1. How many 3 digit numbers containing only the numbers $1, 2, 3$ and 4 as digits are greater than the number formed by reversing its digits?

   Proposed by: Mason Yu

   Answer: $\boxed{24}$

   Solution: There are 64 3-digit numbers containing only 1, 2, 3, and 4 as digits since there are 4 ways to choose 4 digits. 16 of them have the form *abba* (4 ways to choose a and 4 ways to choose b) and are equal to their reverse. Out of the remaining 48, half of them are greater than their reverse and half of them are less than their reverse. The desired count is thus $\frac{48}{2} = 24$.

2. 15 friends want to play a game 3 times. Each game seats exactly 10 players, and the other 5 must sit out. They want to make sure everyone plays at least once. What is the smallest possible number of people who must play 2 or more games for all 15 friends to play at least once across the three games?

   Proposed by: Jonathan Xu

   Answer: $\boxed{8}$

   Solution: There are 30 total "seats" across the three games and 15 people. Giving everyone one game uses 15 seats, leaving 15 extra appearances that must be distributed. To minimize how many people play at least twice, pack these extras into as few people as possible by letting some play three times (each such person contributes 2 extras). Seven people playing three times supply 7 x 2 = 14 extras; you still need 1 more extra, which forces one additional person to play a second time. Thus 7 people play three times and 1 person plays twice, totaling 8 people who play at least twice.

3. Ethan is at the point $(7, 24)$ in the xy-plane. After being reflected across a line, Ethan is at the point $(8, 6)$. What is the y-intercept of the line of reflection?

   Proposed by: Lily Bowen

   Answer: $\boxed{\dfrac{175}{12}}$

   Solution: The line through $(7, 24)$ and $(8, 6)$ has a slope of $(6-24)/(8-7) = -18$. The line of reflection is perpendicular to this line, so it has a slope of $1/18$. The line of reflection also goes through the midpoint of $(7, 24)$ and $(8, 6)$, which is the point $(15/2, 15)$. Thus our equation for the line of reflection is $y - 15 = (1/18)(x - 15/2)$. Plugging in 0 for x, we get that the y-intercept is $(1/18)(-15/2) + 15 = 175/12$.

4. This fall, Alice is busy raking the leaves in her yard. If no new leaves fall, she can finish the job in x hours. However, when leaves start falling steadily from the trees, it takes her $x + 2$ hours to finish. Later, her mischievous neighbor Malice begins secretly tossing in more leaves at the same rate that they fall from the trees, and now it takes Alice $x + 5$ hours to get the yard clean. What is the value of x?

   Proposed by: Ryan Lee

   Answer: $\boxed{10}$

   Solution: If Alice can clear the yard in x hours, it means that she rakes the leaves at a rate of $\frac{1 \text{ yard}}{x \text{ hours}}$. Therefore, $\frac{1}{x} -$ (the rate that the leaves fall from the trees) $= \frac{1}{x+2}$, so the rate that the leaves fall from the trees is $\frac{1}{x} - \frac{1}{x+2}$. Similarly, the rate that Malice tosses in leaves is $\frac{1}{x+2} - \frac{1}{x+5}$.

Since the rate that leaves fall from the trees equals the rate that Malice tosses in leaves, we have $\frac{1}{x} - \frac{1}{x+2} = \frac{1}{x+2} - \frac{1}{x+5}$. Multiplying by $x(x+2)(x+5)$, we get

$$\begin{aligned}
(x+2)(x+5) - x(x+5) &= x(x+5) - x(x+2) \\
(x^2+7x+10) - (x^2+5x) &= (x^2+5x) - (x^2+2x) \\
2x+10 &= 3x \\
10 &= x.
\end{aligned}$$

5. In stormy cyber-Providence, William glances up at a holo-billboard that shows a four-digit positive integer N. Suddenly, the billboard is struck by lightning and glitches out: the integer gets divided by 9, then its digits get reversed, but the display still shows the same number N. What is N?

