$. This yields

$$102^{103}\ln(101) > 102^{101}\ln(103).$$

Taking logarithms again yields

$$103\ln(102) + \ln(\ln(101)) > 101\ln(102) + \ln(\ln(103)),$$

or

$$2\ln(102) > \ln(\ln(103)) - \ln(\ln(101)).$$

But note that $2\ln(102) \approx 8$ and $\ln(\ln(103))$ is already less than $\ln(5) < 2$. Thus the original statement must be false.

9. Stirling approximate both sides:

$$67! \cdot 68! \cdot 69! \approx \sqrt{2\pi \cdot 67}\left(\frac{67}{e}\right)^{67} \cdot \sqrt{2\pi \cdot 68}\left(\frac{68}{e}\right)^{68} \cdot \sqrt{2\pi \cdot 69}\left(\frac{69}{e}\right)^{69}$$

$$\approx \sqrt{2\pi \cdot 68}^3\left(\frac{68}{e}\right)^{204} < \sqrt{2\pi \cdot 180}\left(\frac{68}{e}\right)^{207}$$

Whereas

$$180! \approx \sqrt{2\pi \cdot 180}\left(\frac{180}{e}\right)^{180} > \sqrt{2\pi \cdot 180}\left(\frac{68}{e}\right)^{360}.$$

It is clear that 180! is greater and thus this inequality is false.

10. Again take logarithms of both sides yields

$$250! \ln(100) > 100^{250} \ln(250).$$

Taking logarithms again and Stirling approximating yields

$$\ln(250!) + \ln(\ln(100)) \approx 250 \ln\left(\frac{250}{e}\right) + \ln(\ln(100)) > 250 \ln(100) + \ln(\ln(250))$$

and note that because $\ln(250) > \ln(100)$ and $100 > \frac{250}{e}$ the inequality is false.

. . . .

Set 7

19. [**11**] Let $\triangle ABC$ be an equilateral triangle with side length 1. Construct its circumcircle and select point D from minor arc AC such that $AD \times DC = \frac{1}{26}$. Compute $AD^2 + CD^2$.

Proposed by: Adeethyia Shankar

Answer: $\boxed{\dfrac{25}{26}}$

Solution: $ABCD$ is a cyclic quadrilateral, so $\angle ABC + \angle CDA = 180°$. Since $\angle ABC = 60°, \angle CDA = 120$. By the law of cosines,

$$\begin{aligned}
\overline{AC}^2 &= \overline{AD}^2 + \overline{CD}^2 - 2 \times \overline{AD} \times \overline{CD} \times \cos(\angle CDA) \\
&= \overline{AD}^2 + \overline{CD}^2 - 2 \times \frac{1}{26} \times \cos(120°) \\
&= \overline{AD}^2 + \overline{CD}^2 - 2 \times \frac{1}{26} \times \left(-\frac{1}{2}\right) \\
&= \overline{AD}^2 + \overline{CD}^2 + \frac{1}{26}.
\end{aligned}$$

Since $\overline{AC} = 1, \overline{AD}^2 + \overline{CD}^2 = 1 - \frac{1}{26} = \boxed{\dfrac{25}{26}}$.

20. [**11**] Jonathan bought an egg cooker that can cook eggs in batches of at most 6 eggs. He can buy eggs in packs of 10 or 14. Suppose that he bought n packs of eggs, cooked $2n - 3$ batches, and had 2 eggs left over, where n is a positive integer. What is the least possible value of n?

Proposed by: Mason Yu

Answer: $\boxed{8}$

Solution: Given some n, the maximum number of eggs Jonathan cooked in total is $6(2n-3) = 12n - 18$, and the minimum number is $2n - 3$.

The maximum number of eggs Jonathan bought is $12n$, and the minimum is $8n$.

So for all n, there must be a number k where the following inequalities hold:

$2n - 1 \leq k \leq 12n - 16$

$10n \leq k \leq 14n$

So the only possible contradiction is $10n > 12n - 16$ so $n < 8$.

So therefore $n = \boxed{8}$ works and is minimal.

21. [**11**] Let $ABCD$ be a square, and furthermore, let ω_1 be the semicircle in $ABCD$ with diameter AB, and ω_2 be the quarter-circle in $ABCD$ centered at B with radius AB. A line passing through B intersects ω_1, ω_2, and $\overline{AD}$ at E, F, and G, respectively. Suppose $BE = 5$ and $FG = 2$. What is EF?

