Set 1

1. [**6**] Vincent is rolling a modified 6-sided die. His die only has faces labeled with 1s and 2s. If the probability that he rolls a 1 is twice the probability that he rolls a 2, how many faces are labeled with 1?

   Proposed by: Mason Yu

   Answer: $\boxed{4}$

   Solution: Let i the probability of rolling a 1 and j the probability of rolling a 2. We need that $i + j = 1$ and $i = 2j$. Solving this system yields $i = \frac{2}{3}, j = \frac{1}{3}$. Since there are 6 faces of the die, $\frac{2}{3} \cdot 6 = \boxed{4}$ faces.

2. [**6**] Faizaan's dorm number is a two-digit number. When you divide it by 7 or 9, the remainder is 2. What is his dorm number?

   Proposed by: Amber Bajaj

   Answer: $\boxed{65}$

   Solution: Let n be the missing number. We have $n = m + 2$, where m is divisible by both 7 and 9. The LCM of 7 and 9 is 63. This is the only 2-digit number that is divisible by both 7 and 9. Thus, $m = 7 \times 9 = 63$ and $n = 63 + 2 = \boxed{65}$.

3. [**6**] Five first-year students take the MATH 350 midterm and receive positive integer scores. The mean test score is twice the median, and no one got the same score. What is the minimum possible value of the highest of their scores?

   Proposed by: Mason Yu

   Answer: $\boxed{13}$

   Solution: We want to minimize how large their scores are, so we minimize the median. We can do this by setting the lowest scores as $1, 2, 3$. Then the median is 3. So the mean must be 6, so the sum of the scores is 30. Therefore, the remaining two scores must add to 24. Since the scores must be distinct, we minimize the largest score by setting the remaining scores as 11 and 13. So our desired largest score is $\boxed{13}$.

. . . .

Set 2

4. [**7**] Bruno needs his two-digit security code to withdraw a 1-dollar bill from an ATM at Eastside Marketplace. His code is the last two digits of $20^{20} + 26^{26}$. What two-digit number does Bruno need to enter?

   Proposed by: Mason Yu

   Answer: $\boxed{76}$

   Solution: Note that 20^{20} ends with 20 zeroes so we only need to check 26^{26}. Note that $26 \cdot 26 = 676$, and $676 \cdot 26 = 1976$. Therefore all further multiplications by 26 just keep the last two digits as $\boxed{76}$.

5. [**7**] Bruno walks to Trader Joe's and pays for his groceries with a 1-dollar bill. He gets back 3 coins in change. He forgot the total price of his groceries, but he knows it's less than $1 and contains at least one 7 in it. How many possible amounts could he have received in change?

   Proposed by: Ryan Lee

   Answer: $\boxed{5}$

   Solution: If the 7 is in the tens place, then the amount of change is between 21 and 30 (inclusive). The possibilities are 30 (3 dimes), 27 (2 pennies + quarter), 25 (2 dimes + nickel), 21 (2 dimes + penny).

   If the 7 is in the ones place, then the amount of change must end in 3. This means it must be 3 pennies.

   Our total possibilities are 3, 21, 25, 27, 30, so $\boxed{5}$ possibilities.

6. [**7**] Back in his apartment, Bruno bakes a square pizza with area 9. He cuts it into 16 identical smaller squares. What is the sum of the perimeters of all of these squares?

   Proposed by: Mason Yu

   Answer: $\boxed{48}$

   Solution: Each square has area 9 and therefore side length 3. Since $4^2 = 16$, each smaller square has side length $\frac{3}{4}$. Therefore, each smaller square has perimeter 3. Then since there are 16 of these squares, the total perimeter is $3 \cdot 16 = \boxed{48}$.

. . . .

Set 3

7. [**9**] Sunny draws a triangle with side lengths $8, 15$, and 17. He then draws a rectangle with the same perimeter and area as the triangle. If the diagonal of the rectangle has length L, what is L^2?

   Proposed by: Mason Yu

   Answer: $\boxed{280}$

   Solution: Let the side lengths be a and b. Then $ab = 60$, $2a+2b = 8+15+17 = 40$ so $a+b = 20$. By the Pythagorean Theorem, $L = \sqrt{a^2+b^2}$, so we want to find $a^2 + b^2 = (a+b)^2 - 2ab = 400 - 120 = \boxed{280}$.