   Proposed by: Jaeho Lee

   Answer: $\boxed{9801}$

   Solution: Let the four-digit number be N. We can write its digits as $\overline{abcd}$, which stands for $1000a + 100b + 10c + d$. The reverse of N is $\overline{dcba}$, which means $1000d + 100c + 10b + a$. The problem says that N divided by 9 equals its reverse. This gives us the equation:

   $$\frac{N}{9} = \overline{dcba}$$

   Multiplying by 9, we get:

   $$N = 9 \times \overline{dcba}$$

   Now, let's write this out using the place values for each digit:

   $$1000a + 100b + 10c + d = 9 \times (1000d + 100c + 10b + a)$$

   When we multiply the 9 through on the right side, we get:

   $$1000a + 100b + 10c + d = 9000d + 900c + 90b + 9a$$

   Since N is a four-digit number, a cannot be 0. Since $N/9$ is also a four-digit number (because its reverse, N, is), d cannot be 0. Let's look at the $1000a$ on the left and the $9000d$ on the right. Since N is a four-digit number, it must be less than 10,000. If d were 2 or more, the right side ($9000d + \dots$) would be at least 18,000, which is impossible. Therefore, the digit d must be 1.

   Now that we know $d = 1$, the right side of the equation starts with 9000. This means N itself must be a number in the 9000s. For N to be in the 9000s, its first digit, a, must be 9.

   Let's plug $a = 9$ and $d = 1$ back into our equation:

   $$1000(9) + 100b + 10c + 1 = 9000(1) + 900c + 90b + 9(9)$$

   $$9000 + 100b + 10c + 1 = 9000 + 900c + 90b + 81$$

$$9001 + 100b + 10c = 9081 + 90b + 900c$$

Now we just need to solve for b and c. Let's move all the b's and c's to the right side and the numbers to the left side:

$$9001 - 9081 = (900c - 10c) + (90b - 100b)$$

$$-80 = 890c - 10b$$

Let's divide the whole equation by 10 to make it simpler:

$$-8 = 89c - b$$

Or, by multiplying by -1, we get $8 = b - 89c$. Since b and c must be digits from 0 to 9, we can test values for c. If c was 1, b would have to be $8 + 89 = 97$, which is not a single digit. So, the only possibility is that c must be 0. If $c = 0$, the equation becomes $8 = b - 0$, which means b must be 8.

So, we found $a = 9$, $b = 8$, $c = 0$, and $d = 1$. The number N is 9801.

Let's check our answer: $9801 \div 9 = 1089$. The reverse of 1089 is 9801. It works!

6. Noboru uses equally sized unit matchsticks to create regular unit hexagons on a table. If hexagons can share sides, what is the maximum number of unit hexagons he can create with 89 matchsticks?

   Proposed by: Ryan Lee

   Answer: $\boxed{24}$

   Solution: Create a cluster!

   1st layer: 6 matchsticks.

   Notice that there's an interesting pattern with the next couple: for the nth layer, the number of additional matchsticks needed for the layer per hexagon (excluding overlaps) is

$$5, \underbrace{4, \ldots, 4}_{n-2} ..., 5, \underbrace{4, \ldots, 4}_{n-2},$$

   where there are $n - 2$ 4s in each group. (visualizing this, 5s are corner hexagons, while 4s are edge hexagons). However, edge overlaps are counted twice, so we subtract 1 per hexagon at the end.

   2nd layer: $5 \times 6 - 6 = 24$ (total 30).

   3rd layer: $(5 + 4) \times 6 - 12 = 42$ (total 72).

   Now we have 17 matchsticks left. We are in the 4th layer. so our pattern is 5, 4, 4, 5, 4, 4, ... To minimize it seems best to start on an edge hexagon, so we use 4 matchsticks for our first one. Then, our next one can share a side with this first one, so it only needs 3. Then, any hexagon sharing a side with either of these two has to be a corner, so we need 4 more matchsticks for that. We have 6 matchsticks left, which is perfect for 2 more edge hexagons adjacent to that corner. In total we made 5 hexagons in this layer.

   Thus our total number of hexagons is $1 + 6 + 12 + 5 = 24$.

7. Two integers a, b are considered relatively prime when their greatest common divisor is 1. Let an integer x be considered "good" if and only if $(x^2+x)^2$ is relatively prime to 75. How many good integers are there between 1 and 150, inclusive?