Proposed by: Cody Cheng

Answer: $\boxed{\dfrac{\sqrt{65}-5}{2}}$

Solution: Claim: $\overline{AF}$ is the angle bisector of $\angle EAG$.

Proof: in ω_1, $\angle EAG = \angle EBA$ by the tangent-chord theorem. Then, looking at ω_2, $\angle GAF = \frac{1}{2}\angle FBA = \frac{1}{2}\angle EAG$, done.

The rest is just algebra. Call $x = EF$, and then similar triangles in $\triangle GAE$ and $\triangle ABE$ gives $AE = \sqrt{5x+10}$, so $AG = \sqrt{x^2+9x+14}$. Then, by the angle bisector theorem, $\frac{5x+10}{x^2+9x+14} = \frac{x^2}{4}$, which looks like a quartic but note that you can factor out $(x+2)$ on the left, and after cross-multiplying another $(x+2)$ cancels out. You end up getting $x^2 + 5x - 10 = 0$, which you just take the positive root of to get $\boxed{\dfrac{\sqrt{65}-5}{2}}$.

Set 8

22. [**12**] Michael randomly chooses a complex number z inside the unit disk in the complex plane. What is the probability that the real part of $(z-1)^3$ is positive?

Proposed by: Ethan Bove

Answer: $\boxed{\dfrac{2}{3} - \dfrac{\sqrt{3}}{2\pi}}$

Solution: If we write $z = x + iy$ for real x, y, then if follows that $(z-1)^3$ has real part:

$$\begin{aligned}(x-1)^3 - 3(x-1)y^2 &= (x-1)((x-1)^2 - 3y^2)\\ &= (x-1)(x-1-\sqrt{3}y)(x-1+\sqrt{3}y).\end{aligned}$$

In the desired domain, $x-1$ is always negative, we we need exactly one of:

$$x - 1 \pm \sqrt{3}y \leq 0.$$

These two equations describe disjoint segments of the circle, each of $2\pi/3$ radians. This is a standard calculation, and we find that each radian has area:

$$2\text{Area(segment)} = \theta - \sin\theta = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

We divide by the total area of the unit circle π, we get a probability of:

$$\boxed{\frac{2}{3} - \frac{\sqrt{3}}{2\pi}}.$$

23. [**12**] Leo has 10 distinct positive integers $n_1, n_2, \ldots, n_{10}$ such that each number divides the product of the other numbers. What is the smallest possible value of $n_1 + n_2 + \cdots + n_{10}$?

Proposed by: Joshua Kou

Answer: $\boxed{60}$

Solution:

We can do casework on the number of distinct primes r dividing $n_1 \times n_2 \times \cdots \times n_{10}$. Our method will be to first try taking the 10 smallest numbers that only have the first r primes as divisors, and then checking if it satisfies the condition.

For $r = 1$, we want to choose the smallest prime 2, so we have

$$1, 2, 4, 8, 16, 32, \ldots, 512.$$

This satisfies the condition.

For $r = 2$, we want to choose the two smallest primes 2 and 3, so we have

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18.$$

This satisfies the condition.

For $r = 3$, we have

$$1, 2, 3, 4, 5, 6, 8, 9, 10, 12.$$

This satisfies the condition.

For $r = 4$, we initially try

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10,$$

however, this fails since we only have a single multiple of 7. Thus the smallest set that satisfies the condition is

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 14.$$

Since the minimal sum when $r = 4$ is already worse than the minimal sum when $r = 3$, we should be led to believe that there is no need to consider higher r. An exhaustive search up to $r = 10$ verifies this assumption. Therefore the answer is

$$1+2+3+4+5+6+8+9+10+12 = (1+2+3+4+5+6+7+8+9+10)+5 = 55+5 = 60.$$

24. [**12**] Let $\triangle ABC$ be a triangle with $AB = 5, BC = 13$, and $\angle ABC = 2\angle ACB$. Compute the area of $\triangle ABC$.