8. [**9**] Letters A_1, A_2, A_3, A_4, A_5, and A_6 sit in a circle in clockwise order. Letter A_1 starts with a paintbrush at minute 0. Every minute, the letter with the brush swaps their position with the letter sitting directly across from them, then passes the brush clockwise to the left. After 2026 minutes, for which i does A_i have the brush?

   Proposed by: Alice Wang

   Answer: $\boxed{B}$

   Solution: Note that this action is periodic with order 6. That is, after every 6 minutes, the letters go back to their original position. Therefore, the positions of letters after 2026 minutes are equivalent to the positions of letters after 4 minutes, since $2026 \equiv 4 \pmod 6$.
   It is then easy to calculate that on the 4-th action, A_1 and A_4 swapped positions with A_4 holding the baton, therefore letter A_2 has the baton after the 2026 minutes. Thus $i = \boxed{2}$.

9. [**9**] Jake has 10 pastels, each either red or blue. He then adds an unknown number (possibly zero) of red pastels such that the original probability of randomly picking a red pastel is equal to the probability of randomly picking a blue pastel after adding in the red pastels. What is the sum of all possible amounts of red pastels that Jake added?

   Proposed by: Jerry Sun

   Answer: $\boxed{115}$

   Solution: Let r be the number of red marbles originally, where $10 - r$ is the number of blue marbles. For the given probability to be the same, we must have $\frac{r}{10} = \frac{10-r}{10+s}$, where s is the number of added red marbles. This yields $rs + 10r = 100 - 10r$ which becomes $r(s + 20) = 100$. Then, possible pairs (r, s) are $(1, 80), (2, 30), (4, 5), (5, 0)$. Summing the s values yields $80 + 30 + 5 + 0 = \boxed{115}$.

. . . .

Set 4

10. [**11**] Josiah chooses a random integer b from 0 to 10, inclusive. Sharpe then chooses a random integer c from 0 to 20, inclusive. What is the probability that $x^2 - bx + c$ has exactly one positive root? Express your answer as a fraction reduced to lowest terms.

    Proposed by: Jake Rosenberg

    Answer: $\boxed{\dfrac{2}{33}}$

    Solution: If $c = 0$, then the polynomial will have roots at $x = 0$ and $x = b$, which gives us one positive root unless $b = 0$. The remaining possibilities are when $x^2 - bx + c$ has only one unique root, and these occur at $(b = 2, c = 1)$, $(b = 4, c = 4)$, $(b = 6, c = 9)$, and $(b = 8, c = 16)$. Thus there are $4 + 10 = 14$ possible ways for $x^2 - bx + c$ to have exactly one positive root.

    There are $(10 + 1)(20 + 1) = 231$ ways to choose b and c, so the probability that $x^2 - bx + c$ has exactly one positive root is $\frac{14}{231}$, which reduces to $\boxed{\dfrac{2}{33}}$.

11. [**11**] When viewed from above, the four towers of Grad Center appear to be similar, non-congruent triangles. They each have integer side lengths, and their perimeters add up to 143. What is the greatest possible length of the smallest triangle's longest side?

    Proposed by: Mason Yu

    Answer: $\boxed{6}$

    Solution: Note that since the four triangles are similar, their perimeters must therefore be proportional to their side lengths. Therefore there must exist a minimal triangle with perimeter p such that the gcd of its side lengths is 1 and all other triangles are scaled up from p. Let the perimeter be p. Then, notice that 143 has factors 13 and 11. So p must be either 11 or 13 since all side lengths are integers. Thus we need four distinct integer "scaling factors" $k_1 < k_2 < k_3 < k_4$ of p, and we have the equation $(k_1 + k_2 + k_3 + k_4)p = 143$. Note that $p = 11$ and $p = 13$ both necessitate that k_1 is 1 since the scaling factors need to be unique, and if k_1 is 2 or greater, then $k_1 + k_2 + k_3 + k_4 \geq 14$. Therefore, we choose $p = 13$ since we want to maximize the smallest triangle. Using triangle inequality, we have the maximal triangle has side lengths $6, 6, 1$ and thus our desired side length is $\boxed{6}$.