   Proposed by: Sebastian Weinberger

   Answer: $\boxed{30}$

   Solution: An integer is "good" if and only if $(x^2+x)^2$ is relatively prime to 75, which happens if and only if neither 3 nor 5 divides $(x^2+x)^2 = x^2(x+1)^2$, which occurs if and only if neither 3 nor 5 divides either x or $x+1$. For this, only the remainders of x and $x+1$ upon division by 3 or 5 matter; in particular, we can find that x is good if and only if x has a remainder of 1 upon division by 3 and a remainder of $1, 2,$ or 3 upon division by 5. Equivalently, x is good if and only if it has a remainder of $1, 7,$ or 13 upon division by 15. There are $3 \times 10 = 30$ such numbers between 1 and 150, inclusive, so the answer is 30.

8. Let S be the set of integers n with $1 \leq n \leq 2025$ such that $\operatorname{lcm}(n, 84) + \gcd(n, 84) = n + 84$. Compute the number of elements in S.

   Proposed by: Jaeho Lee

   Answer: $\boxed{35}$

   Solution: Use the identity $\operatorname{lcm}(a, b) = \dfrac{ab}{\gcd(a, b)}$. Substituting $a = n$ and $b = 84$ into the given condition gives $\dfrac{84n}{\gcd(n, 84)} + \gcd(n, 84) = n + 84$. Multiply through by $\gcd(n, 84)$ to get $84n + (\gcd(n, 84))^2 = (n + 84)\gcd(n, 84)$. Rearranging yields

   $$n(84 - \gcd(n, 84)) = \gcd(n, 84)(84 - \gcd(n, 84)).$$

   If $\gcd(n, 84) = 84$, the equation is automatically true, so every multiple of 84 qualifies. If $\gcd(n, 84) \neq 84$, then $84 - \gcd(n, 84) \neq 0$ and we can divide both sides to get $n = \gcd(n, 84)$, which means precisely that n is a divisor of 84. Thus, all solutions in $[1, 2025]$ are the divisors of 84 together with the multiples of 84 up to 2025. Since $84 = 2^2 \cdot 3 \cdot 7$ has $(2+1)(1+1)(1+1) = 12$ divisors, and there are $\left\lfloor \dfrac{2025}{84} \right\rfloor = 24$ multiples of 84 in the range, but 84 itself has been counted twice, the total is $12 + 24 - 1 = 35$.

9. Let a, b, c be the three roots of the equation

   $$x^3 + 6x^2 - 52x + 8 = 0.$$

   Find the value of

   $$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

   Proposed by: Leo Zhang

   Answer: $\boxed{0}$

   Solution: From the equation

   $$x^3 + 6x^2 - 52x + 8 = 0,$$

   we can rewrite it as

   $$(x+2)^3 = 64x.$$

Hence,

$$4\sqrt[3]{x} = x + 2.$$

Therefore,

$$4\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}\right) = (a + b + c) + 6 = -6 + 6 = 0.$$

Thus,

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = 0.$$

10. Let $ABCD$ be a convex quadrilateral such that $\angle BAD = 45^\circ$ and $BC = 4$. Let ω_1 be the circle centered at B passing through A and C and ω_2 be the circle centered at C passing through B and D such that ω_1 and ω_2 intersect at points P and Q, where P lies on segment AD and $P \neq D$. If lines AB and DC intersect at point X, the length of QX can be expressed as $\frac{a}{\sqrt{b}}$ for positive integers a and b, where b is not divisible by a square greater than 1. Find $a + b$.

    Proposed by: Anthony Yang

    Answer: $\boxed{7}$

    Solution: Since A, P, C, Q all lie on circle ω_1, we know $BA = BP = BQ = BC = 4$. Similarly, we find that $CD = CP = CX = CB = 4$. Since $BC = CP = PB$, we know that ΔBPC is equilateral which means $\angle BCP = \angle CPB = \angle PBC = 60^\circ$. Further, since $BA = BP$ we find that

    $$\angle ABP = 180^\circ - 2\angle BAP = 180^\circ - 2\angle BAD = 90^\circ$$

    Now, note that

    $$\angle CPD = 180^\circ - \angle CPB - \angle BPA = 180^\circ - 60^\circ - 45^\circ = 75^\circ$$

    Since $CD = CP$, we get that

    $$\angle PCD = 180 - 2\angle CPD = 30^\circ$$

    Then, by inscribed angle theorem, we get

    $$\angle AQP = \frac{\angle ABP}{2} = 45^\circ \qquad \angle PQD = \frac{\angle PCD}{2} = 15^\circ$$