Proposed by: Michael Lu

Answer: $\boxed{\frac{39}{2}}$

Solution: Let $\angle ACB = \theta$ and $\angle ABC = 2\theta$. Take D on BC such that $AD = AB$. Then $\angle ADB = 2\theta$ so $\angle ADC = 180 - 2\theta$ and $\angle DAC = \theta$. Then $\triangle ADC$ and $\triangle ADB$ are both isosceles so we get $AB = AD = DC = 5$. That means $DB = CB - CD = 8$ and we can find that the altitude from A to BC has length 3 so we get an area of $\frac{1}{2} \cdot 13 \cdot 3 = \boxed{\frac{39}{2}}$.

. . . .

Set 9

25. [**13**] Let ABC be an isosceles triangle with $AB = AC$. Let M be the foot of the altitude from A to $\overline{BC}$, and let D be the foot of the altitude from C to $\overline{AB}$. Let H be the intersection of these two altitudes. Given that $AH = HM = 2$, compute MD.

Proposed by: Max Liu

Answer: $\boxed{2\sqrt{2}}$

Solution: Solution 1:
We show that $\overline{MD}$ is tangent to the circle with diameter $\overline{AH}$. Note that D lies on the circle with diameter $\overline{AH}$ since $\angle ADH = 90^\circ$, so we just need to show that $\angle HAD = \angle MDH$. This is equivalent to $\angle MAB = \angle MDC$.

Consider the right triangle $\triangle BDC$. Since M is the midpoint of hypotenuse $\overline{BC}$, we know that $MB = MD = MC$, so $\triangle MBD$ is isosceles with $\angle MBD = \angle MDB$. Then, $\angle MAB = 90^\circ - \angle MBD = 90^\circ - \angle MDB = \angle MDC$, so $\overline{MD}$ is tangent to the circle with diameter AH.

By power of a point, $MD^2 = MH \cdot MA = 2 \cdot 4$, so $MD = \boxed{2\sqrt{2}}$.

Solution 2:
We show that $\triangle MHC \sim \triangle MBA$. We know that $\angle CMH = \angle AMB = 90^\circ$, so we just need to show that $\angle MAB = \angle MCH$. We have

$$\angle MAB = 90^\circ - \angle MBA = 90^\circ - \angle CBD = \angle BCD = \angle MCH,$$

as desired.

Let $MB = MD = MC = x$. By the Pythagorean theorem, we have $AB = \sqrt{x^2 + 4^2}$ and $HC = \sqrt{x^2 + 2^2}$. Using the fact that $\triangle MHC \sim \triangle MBA$, we have $\frac{MH}{HC} = \frac{MB}{BA}$. Plugging in known values gives us

$$\frac{2}{\sqrt{x^2+4}} = \frac{x}{\sqrt{x^2+16}}.$$

Cross-multiplying yields $2\sqrt{x^2+16} = x\sqrt{x^2+4}$. Squaring gives us $4(x^2 + 16) = x^2(x^2 + 4)$. This simplifies to $64 = x^4$, so $x = 2\sqrt{2} \implies MD = \boxed{2\sqrt{2}}$.

26. [**13**] As all gacha game players know, the fifth perfect number P_5 is $2^{12}(2^{13} - 1) = 33550336$, meaning that all proper divisors of P_5 (divisors not equal to itself) sum to P_5. Given $2^{13} - 1$ is prime, how many positive integers n satisfy $n < P_5$, and three distinct proper divisors of n multiply to P_5?

Proposed by: Mason Yu

Answer: $\boxed{127}$

Solution: Let n be any number that satisfies the condition. Using that $2^{13}-1$ is prime, we need that n divisible by $2^{13}-1$. Then the 2^{12} factor of P_5 must be split across the three distinct divisors. This can done by giving one factor the 2^5, one factor the 2^4, and one factor the 2^3 to multiply to 2^{12}. Note that there must be at least a 2^5 factor, otherwise the divisors wouldn't be distinct. So there must be a 2^5 factor and a $2^{13}-1$ factor. Therefore, there are 127 possible numbers because we count $1 \cdot 2^5(2^{13}-1), 2 \cdot 2^5(2^{13}-1), \cdots (2^7-1) \cdot 1 \cdot 2^5(2^{13}-1)$ as possible numbers less than P_5, and $2^7-1 = \boxed{127}$.

27. [**13**] Let $(g_n)_n$ be a geometric sequence, and $(a_n)_n$ be a sequence of real numbers such that $a_1 = a_4$ and $|a_n - g_n| \leq 4$ for all positive integers n. Let $r = a_3 - a_2$. What is the difference between the least and greatest possible values of r?