12. [**11**] Four juniors, A, B, C, and D are playing a game. Independently, they each randomly pick a point on the circumference of a circle. D is the game master, and whoever's point is closest to D's point wins (ties count as a loss). However, A decides to cheat, secretly looking at B and Cs points before choosing their own point to maximize their chance of winning. How many times more likely is A to win if they cheat compared to if they do not cheat?

    Proposed by: Joshua Kou

    Answer: $\boxed{\frac{9}{8}}$

    Solution: Refer to A's point as A, and similarly for B and C.

    If A does not cheat, the game is symmetric so each player has a $1/3$ chance of winning.

    If A does cheat, their chance of winning only depends on the distance between B and C. In particular, B and C divide the circumference into two arcs α, β. If A lies on α, their probability of winning is the fraction of the circumference given by $\alpha/2$, and similarly for β. Hence A maximizes their probability of winning by placing their point anywhere on the major arc between B and C, in which case their probability of winning is $\frac{|\widehat{BC}|}{4\pi}$. Since the length of the major arc $|\widehat{BC}|$ ranges uniformly between $[\pi, 2\pi]$, A wins with probability

$$\frac{3\pi/2}{4\pi} = \frac{3}{8}.$$

    The ratio between the probability of winning when cheating and when not cheating is

$$\frac{3/8}{1/3} = \boxed{\frac{9}{8}}.$$

....

Set 5

13. [**13**] Anthony wants to ride a bus that is supposed to come every 10 minutes, starting from 8:00AM. However, each bus has $\frac{1}{3}$ chance of not coming at all. If Anthony arrives at the bus station at a uniformly random time between 8:00AM to 9:00AM, what is Anthony's average wait time in minutes?

    Proposed by: Alice wAng

    Answer: $\boxed{10}$

    Solution: Note that Mason's wait time should be the number of minutes to the first scheduled bus, plus 10 times each missed bus. Mason's original wait time, if all buses come on time, is $\frac{10}{2} = 5$. The expected number of missed buses before the first one comes is $\frac{1/3}{2/3} = \frac{1}{2}$, which will on average result in $10 \cdot \frac{1}{2} = 5$ more minutes of wait time. Thus the total expected wait time is $5 + 5 = \boxed{10}$.

14. [**13**] There are 3 trains from Boston to Providence labeled T_1, T_2, and T_3, with exactly 2 students on each train. Each train independently has probability $\frac{1}{2}$ of being cancelled. If train T_i is cancelled, those students take the next uncancelled train T_j with $j > i$. If no such train exists, those students are stuck in Boston. The students are randomly divided into 2 teams of 3. Compute the probability that everyone on team 1 successfully reaches Providence.

   Proposed by: Peter Wang

   Answer: $\boxed{\dfrac{11}{20}}$

   Solution: When computing the expected probability that all of team 1 make it, we condition on the pattern of cancellations. Each train F_1, F_2, F_3 independently cancels with probability $\frac{1}{2}$. If flight i is cancelled, its students are rebooked on the lowest-numbered non-cancelled train $j > i$; if no such train exists, they cannot attend. Let t be the number of consecutive cancelled trains at the end ($t \in \{0, 1, 2, 3\}$). Then only $6 - 2t$ of the six students reach the competition. Since cancellations are independent,

$$P(t = 0) = \tfrac{1}{2}, \quad P(t = 1) = \tfrac{1}{4}, \quad P(t = 2) = \tfrac{1}{8}, \quad P(t = 3) = \tfrac{1}{8}.$$

   If $S = 6 - 2t$ students succeed, the probability that all three students on team 1 do is

$$P(\text{all make it} \mid S) = \frac{\binom{S}{3}}{\binom{6}{3}}.$$

   Hence,

$$\begin{aligned} P(\text{all make it}) &= \sum_t P(t) \cdot \frac{\binom{6-2t}{3}}{\binom{6}{3}} \\ &= \tfrac{1}{2} \cdot \frac{\binom{6}{3}}{\binom{6}{3}} + \tfrac{1}{4} \cdot \frac{\binom{4}{3}}{\binom{6}{3}} + \tfrac{1}{8} \cdot 0 + \tfrac{1}{8} \cdot 0 \\ &= \tfrac{1}{2} + \tfrac{1}{4} \cdot \tfrac{4}{20} = \boxed{\tfrac{11}{20}}. \end{aligned}$$

15. [**13**] Outside the Capitol building, Efe polls a group of unknown size about ice cream flavors. The poll shows that 6% prefer strawberry, 7% prefer chocolate, 67% prefer vanilla, and the rest have no preference. However, Efe knows that the poll rounds percentages to the nearest integer (with 0.5 rounding up). What is the least possible number of people in the group that have no preference?