    This means $\angle AQD = \angle AQP + \angle PQD = 60^\circ$. Now, note that

    $$\angle AXD = 180^\circ - \angle XAD - \angle XDA = 180^\circ - 45^\circ - 75^\circ = 60^\circ$$

    Then, since $\angle AQD = \angle AXD$, we know $AQXD$ is cyclic. We now wish to use Ptolemy's theorem on $AQXD$. Since $BA = BP = 4$ and $CP = CD = 4$, we get $AP = 4\sqrt{2}$ and $PD = 2\sqrt{6} - 2\sqrt{2}$. Then, we know

    $$AD = AP + PD = 2\sqrt{2} + 2\sqrt{6}$$

    Now, note that ΔQBC is equilateral which means $\angle QBC = 60^\circ$. This means that

    $$\angle QBA = 360^\circ - \angle QBC - \angle CBP - \angle PBA = 150^\circ$$

    Since $BQ = BA$, we know ΔQBA is isosceles so $\angle BAQ = 15^\circ$. Then, from $BA = 4$, we know $QA = 2\sqrt{6} + 2\sqrt{2}$. Note that $QA = AD$, which means ΔQAD is isosceles. However, we previously found that $\angle AQD = 60^\circ$ so ΔQAD must be equilateral. Thus, we find that

    $$QA = AD = DQ = 2\sqrt{6} + 2\sqrt{2}$$

By Ptolemy's theorem, we then have

$$QX \cdot AD + AQ \cdot DX = QD \cdot XA$$

Note that $QA = AD = DQ$, so we can divide this equation by QA to get

$$QX + DX = XA$$

By Law of Sines on ΔAXD, we have

$$\frac{AD}{\sin \angle AXD} = \frac{AX}{\sin \angle ADX} = \frac{DX}{\sin \angle DAX}$$

We already know $\angle AXD = 60°$, $\angle ADX = 75°$, and $\angle DAX = 45°$, so solving this equation gives $AX = 4 + \frac{8}{\sqrt{3}}$ and $DX = 4 + \frac{4}{\sqrt{3}}$. Thus, we have

$$QX = AX - DX = \frac{4}{\sqrt{3}}$$

so our answer is $4 + 3 = \boxed{7}$.

11. Let $f(x)$ be a third degree monic polynomial over the integers; that is, $f(x) = x^3 + ax^2 + bx + c$ where a, b, c are integers. Suppose also that $0 \leq f(i) \leq 10$ for $i \in \{1, 3, 5\}$. Find the maximum possible value of $f(7)$.

Proposed by: Sebastian Weinberger

Answer: $\boxed{84}$

Solution: The key is to define $g(x) := f(x + 3)$ and note that g is a monic polynomial over the integers if and only if f is. Then the problem reduces to finding the maximum possible value of $g(4)$ given that g is a third-degree monic polynomial over the integers subject to $0 \leq g(i) \leq 10$ for $i \in \{-2, 0, 2\}$. Writing $g(x) = x^3 + a_0x^2 + b_0x + c_0$, these conditions are equivalent to the following:

$$0 \leq -8 + 4a_0 - 2b_0 + c_0 \leq 10,$$

$$0 \leq c_0 \leq 10,$$

and

$$0 \leq 8 + 4a_0 + 2b_0 + c_0 \leq 10.$$

Adding the first and last inequalities, we find that $0 \leq 8a_0 + 2c_0 \leq 10$; since $0 \leq c_0 \leq 10$ this implies that $-20 \leq 8a_0 \leq 20$. That is to say, $a_0 \in \{-2, -1, 0, 1, 2\}$.

Meanwhile, subtracting the first inequality from the last we find that $-10 \leq 16+4b_0 \leq 10$. That is to say, $b_0 \in \{-6, -5, -4, -3, -2\}$.

Intuitively, to maximize $g(4)$, we want to pick the maximum possible values for a_0 and b_0; that is, $a_0 = 2$ and $b_0 = -2$ (this can be proven but is a pain). Doing so and looking at the inequalities, we find that $0 \leq -8+4(2)-2(-2)+c_0 = c_0+4 \leq 20$ and $0 \leq 8+4(2)+2(-2)+c_0 = c_0+12 \leq 10$. Since $c_0 > 0$ this is impossible, so we can then check the next-best case for b_0, where $b_0 = -3$. Then we find $0 \leq -8+4(2)-2(-3)+c_0 = c_0+6 \leq 20$ and $0 \leq 8+4(2)+2(-3)+c_0 = c_0+10 \leq 10$. That is, to maximize c_0 we ought to pick $c_0 = 0$. We can verify that $g(x) = x^3 + 2x^2 - 3x$ indeed satisfies the given conditions.