Proposed by: Mason Yu

Answer: $\boxed{32}$

Solution: Essentially we want to maximize $|g_3 - g_2|$ but note that if g is a geometric sequence, then $-g$ is also a geometric sequence, so we can just maximize $g_3 - g_2$ and take the other direction. By testing boundary values, we can get the geometric series $4, -4, 4, -4$ and $-4, 4, -4, 4$. The associated sequence is $8, -8, 8, -8$ and $-8, 8, -8, 8$ which lead to an answer of 32.

Now a proof that this is in fact optimal: Suppose there exists a sequence such that $|g_3 - g_2| > 8$ but $|g_4 - g_1| \leq 8$. Setting $g_n = ax^{n-1}$, this is equivalent to $a|x^2 - x| > 8$ and $a|x^3 - 1| \leq 8$ which is just $|x^3 - 1| < |x^2 - x|$ so $|x^2 + x + 1| < |x|$ which is false. Therefore our answer of $\boxed{32}$ we found was optimal.

Set 10

28. [**15**] Find the sum of all positive integers n satisfying

$$\left| \sum_{x=1}^{n} \log_{28}(x^2 + 7x + 12)^{\cos(\pi x)} \right| = 1.$$

Proposed by: Leo Zhang

Answer: $\boxed{111}$

Solution: It is easy to see that for any integer x,

$$f(x) = [(x+3)(x+4)]^{(-1)^x}.$$

Hence: When n is odd,

$$\left| \sum_{k=1}^{n} \log_{28} f(k) \right| = \log_{28} 4(n+4);$$

When n is even,

$$\left| \sum_{k=1}^{n} \log_{28} f(k) \right| = \log_{28} \frac{(n+4)}{4}.$$

Thus, the values of n satisfying the condition are $n = 3$ or $n = 108$, and their sum is $\boxed{111}$.

29. [**15**] Let $f(x) = x^3 + ax^2 + bx + c$, where a, b, c are distinct nonzero integers. If $f(a) = a^3$ and $f(b) = b^3$, find the value of c.

Proposed by: Leo Zhang

Answer: $\boxed{16}$

Solution: According to the problem, a and b are two roots of the equation

$$ax^2 + bx + c = 0.$$

Hence we have

$$\begin{cases} a + b = -\dfrac{b}{a}, \\ ab = \dfrac{c}{a}. \end{cases}$$

Since a, b are both nonzero integers, we must have $b = ka$ for some nonzero integer k. Then the first equation gives

$$a + b = a + ka = -\frac{b}{a} = -k \quad \Rightarrow \quad k(a+1) = -a.$$

Thus, $k \mid a$ and there exists some nonzero integer m such that $a = mk$. Then

$$mk + 1 = -m \quad \Rightarrow \quad m(k+1) = -1.$$

Taking nonzero integer solutions gives $m = 1$ and $k = -2$. Hence $a = -2$, $b = 4$. Finally,

$$c = aba = (-2) \times 4 \times (-2) = 16.$$

Therefore,

$$c = \boxed{16}.$$

30. [**15**] Consider the sequence starting with $1, 2, 3$, and every subsequent term is the sum of three previous numbers in the sequence, randomly chosen without replacement. What is the expected value of the 200th term in this sequence?

Proposed by: Mason Yu

Answer: $\boxed{12060}$

Solution: Let the sequence of expected values of the nth term be E_n. Then $E_1 = 1, E_2 = 2, E_3 = 3, E_4 = 6$. Since each term has an equal chance $\frac{3}{n-1}$ of being chosen,

$$E_n = \sum_{i=1}^{n-1} \frac{3}{n-1} E_i = \frac{3}{n-1} \left(E_{n-1} + \sum_{i=1}^{n-2} E_i \right)$$

$$= \frac{3}{n-1} \left(E_{n-1} + \frac{n-2}{3} E_{n-1} \right) = E_{n-1} \cdot \frac{3}{n-1} \cdot \frac{n+1}{3}$$

$$= \frac{n+1}{n-1} E_{n-1}.$$

Therefore, $E_{200} = 6 \cdot \frac{6}{4} \cdot \frac{7}{5} \cdot \frac{8}{6} \cdots \frac{200}{198} \cdot \frac{201}{199} = 6 \cdot \frac{200 \cdot 201}{4 \cdot 5} = 6 \cdot 10 \cdot 201 = \boxed{12060}$.