   Proposed by: Mason Yu

   Answer: $\boxed{11}$

   Solution: We test the first two conditions. We need that there are two integers A, B such that

$$0.055n \le A < 0.065n \le B < 0.075n.$$

   Therefore we only need to test $0.075n - 0.055n \ge 2$. We test from 50, the smallest possible value where this holds. Then we have $2.75 \le A < 3.25 \le B < 3.75$ which fails on the B side of the inequality. So we take the least possible n such that the highest constraint is greater than 4. This is 54, which yields a constraint of $2.97 \le A < 3.51 \le B < 4.05$. This suffices for

$A = 3, B = 4$.. Then if we have 54 people, then 3 people prefer strawberry and 4 people prefer vanilla. We check that 36 people prefer vanilla ice cream, which is $\frac{2}{3} \approx 67\%$ of 54. This leads to $\boxed{11}$ people with no preference.

....

Set 6

16. [**16**] The cloud service *Oscar* manages 20 virtual machines, each handling 100 requests. At time 0, there are no requests. Every second starting from time 1, 21 new requests arrive. Whenever total usage exceeds 50% capacity at time t, one new machine is queued up to be added at time $t+5$. From time t to $t+4$ (inclusive), no other computers are queued up. The system crashes at time T when it runs out of capacity. Compute T (in seconds).

    Note: if a machine is added at time t, it is added before the requests come at time t.

    Proposed by: Faizaan Siddique

    Answer: $\boxed{962}$

    Solution: The system can handle a total of $20 \times 100 = 2000$ user requests. Observe that each time a new virtual computer is created, the threshold for triggering the next usage warning is raised by $\frac{100}{2} = 50$. However, in this time $21 \cdot 5 = 105 > 50$ new requests will have been made anyway. Thus, after the very first warning is triggered, Brumonetes will be constantly creating new virtual computers every 5 seconds to handle the incoming user load. Thus, we desire the smallest T where

$$21T > 2000 + 100 \cdot \left\lceil \frac{T - (\lceil \frac{1000}{21} \rceil + 5) + 1}{5} \right\rceil = 2000 + 100 \cdot \left\lceil \frac{T - 52}{5} \right\rceil.$$

    For $m = 0, 1, 2$, we have $\left\lceil \frac{T-52}{5} \right\rceil = n - 10$. For $m = 3, 4$, we have $\left\lceil \frac{T-52}{5} \right\rceil = n - 9$.

    Let $T = 5n + m$, where $0 \le m \le 4$. The inequality becomes

$$5n + 21m > \begin{cases} 1000 \text{ if } m = 0, 1, 2 \\ 1100 \text{ if } m = 3, 4 \end{cases}.$$

    We can check that $n = 192, m = 2$ yields the smallest solution $T = \boxed{962}$.

17. [**16**] Each machine in *Oscar* is in the shape of a convex quadrilateral $ABCD$ with $\angle A = 90°$ and $\angle B = 150°$. Furthermore, we are given that $\angle CAB = 18°$ and $AD = BC$. What is the measure of $\angle D$?

    Proposed by: Cody Cheng

    Answer: $\boxed{72°}$

    Solution: Let P be a point in $ABCD$ such that $\triangle APD$ is equilateral. Then $AP = AD = BC$, and furthermore, $\angle BAP = 30°$, so $AP \| BC$ and $APCB$ is a parallelogram. As a result, $AB = PC$. Also, $PD = PA$ by construction and $\angle DPC = \angle APC = 150°$ so by SAS, $\triangle DPC \cong \triangle APC$. Therefore, $\angle CDA = \angle CDP + \angle PDA = 12° + 60° = \boxed{72°}$.

18. [**16**] In the SciLi, windows are arranged in a 10×10 grid. A screen is attached to each window, and a cable connects adjacent screens in each row and column. Screen A is at the top-left, and screen B is at the bottom-right. Bruno is playing a game and wants to move his character from A to B. He can move down or right to adjacent screens if they are connected by a cable.