So the maximal possible $g(4) = f(7)$ occurs when

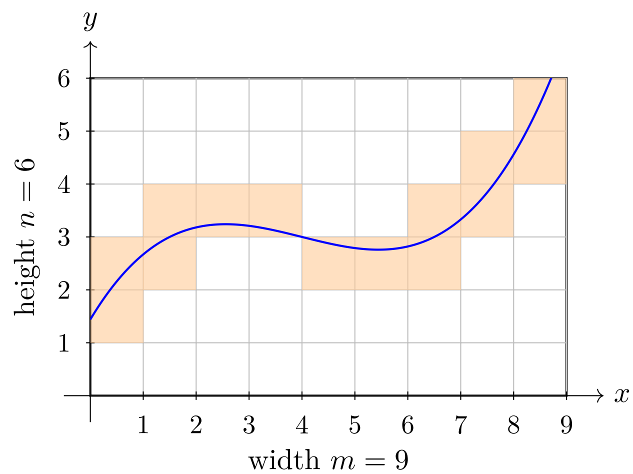
$$g(x) = x^3 + 2x^2 - 3x.$$

Then the maximum value for $f(7) = g(4)$ is

$$4^3 + (2)4^2 - 3(4) = 84,$$

our final answer.

12. In the xy-plane, a rectangular unit grid has width m and height n, parallel to the x and y axes respectively. We call a function "busy" if it intersects at least a third of the squares in the grid. What is the sum of all values of m such that for some n, there exists a "busy" cubic function but not a "busy" quadratic function? (Note that here, only intersecting a vertex of the square does not count as intersecting the square.)



Proposed by: Mason Yu

Answer: $\boxed{357}$

Solution: Notice that a cubic has at most three roots. Therefore it can cross any horizontal line at most three times. Now, to maximize the number of intersected unit squares, we start at the leftmost column. Note that if we do not cross any horizontal line, we just have a maximum of m squares intersected, one for each column, but if we cross a horizontal line we can increase one unit square intersected. We can cross $n-1$ horizontal lines each 3 times by setting four points to create the cubic, two on the first row and two on the last row, so we go back and forth between all rows 3 times. Then we have a maximum of $m+3(n-1)$ squares intersected. We want $m+3(n-1) \geq \frac{mn}{3} \Leftrightarrow 3m+9(n-1)-mn \geq 0 \Leftrightarrow -mn+3m+9n-9 \geq 0 \Leftrightarrow (m-9)(n-3) \leq 18$.

Similarly, a quadratic has only two roots, so we have a maximum of $m+2(n-1)$ squares intersected, so $m+2(n-1) < \frac{mn}{3} \Leftrightarrow -mn+3m+6n-6 < 0 \Leftrightarrow (m-6)(n-3) > 12$.

Note that this implies $m \neq 6$. And also $m \not< 6$ because if so $(m-6)(n-3)$ would never be large enough. Then this implies $n > 3$.

Combining these inequalities and solving for m, we have $6 + \frac{12}{n-3} < m \leq 9 + \frac{18}{n-3}$, or

$$6\left(1+\frac{2}{n-3}\right) < m \leq 9\left(1+\frac{2}{n-3}\right).$$

First, we check what happens when n gets very large. Then the inequality becomes approximately $6 < m \leq 9$. So $m = 7, 8, 9$ works.

$n = 4 \implies 18 < m \leq 27$

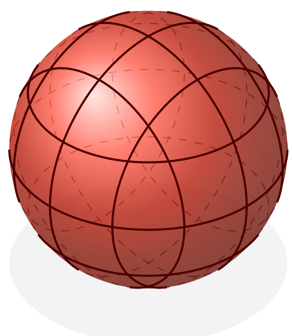
$n = 5 \implies 12 < m \leq 18$

$n = 6 \implies 10 < m \leq 15$

$n = 7 \implies 9 < m \leq 13.5$

and since this interval just keeps decreasing in bounds, we have covered every case from $7 \leq m \leq 27$. The sum of all integers from 7 to 27 inclusive is 357.