31. [**17**] $ABCD$ is a rectangle, E lies on AD in between A and D. Let l be the perpendicular bisector of line segment EC. Lines l and EC split $ABCD$ into four regions whose areas in some order are $52, 92, 117$, and 369. What is the perimeter of $ABCD$?

Proposed by: Mason Yu

Answer: $\boxed{59\sqrt{3}}$

Solution: Let M be the intersection of l and EC, P intersection of l closest to B and Q intersection of l closest to D There are a few cases: Case 1: l intersects AD and BC. But this is illegal because then EMQ and CMP would be congruent.

Case 2: l intersects AB and DC. But this is illegal because $AEMP$, the largest region, is bounded by $3 \cdot MPBC$, which is clearly false because of the magnitude of 495.

Case 3: l intersects BC and DC. Then drop the perpendicular from P to AD and call it R. Since EDC has area 144 and FCP has area 169, side lengths of $PRDC$ are in ratio $12 : 13$. Since CMP has area 117 and EMP is congruent, ECP has area 234 and therefore $PRDC$ has area 468 so $PC = 13\sqrt{3}, DC = 12\sqrt{3}$. Then, $EMPR$ has area 207. So we need another area 162 which is $ABPR$. Since $PR = 12\sqrt{3}$ then $BP = \frac{9\sqrt{3}}{2}$. So $AB = CD = 12\sqrt{3}$ and $BC = AD = BP + PC = \frac{35\sqrt{3}}{2}$. Therefore the perimeter is just $\boxed{59\sqrt{3}}$.

32. [**17**] Sebastian has an 8×8 checkerboard which consists of black and white squares. Every turn, they draw a square grid on the checkerboard, and swap the color of all squares inside that grid. What is the fewest number of turns they need to change every square to the same color?

Proposed by: Mason Yu

Answer: $\boxed{14}$

Solution: Proof by induction that the solution has a minimum of $2(n-1)$ for side length n:

The base case is 2 which the only way is to achieve by changing both of the corners.

Then suppose $n-1$ has a minimum of $2n-4$, then for n take any subsquare of size $n-1$. Any valid configuration for the n square must have that subsquare already covered by a $n-1$ square. You now need at least two more squares: one to differentiate the color of the subsquare from the big square, and another to differentiate the single corner not touched from the subsquare.

This construction can actually be achieved by taking opposite corners, then building squares of side length $1, 2, 3 \ldots 7$ lined up in each of these corners. Therefore $\boxed{14}$ is the minimum and it is achievable.

33. [**17**] Suppose positive integers a, b, and c satisfy $4^a = b^4 + c! - 18$ and $a + b + c = S$. What is the sum of all possible values of S?

Proposed by: Mason Yu

Answer: $\boxed{13}$

Solution: Taking the entire equation $\pmod 5$ yields that $c < 5$. Taking the entire equation $\pmod 3$ yields that $c \neq 2$.

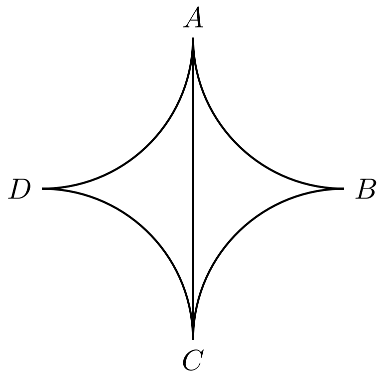
So we only have three cases to test. $c = 1$ yields that $4^a + 17 = b^4$. We rearrange this as a difference of squares: $b^4 - 4^a = (b^2 - 2^a)(b^2 + 2^a) = 17$, so we need $b^2 = 9$ and $2^a = 8$. So here we have the solution $(3, 1, 3)$ $c = 3$ yields that $(b^2 - 2^a)(b^2 + 2^a) = 12$, so $b^2 = 4$ and $2^a = 2$. Then we have the solution $(1, 3, 2)$. $c = 4$ yields that $(2^a - b^2)(2^a + b^2) = 6$ which has no integer solutions because all pairs of factors have different parity.

For the two solutions, $k = 6$ or 7. The sum is 13.

....