A recent meltdown destroyed one cable in such a way that maximized the number of remaining possible routes from A to B. Then, a party destroyed another cable in such a way that minimized the number of remaining possible routes from A to B. How many routes remain from A to B?

Proposed by: Adeethyia Shankar

Answer: $\boxed{24309}$

Solution: For the first road, we must minimize the number of routes that depend on it, so we should select a road in the bottom right or top left of Brunotopia so that we only lose one route. Without loss of generality, let's select the horizontal road in the bottom left that is parallel to Bruno's house.

For the second road, we should select a road connected to one of the bears' houses, with the goal of maximizing the number of blocked routes. Without considering the first blocked road, each of these four options eliminates exactly half of the routes by symmetry. Applying the principle of inclusion-exclusion, maximizing the total number of blocked routes is equivalent to minimizing the number of routes that require both roads. Selecting either the horizontal road connected to Aruno's house or the vertical road connected to Bruno's house achieves zero intersection.

As traversing 9 horizontal and vertical roads each allows for $\binom{18}{9}$ choices, the number of remaining routes is

$$\frac{1}{2}\binom{18}{9} - 1 = 24310 - 1 = \boxed{24309}.$$

. . . .

Set 7

19. [**20**] Let $f(x)$ be a function over the positive integers, such that for any two distinct positive integers m and n, the least common multiple of $f(m)$ and $f(n)$ is strictly less than $m+n$. Suppose $f(50) = 49$. How many possible values are there for the sum $f(1) + f(2) + f(3) + \cdots + f(49)$?

Proposed by: Mason Yu

Answer: $\boxed{96}$

Solution: Note that by plugging in the inequality, $f(n) \leq \operatorname{lcm}(f(n), f(1)) < n + 1$. Therefore, $f(n) \leq n$ for all $n \neq 1$.

By the initial condition, $\operatorname{lcm}(49, f(n)) < 50 + n$ for all n. The LHS must be of the form $49k$ for some positive integer k. Then, if $n \leq 48$, then $k = 1$. So $f(n)$ must be a power of 7 for all $n \leq 48$.

Therefore, all our possible values for $f(1), f(2), f(3) \ldots f(49)$ are that $f(1) = f(2) = \cdots = f(6) = 1$ and $f(7), f(8), f(9) \ldots f(48)$ are all either 1 or 7. These values work because for $1 \leq n, m \leq 6, \operatorname{lcm}(f(m), f(n)) = 1 < m + n$, and for all other $1 \leq n, m \leq 48, \operatorname{lcm}(f(m), f(n)) = 7 < m + n$.

Now, we just need to analyze all values of $f(49)$. Note $\text{lcm}(49, f(49)) < 99$, so $\text{lcm}(49, f(49)) = 49$ or 98. Therefore, $f(49)$ can take values $1, 2, 7, 14$, or 49. All of these values work, because for all $n \leq 48, \text{lcm}(f(n), f(49)) \leq 49 < 49 + n$.

Therefore, the possible values of the sum $f(1) + f(2) + f(3) + \cdots + f(49)$ is $6 + 42 + 6j + f(49)$ for some nonnegative integer $j \leq 42$. Note that for $f(49) = 1, 7, 49$, this sum is always $1 \pmod 6$ and for $f(49) = 2, 14$ this sum is always $2 \pmod 6$. For the former case, the possible values of the sum are $98, 104, 110, \ldots 398$. There are $\frac{300}{6} + 1 = 51$ values here. For the latter case, the possible values of the sum are $99, 105, 111, \ldots 363$. There are $\frac{264}{6} + 1 = 45$ values here. The total number of values is $45 + 51 = \boxed{96}$.

20. [**20**] Cody manufactures 16 rings in 8 pairs, with each pair having a different design. He arranges them such that 4 of the pairs are separated by an even number of rings and the other 4 are separated by an odd number. Let N be the number of possible arrangements of rings. If the prime factorization of N is $p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$, then calculate $p_1 + p_2 + p_3 + p_4 + n_1 + n_2 + n_3 + n_4$.