13. Chef Joshua is dicing a perfectly spherical tomato on a rectangular cutting board. He makes 3 cuts parallel to the plane board, 3 cuts orthogonal to the length of the board, and the plane of the board. He makes the last 3 cuts orthogonal to the previous 6 for a total of 9 cuts. Given that every cut passes through some part of the tomato, the expected number of pieces Chef Joshua dices the tomato into is a number of the form $a + \frac{b\pi}{c}$. Calculate $a + b + c$.



Proposed by: Vincent Lee

Answer: $\boxed{59}$

Solution: Forgetting about the tomato for now, let's consider the 9 cuts as planes in 3-dimensional space. Let A be the set of planes. Let L be the set of lines where two planes of A intersect. Note that every pair of non-parallel planes intersect at a line in 3-dimensional space. Thus $|L| = 3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 = 27$. Let P be the set of points where three planes of A intersect. Every triple of pairwise non-parallel planes concur at a point in 3-dimensional space. Thus $|P| = 3 \cdot 3 \cdot 3 = 27$. The key insight is to note that the number of regions the tomato is divided into is determined by the number of the aforementioned lines and points that intersect the tomato. In particular, the number of regions is given by

$$1 + \#\{A \text{ intersecting tomato}\} + \#\{L \text{ intersecting tomato}\} + \#\{P \text{ intersecting tomato}\}.$$

It's difficult to visualize this in 3-dimensions, so to convince yourself, it may be helpful to consider the toy case of lines dividing a circle, where it the number of regions of the circle is given by $1 + \#\{\text{lines intersecting circle}\} + \#\{\text{points inside circle}\}$.

Now with the formula in hand, it remains to estimate

$$1 + \#\{A \text{ intersecting tomato}\} + \#\{L \text{ intersecting tomato}\} + \#\{P \text{ intersecting tomato}\}.$$

Since it is given that every cut intersects the tomato $\#\{A \text{ intersecting tomato}\} = 9$. We next look to the points. Let C be the axis-aligned cube with faces tangent to the tomato. Now we note that every point in C corresponds to a unique triple of axis aligned cuts intersecting the tomato, and moreover, a uniformly random choice of point in C corresponds to a uniformly random choice of 3 cuts. Hence the probability that a point lies in the tomato is equal to the ratio

$$\frac{\text{volume of tomato}}{\text{volume of } C} = \frac{\frac{4}{3}\pi r^3}{(2r)^3} = \frac{\pi}{6}.$$

Since there are a total of 27 points in P, by linearity of expectation, the expected number of points intersecting the tomato is

$$27 \cdot \frac{\pi}{6} = \frac{9\pi}{2}.$$

Finally, we look at the lines. Suppose we have a line formed by the intersection of 2 orthogonal vertical cuts. Then viewing the tomato from a bird's eye view, i.e., considering the projection down to 2-dimensional space, the vertical line of intersection becomes a point, and the sphere becomes a circle. In particular, in 2-dimensional space, let S be the axis aligned square with sides tangent to the circle. Now we note that every point in S corresponds to a unique pair of axis aligned vertical cuts of the tomato such that their line of intersection projects to that point. Moreover, a uniformly random choice of point in S corresponds to a uniformly random choice of the two cuts. Hence the probability that a line intersects the tomato is equal to the ratio

$$\frac{\text{area of circle}}{\text{area of } S} = \frac{\pi r^2}{(2r)^2} = \frac{\pi}{4}.$$

The same logic holds for pairs of horizontal and vertical cuts. Since there are a total of 27 lines in L, by linearity of expectation, the expected number of lines intersecting the tomato is

$$27 \cdot \frac{\pi}{4} = \frac{27\pi}{4}.$$

Taken together, the expected number of regions the 9 cuts divide the tomato into is

$$\begin{aligned} & 1 + \#\{A \text{ intersecting tomato}\} + \#\{L \text{ intersecting tomato}\} + \#\{P \text{ intersecting tomato}\} \\ =& 1 + 9 + \frac{27\pi}{4} + \frac{9\pi}{2} \\ =& 10 + \frac{45\pi}{4}. \end{aligned}$$

In the requested answer format, the sum is

$$10 + 45 + 4 = 59.$$

14. For a given integer m, let $l_m(n)$ be the smallest integer $k \geq 1$ such that $n^{2^k} \equiv n \pmod{m}$, or 0 if no such k exists. For example $l_7(2) = 2$ as $2^{2^2} \equiv 2 \pmod{7}$. What is the maximum value of

$$\sum_{n=1}^{m} l_m(n)$$

among all two-digit positive integers m?