Set 12

34. [**20**] The following shape $ABCD$ is created from the space between four tangent unit circles. 299 line segments with endpoints on the shape lie parallel to each other and to segment AC, and split $ABCD$ into 300 pieces of equal area. Estimate the sum of the lengths of all these segments. Express your answer as a base 10 decimal.



If you are within 2% of the right answer, you get 20 points. Every additional 1% error deducts one point (minimum zero points).

Proposed by: Mason Yu

Answer: $\boxed{268.07}$

Solution: Let N be the number of line segments. To estimate the sum of the lengths, note that since the pieces are equal area, the density of the segments (inverse of width) is approximately proportional to the height of the figure at that point (which is also the length of the segment by symmetry). So therefore the average length of a segment is the weighted sum $\frac{\sum_{n=1}^{N} l_n^2}{\sum_{n=1}^{N} l_n}$ where l_n is the length of the nth segment. Now we consider behavior when N large.

The numerator can be approximated by considering the volume of a solid formed with the base of the original shape and the height of the corresponding line segment at that point. So, consider a circle of radius 1 cut in two pieces tangent to each other. The length of a segment at any height is the width of the square minus the width of this circle. Let the corresponding width of the circle be ω_n.

Then, $l_n = 2 - \omega_n$, and $l_n^2 = 4 - 4\omega_n + \omega_n^2$.

Since the solid has width 2, the first term contributes 8. The second term contributes -4π since the sum of the widths of the circle is just the area of the circle. The third term can be rewritten as $\frac{4}{\pi} \cdot \frac{\pi}{4}\omega_n^2$, which is $\frac{4}{\pi}$ times the volume of a sphere, equal to $\frac{16}{3}$. Therefore the entire numerator sums to $\frac{40}{3} - 4\pi$.

The denominator can just be approximated by the area of the shape, which is $4 - \pi$. Therefore the average length of a segment is approximately $\frac{\frac{40}{3} - 4\pi}{4-\pi}$. However, we assumed an extra "zero" segment because we were summing over all of the areas, so we need to add another segment and multiply by 300. This gives $\frac{4000-1200\pi}{4-\pi} \approx 268.04$, which is more than close enough for full credit (the actual sum is $\boxed{268.07}$.)

35. [**20**] Estimate:

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{ab-2|a-2b|+\max(a,b)}{2^a+2^b}.$$

Express your answer as a base 10 decimal.

If you are within 2% of the right answer, you get 20 points. Every additional 1% error deducts one point (minimum zero points).

Proposed by: Mason Yu

Answer: $\boxed{12.348}$

Solution: We first note the use of the series

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=1,\sum_{n=1}^{\infty}\frac{n}{2^n}=2,\sum_{n=1}^{\infty}\frac{n^2}{2^n}=6$$

The first two are relatively well known, and the second can either be done using integration of the others, or by the following trick:

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}=\sum_{n=1}^{\infty}\frac{n(n+1)}{2^n}-\sum_{n=1}^{\infty}\frac{n}{2^n}=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{2k}{2^n}-2$$

$$=\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{2k}{2^n}-2=\sum_{k=1}^{\infty}\frac{2k}{2^{k-1}}-2=4\sum_{k=1}^{\infty}\frac{k}{2^k}-2=8-2=6.$$

We evaluate

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{|a-2b|}{2^a+2^b}.$$

The key insight is to split the summation region into three regions: where $a>b, a=b$, and $a<b$.

$a>b$: set $a=b+k$. The fraction becomes

$$\sum_{k=1}^{\infty}\sum_{b=1}^{\infty}\frac{|b-k|}{2^{b+k}+2^b}=\sum_{k=1}^{\infty}\frac{1}{2^k+1}\sum_{b=1}^{\infty}\frac{|b-k|}{2^b}$$

$$=\sum_{k=1}^{\infty}\frac{1}{2^k+1}\left(\sum_{b=1}^{k}\frac{k-b}{2^b}+\sum_{b=k+1}^{\infty}\frac{b-k}{2^b}\right)$$

$$=\sum_{k=1}^{\infty}\frac{1}{2^k+1}\left(k\cdot\sum_{b=1}^{k}\frac{1}{2^b}-\sum_{b=1}^{k}\frac{b}{2^b}+\sum_{c=1}^{\infty}\frac{c}{2^{(c+k)}}\right)$$

$$=\sum_{k=1}^{\infty}\frac{1}{2^k+1}\left(k\cdot\left(1-\frac{1}{2^k}\right)-\left(2-\frac{k+2}{2^k}\right)+\frac{2}{2^k}\right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^k+1}\left(k-2+\frac{4}{2^k}\right).$$