Proposed by: Mason Yu

Answer: $\boxed{40}$

Solution: Split the line of rings into two distinct groups, one having all the odd indices, and one having all the even indices. Four of the pairs must have one ring in each group (since any two of these are separated by an even number of balls). So the other four pairs must be split evenly with two pairs in each group. There are $\frac{8!}{4!2!2!}$ ways to partition the colors into 3 groups (odd only, even only, odd-even)

There are $\frac{8!}{2!2!}$ ways to order the colors within the odd component (since there are two repeats) and same with the even component.

So the answer is $\frac{8!}{4!2!2!} \frac{8!}{2!2!} \frac{8!}{2!2!} = 2^{12} \cdot 5^3 \cdot 3^5 \cdot 7^3$.

Therefore our answer is $2 + 5 + 3 + 7 + 12 + 3 + 5 + 3 = \boxed{40}$.

21. [**20**] A factory robot takes a jewel shaped as an acute triangle ABC and marks a point D such that

$$AB \cdot CD = AC \cdot BD = BC \cdot AD.$$

Given that $\angle ACB = 55°$, what is the value of $\angle ADB$ in degrees?

Proposed by: Sunny Xu

Answer: $\boxed{115°}$

Solution: Since $AC \cdot BD = BC \cdot AD$, $\frac{AC}{BC} = \frac{AD}{BD}$. Since D is inside $\triangle ABC$, $\angle ADB > \angle ACB$, construct B' such that $\angle ADB' = \angle ACB$, $\angle ADB' + \angle B'DB = \angle ADB$, and $B'D = BD$. As a result, $\frac{AC}{BC} = \frac{AD}{BD} = \frac{AD}{B'D}$, and since $\angle ACB = \angle ADB'$,

$$\triangle ACB \sim \triangle ADB'.$$

Hence, $\angle CAB = \angle DAB'$, so $\angle CAB - \angle DAB = \angle DAB' - \angle DAB$, which is $\angle CAD = \angle BAB'$. Since $\triangle ACB \sim \triangle ADB'$, $\frac{AC}{AD} = \frac{AB}{AB'}$, and since $\angle CAD = \angle BAB'$,

$$\triangle ACD \sim \triangle ABB',$$

so $\frac{AB}{AC} = \frac{BB'}{CD}$. Since $AB \cdot CD = AC \cdot BD$,

$$BB' = BD.$$

Since $BD = B'D$, $\triangle BDB'$ is an equilateral triangle, so $\angle BDB' = 60^\circ$. Hence,

$$\angle ADB - \angle ACB = \angle ADB - \angle ADB' = \angle BDB' = 60^\circ.$$

Finally, we have $\angle ADB = 60^\circ + \angle ACB = \boxed{115^\circ}$.

....

Set 8

22. [**24**] For some reason, the route numbers for RIPTA buses at Kennedy Plaza are represented by complex numbers. Jaeho is trying to find the bus to Woonsocket, but he only knows that its route number is a complex number z such that the following equation has a real root:

$$x^2 + zx + 4 + 3i = 0.$$

Find the minimum possible value of $|z|$.

Proposed by: Leo Zhang

Answer: $\boxed{3\sqrt{2}}$

Solution: From the equation

$$x^2 + zx + 4 + 3i = 0,$$

we can express this as

$$zx = -\big(x^2 + 4 + 3i\big).$$

Clearly, $x = 0$ is not a solution. Hence $x \neq 0$, and thus

$$z = -\left(\frac{x^2+4}{x} + \frac{3i}{x}\right).$$

If the equation $x^2 + zx + 4 + 3i = 0$ has a real root x, then

$$z = -\left(x + \frac{4}{x}\right) - \frac{3}{x}i, \quad x \in \mathbb{R}.$$

Therefore,

$$|z| = \sqrt{\left[-\left(x + \frac{4}{x}\right)\right]^2 + \left(-\frac{3}{x}\right)^2} = \sqrt{x^2 + \frac{25}{x^2} + 8}.$$

By the AM–GM inequality,

$$x^2 + \frac{25}{x^2} \geq 2\sqrt{x^2 \cdot \frac{25}{x^2}} = 10.$$

Thus,

$$|z| \geq \sqrt{10+8} = \sqrt{18} = 3\sqrt{2}.$$

Equality holds when $x = \pm\sqrt{5}$. So the minimum value of $|z|$ is $\boxed{3\sqrt{2}}$.