Proposed by: Joshua Kou

Answer: $\boxed{802}$

Solution: It is possible verify that a sum of 802 when $m = 83$ is maximal by computing the sum for all two-digit m. The rest of the solution will be devoted to giving some intuition on how to arrive at $m = 83$ within the limited time of the competition.

The first insight is that the sum is maximized for some prime m. To see this, consider the map

$$f(n) = n^2 \bmod m.$$

By definition $l_m(n)$ is equal to the length of the cycle containing n under this map, or 0 if n does not lie in a cycle. Note that if a prime p divides both n and m, then p divides all iterates $f^k(n)$ of n. In particular, writing

$$m = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k},$$

for distinct primes $p_1, \dots, p_n$. Let P be the subset of $p_1, \dots, p_k$ dividing n and let N be the subset of $p_1, \dots, p_k$ not dividing n. By the note before, any iterate $f^k(n)$ must be divisible by any prime $p_i \in P$. Furthermore, if n is in a cycle, i.e., some iterate satisfies $f^r(n) = n$, then any iterate $f^k(n)$ must not be divisible by any prime $p_j \in N$, as otherwise letting rs be some multiple of r greater than k, we have that $n = f^{rs}(n)$ would be divisible by p_j, a contradiction. Thus the length of the cycle containing n is at most the number of integers mod$m$ that are divisible by primes in P and not divisible by primes in N, which is given by the formula

$$l_m(n) \leq \phi\left(m/\prod P\right)$$

where ϕ is Euler's totient function and $\prod P$ is the product of the primes in P.

$$\phi\left(m/\prod P\right)$$

is maximized when $\prod P = 1$, so

$$l_m n \leq \phi\left(m/\prod P\right) \leq \phi(m).$$

To maximize the sum $\sum l_m(n)$, we want to sum over many terms $1 \leq n \leq m$, and for each cycle length to be high. This suggests that we want m to be a large prime, since that maximizes $\phi(m)$.

The second insight is that we want $(m-1)/2$ to be prime. Given our first insight, let's assume that m is prime. Then let g be a generator of the multiplicative group $1, \dots, m-1$, so our squaring map f can be rewritten as

$$f(g^k) = g^{2k}.$$

Since $g^k \equiv g^l \bmod m$ if and only if $k \equiv l \bmod (m-1)$, viewing the exponents induces a map h

$$h(k) = 2k \bmod (m-1).$$

Thus since $g^k = f^n(g^k) = g^{2^n k}$ if and only if $k \equiv 2^n k \bmod (m-1)$, the cycle length $l_m(g^k)$ is the cycle length of k under h. If $2^n \equiv 1 \bmod (m-1)$, then certainly

$$k \equiv 2^m k \bmod (m-1).$$

Therefore, $l_m(g^k) \leq n$, and using a similar argument as in the previous section involving the divisibility of iterates, in fact

$$l_m(g^k) \leq n \leq \phi(m-1).$$

Hence to maximize the cycle length of g^k under $f := g^k \mapsto g^{2k} \bmod m$, we want to maximize the cycle length of 2 under $h := k \mapsto 2k \bmod (m-1)$, which is bounded by $\phi(m-1)$. Thus ideally $m-1$ is prime. Unfortunately, if m is a large prime, $m-1$ cannot be prime, since $m-1$ will be even. So at best $m-1 = 2p$ for some prime p. This suggests that we want to look for primes p such that $m = 2p+1$ is also prime (these are called Sophie Germain primes and are of interest in number theory).