$a < b$: set $b = a + k$. The fraction becomes

$$\sum_{k=1}^{\infty}\sum_{b=1}^{\infty} \frac{|-a-2k|}{2^{a+k}+2^a} = \sum_{k=1}^{\infty}\sum_{b=1}^{\infty} \frac{a+2k}{2^{a+k}+2^a}$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^k+1} \sum_{a=1}^{\infty} \frac{a+2k}{2^a}$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^k+1}\left(\sum_{a=1}^{\infty} \frac{a}{2^a} + 2k \cdot \sum_{a=1}^{\infty} \frac{1}{2^a}\right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^k+1}(2+2k) = \sum_{k=1}^{\infty} \frac{2k+2}{2^k+1}.$$

And finally we set $a = b$ to have

$$\sum_{a=1}^{\infty} \frac{|-a|}{2^{a+1}} = \frac{1}{2}\sum_{a=1}^{\infty} \frac{a}{2^a} = 1.$$

Summing these three cases up we have:

$$1 + \sum_{k=1}^{\infty} \frac{1}{2^k+1}\left(2k+2+k-2+\frac{4}{2^k}\right)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{2^k+1}\left(3k+\frac{4}{2^k}\right)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{3k}{2^k+1} + \sum_{k=1}^{\infty} \frac{4}{2^k(2^k+1)}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{3k}{2^k+1} + \sum_{k=1}^{\infty}\left(\frac{4}{2^k} - \frac{4}{2^k+1}\right)$$

$$= 1 + \sum_{k=1}^{\infty} \frac{3k}{2^k+1} + 4 - \sum_{k=1}^{\infty} \frac{4}{2^k+1}$$

$$= 5 + \sum_{k=1}^{\infty} \frac{3k-4}{2^k+1}$$

We can use the similar techniques to evaluate

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{ab+\max(a,b)}{2^a+2^b}.$$

Note that we have significantly less computation here, since the expression is symmetric.

$a < b$: Set $b = a + k$. The summation becomes

$$\sum_{a=1}^{\infty}\sum_{k=1}^{\infty}\frac{a(a+k)+a+k}{2^a+2^{a+k}} = \sum_{k=1}^{\infty}\frac{1}{2^k+1}\sum_{a=1}^{\infty}\frac{a^2+ak+a+k}{2^a}$$

$$= \sum_{k=1}^{\infty}\frac{1}{2^k+1}(6+2k+2+k) = \sum_{k=1}^{\infty}\frac{8+3k}{2^k+1}.$$

$a = b$: The summation becomes

$$\sum_{a=1}^{\infty}\frac{a^2+a}{2\cdot 2^a} = \frac{6}{2}+\frac{2}{2} = 4.$$

Since the $a > b$ case is symmetric, the entire sum is just

$$4 + \sum_{k=1}^{\infty}\frac{6k+16}{2^k+1}.$$

And therefore, we have that

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{ab-2|a-2b|+\max(a,b)}{2^a+2^b}$$

$$= \left(4+\sum_{k=1}^{\infty}\frac{6k+16}{2^k+1}\right) - 2\left(5+\sum_{k=1}^{\infty}\frac{3k-4}{2^k+1}\right)$$

$$= -6 + \sum_{k=1}^{\infty}\frac{24}{2^k+1}.$$

This series converges relatively quickly to approximately $\boxed{12.348}$.

36. [**20**] Let $L_6(n)$ be the number formed by replacing the last digit of n with 6. Let $F_7(n)$ be the number formed by replacing the first digit of n with 7. Suppose $\frac{L_6(n)+F_7(n)}{n}$ is an integer. Additionally, n does not contain the digits 6 or 7, but n is divisible by 6 or 7 (or both).

Find the set S of the 10 smallest positive integers n that satisfy these properties. If you list m distinct integers, and they are all in S, you get $2m$ points.