23. [**24**] A hockey puck of negligible radius sits on the perfectly circular Providence Rink, halfway between the center and the wall. Byron hits the hockey puck such that the 999th time it bounces off the circular wall is precisely at the point on the wall closest to where it started. How many possible directions could Byron have hit the hockey puck in?

Proposed by: Mason Yu

Answer: $\boxed{667}$

Solution: Let the point 0 be the point on the wall closest to where it starts. Suppose the hockey puck is hit towards point θ on the unit circle. By Law of Sines, the angle from ϕ satisfies $\sin(\phi) = \frac{\sin(\phi+\theta)}{2}$. And also the angular displacement each bounce is $\pi - 2\phi$. So then we want $\theta + 998(\pi - 2\phi) = 2\pi k$ for some integer k. Using these two equations, we get that $\sin(((2k-998)\pi + 1996\phi) + \phi) = 2\sin(\phi)$, which simplifies to $\sin(1997\phi) = 2\sin(\phi)$. Since ϕ uniquely determines θ, we just have to find all the solutions for this relation of ϕ between 0 and $\frac{\pi}{6}$, which is the maximum angle of ϕ. We know that $2\sin(\phi)$ is increasing from 0 to $\frac{\pi}{6}$, and $\sin(1997\phi)$ is a rapidly oscillating function with some number of peaks there. Note that every peak is of the form $\frac{(1+4k)\pi}{3994}$, so every k from 0 to 166 works. This generates 333 solutions, since every k generates two solutions except for 0 (since we don't count the intersection at zero). Now we multiply by two (to take into account both sides) and then add the zero case to get $\boxed{667}$ solutions.

24. [**24**] Let $a_0, a_1, a_2, \ldots, a_{150}$ be positive integers such that for all $n \geq 1$, the decimal expansion of $\frac{a_n}{a_{n-1}}$ contains the digits forming n directly after the decimal point. For example, $\left\{\frac{a_{15}}{a_{14}}\right\} = .15\ldots.$ What is the least possible value of a_{50}?

Proposed by: Mason Yu

Answer: $\boxed{27}$

Solution: We want $\left\{\frac{a_{51}}{a_{50}}\right\}$ to have decimal digits starting with 0.51. This is equivalent to $a_{50}(m + 0.51) \leq a_{51} < a_{50}(m + 0.52)$ for some integer m. We let $m = 0$ since it just adds an integer to both sides of the inequality, and we want to minimize a_{50}. Then, $a_{50} \cdot 0.51$ and $a_{50} \cdot 0.52$ must contain an integer in between them. Note that for even integers a_{50}, $a_{50} \cdot 0.51$ has decimal representation $\overline{\left(\frac{a_{50}}{2}\right)}.\overline{(a_{50})}$ and $a_{50} \cdot 0.52$ has decimal representation $\overline{\left(\frac{a_{50}}{2}\right)}.\overline{(2a_{50})}$. Due to these decimals carrying over, these yield different integer parts when $a_{50} > 50$. And for odd integers a_{50}, $a_{50} \cdot 0.51$ has decimal representation $\overline{\left(\frac{a_{50}-1}{2}\right)}.\overline{(50 + a_{50})}$ and $a_{50} \cdot 0.52$ has decimal representation $\overline{\left(\frac{a_{50}-1}{2}\right)}.\overline{(50 + 2a_{50})}$. Again, due to carry-over, these yield different integer parts when $a_{50} > 25$. So we can have $a_{50} = 27$.

Now we check whether this sequence works in both directions. For the forward direction, note that we can set m and therefore a_{51} to be arbitrarily large. This means that there will always be some positive integer for which $\frac{a_{52}}{a_{51}}$ has the correct digits after the decimal. This applies to all numbers in the forward direction.

For the backwards direction, note that $a_{49} = 54$ and $a_{48} = 110$ work. Then, since all previous integers can be set to be greater than 100, the range $\frac{a_n}{d} - \frac{a_n}{d+0.01} = \frac{0.01a_n}{d(d+0.01)}$, for the target decimal d is always greater than 1 and therefore there must be an integer a_{n-1} such that $\frac{a_n}{a_{n-1}}$ has the correct decimal d.

Therefore, $\boxed{27}$ works in both directions and it is optimal.