Then one can check that the largest prime satisfying this condition is $p = 41$, $m = 83$. It remains to compute the sum of $l_m(n)$ for $m = 83$. Let $n \neq 0, 1$ be an integer $\mod m$. Then $n = g^k$ for some $k \neq 0$. If k is odd, then since all the iterates of k under h are even, k is not in a cycle, so $l_m(n) = 0$. On the other hand, if k is even, then the cycle length of k under h is the cycle length of 2 under $k \mapsto 2k \bmod 41$. The cycle length is 20. This can be checked manually, but there exists a time-saving tool for determining the cycle length: Lagrange's Theorem tells us that the cycle length must divide $41 - 1 = 40$, so the cycle length is one of $1, 2, 4, 5, 8, 10, 20, 40$. Note that $2^{10} = 1024 \equiv -1 \bmod 41$, so the cycle length is at most 20 and does not divide 10. That leaves $2^8 \not\equiv 1 \bmod 41$, so we conclude that the cycle length is 20. Then the sum is

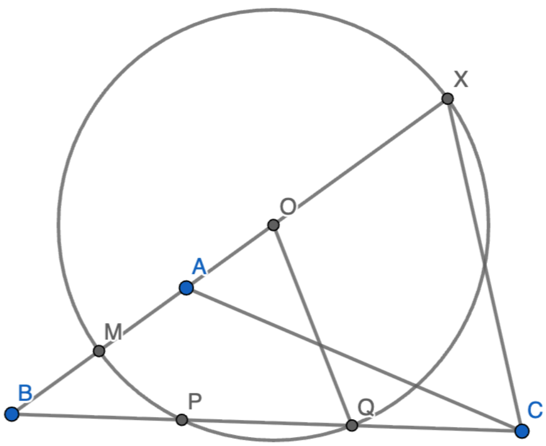
$$\begin{aligned}
\sum_{n=1}^{m} l_m(n) &= l_m(m) + l_m(1) + \sum_{n=2}^{m-1} l_m(n) \\
&= 1 + 1 + \sum_{k=1}^{m-2} l_m(g^k) \\
&= 2 + \sum_{\substack{k=1 \\ k \text{ odd}}}^{m-2} l_m(g^k) + \sum_{\substack{k=1 \\ k \text{ even}}}^{m-2} l_m(g^k) \\
&= 2 + 0 + \sum_{\substack{k=1 \\ k \text{ even}}}^{m-2} 20 \\
&= 2 + 40 \cdot 20 \\
&= 802
\end{aligned}$$

15. Triangle ABC has side length $BC = 3\sqrt{10}$. A circle ω with center O passes through the midpoint of $\overline{AB}$ and trisects $\overline{BC}$ at P and Q, where P is closer to B. X lies on ω such that B, A, X lie on the same line in that order. If $\angle AOQ = \angle BCX$ and $\angle BQO = \angle AXC$, what is the sum of the area of circle ω and the area of ABC?

    Proposed by: Mason Yu

    Answer: $\boxed{16\pi + 3\sqrt{15}}$

    Solution:



We claim O lies on AB. Note that $\angle XBC + \angle BXC + \angle BCX = 180°$, and $\angle QBO + \angle BQO + \angle BOQ = 180°$. Substituting for equalities and rearranging we obtain $\angle XBC - \angle QBO = \angle BOQ - \angle AOQ$. Now if $\angle XBC - \angle QBO$ is positive then A must lie on the opposite side of OB as Q. This directly implies that $\angle BOQ - \angle AOQ$ is negative. An analogous argument shows that $\angle XBC - \angle QBO$ negative implies $\angle BOQ - \angle AOQ$ is positive. Therefore both sides must be zero, and then O must be collinear with B, A, X.

Because $\angle BQO = \angle AXC$, then $XOQC$ is a cyclic quadrilateral. Let its circumcircle be γ. Note that X lies on ω, so X must be the intersection of ω and γ.

Let the midpoint of AB be M. Using power of a point on circle ω, we have $BM \cdot BX = BP \cdot BQ = \sqrt{10} \cdot 2\sqrt{10} = 20$. Also, using power of a point on circle γ, we get $BO \cdot BX = BQ \cdot BC = 3\sqrt{10} \cdot 2\sqrt{10} = 60$. Therefore $BO \cdot BX = 60$ and $BM \cdot BX = 20$. Thus $BO = 3BM$. Therefore BM is half the radius of the circle. Using this relation, $BM \cdot BX = BM \cdot 5BM = 20$ and BM must be 2, then the radius of the circle is 4, and the area of the circle is 16π.

Then since A is collinear with B and O we can use by symmetry the length $OB = OC$ is 6, and the area of ABC is just two thirds of the area of BOC, which is $\frac{9\sqrt{15}}{2}$ by Heron's formula. So the area of ABC is $3\sqrt{15}$.

Therefore our desired sum is $16\pi + 3\sqrt{15}$.