Proposed by: Mason Yu

Answer: $\boxed{252, 2502, 19999, 25002, 39998, 120001, 250002, 2500002, 25000002, 250000002}$

Solution: Let a be the first digit of n, b be the last digit of n, m be all the middle digits of n, and let k be the number of digits of n. Then $\frac{L_6(n)+F_7(n)}{n} = \frac{n-b+6+n+(7-a)10^{k-1}}{n} = 2 + \frac{6-b+(7-a)10^{k-1}}{n}$.

Now, we do casework on the value of this ratio. Note that there are only 8 cases for the ratio (excluding the single digit 1) because the ratio is strictly less than $\frac{n+7n+6}{n} < 9$. (In the highest case scenario it is $\frac{16+70}{10} = 8.6$)

If the ratio is 1, we need $\frac{6-b+(7-a)10^{k-1}}{n} = -1$. But then $a = 8$ or 9 which means that $n \approx 8\cdot 10^{k-1}$. But the magnitude of the denominator is at most $\approx (7-9)10^{k-1} = 2\cdot 10^{k-1}$ since the first two terms are small. So there are no cases. (Manually checking small cases also does not yield any possible solutions.)

If the ratio is 2, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 0$. Therefore we need $a = 7$ which contradicts the initial properties. In the only other case, n would need to be 1 digit which also leads to a contradiction.

If the ratio is 3, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 1$. So $6-b+(7-a)10^{k-1} = n$, and rearranging in terms of m, a, b, we have $6-b+(7-a)10^{k-1} = a\cdot 10^{k-1}+10m+b$, so we have $(7-2a)10^{k-1} = 10m-6+2b$. We know that $10m < 10^{k-1}$ because m has two less digits than n, so therefore $a = 3$. Then we have that $10^{k-1} - 10m = 2b - 6$. We know that the LHS is divisible by 10, and is positive, so $b = 8$, and $10^{k-2} - m = 1$. Therefore m is an arbitrarily long string of 9s, which leads to the set of solutions $39\ldots 8$. (38 is also a solution here because $m = 0, k = 2$ works.)

If the ratio is 4, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 2$. So $6 - b + (7 - a)10^{k-1} = 2n$, and rearranging in terms of m, a, b, we have $6-b+(7-a)10^{k-1} = 2a\cdot 10^{k-1} + 20m + 2b$, so we have $(7-3a)10^{k-1} = 20m - 6 + 3b$. Using the same bounds on m, we must set $a = 2$ and $b = 2$. Then we have that $10^{k-1} - 20m = 0$. Therefore m is of the form $50\ldots 0$, which leads to the set of solutions $250\ldots 2$.

If the ratio is 5, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 3$. So like before, $(7 - 4a)10^{k-1} = 30m - 6 + 4b$. Using the same bounds on m, we must set $a = 1$. Then we have that $10^{k-1} - 10m = \frac{4b-6}{3}$, necessitating $b = 9$ and m is an arbitrarily long string of 9s, which leads to the set of solutions $19\ldots 9$. (19 is also a solution here because $m = 0, k = 2$ works.)

If the ratio is 6, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 4$, and $(7 - 5a)10^{k-1} = 40m - 6 + 5b$. Again $a = 1$ is necessary, so $10^{k-1} - 20m = \frac{5b-4}{2}$, which yields no valid solutions.

If the ratio is 7, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 5$. So like before, $(7-6a)10^{k-1} = 50m-6+6b$. Using the same bounds on m, we must again set $a = 1$. Then we have that $10^{k-1} - 50m = 6b - 6$ necessitating $b = 1$ and m is of the form $20\ldots 0$, which leads to the set of solutions $120\ldots 1$.

If the ratio is 8, we need $\frac{6-b+(7-a)10^{k-1}}{n} = 6$. So, $(7 - 7a)10^{k-1} = 60m - 6 + 7b$. But then $a = 1$ and we have $60m + 7b = 6$. Thus $m = 0$. But then $7b = 6$, contradiction.

Therefore we have all solution families. Note that the $25\ldots 2$ family is always divisible by 6, and no other family is ever. Then we check when each family is divisible by 7. Note that the next largest solution in each family can be obtained by multiplying by 10 and adding a residue $\bmod 7$, and we can check that these operations are cyclic with period 6. Using this fact and manually checking values we can obtain the first 10 values of n, which are $\boxed{252, 2502, 19999, 25002, 39998, 120001, 250002, 2500002, 25000002, 250000002}$.