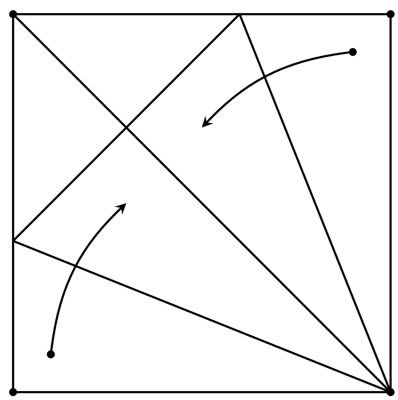
1. Mason has $x^2 + 12x + 38$ apples for some positive integer x. When he splits these apples equally among $x + 2$ friends, he finds that there are 3 apples remaining. Find the sum of all possible values of x.

   Proposed by: Ryan Lee

   Answer: $\boxed{16}$

   Solution: Dividing the polynomials, we get a quotient of $x + 10$ and a remainder of 18. Thus we need to have $18/(x+2)$ have a remainder of 3. So 15 must be a multiple of $x+2$, which yields $x + 2 = 1, 3, 5, 15$, or $x = 1, 3, 13$ (ignoring negative values). But if $x = 1$ then it's impossible to have a remainder of 3. So our possible values are $x = 3, 13$ for a sum of 16.

2. Jerry is folding paper cranes with square papers of side length 1. At one step, he has to form a crease from one corner to the opposite corner, and then fold two edges on either side of the diagonal crease (as shown in the diagram) so that they fully line up with the crease. What is the area of the resulting kite? Express your answer as $a\sqrt{b} + c$, where a, b, c are (not necessarily positive) integers and b contains no square factors other than 1.



   Proposed by: Mason Yu

   Answer: $\boxed{-\sqrt{2} + 2}$

   Solution: Split the kite into two triangles down the diagonal crease. The diagonal has length $\sqrt{2}$ so, since the angle formed by the diagonal and one of the sides is $45°$, the segment that isn't covered by the folded over edge is $\sqrt{2} - 1$ which is also the height of the triangle. Therefore the area of each of the two triangles is $\frac{\sqrt{2}(\sqrt{2}-1)}{2} = 1 - \frac{\sqrt{2}}{2}$ and the combined area of both triangles is twice this which is $-\sqrt{2} + 2$.

3. Amber is presented with the expression 1 ? 2 ? 3 ? 4 ? 5 ? 6 ? 7, where each question mark represents an operation. How many ways are there for Amber to fill in the six question marks with 2 '+'s , 2 '−'s, and 2 '×'s such that the result of the expression is an odd number?

   Proposed by: Jake Rosenberg

   Answer: $\boxed{12}$

   Solution: Because we only care about the result of the expression mod 2, we can reduce it to 1 ? 0 ? 1 ? 0 ? 1 ? 0 ? 1. We also realize that addition and subtraction are equivalent mod 2, so we can reduce again to 4 '+'s and 2 '×'s.

   If the two multiplications are not placed next to each other, then two pairs of 1 and 0 will be reduced to 0s. This means that there will be two 1s remaining, so the result of the expression will be even.

If the two multiplications are placed next to each other, and there is a 0 in between them, then two 1s and one 0 will be reduced to a 0. Again, there will be two 1s remaining, so the result of the expression will be even.

If the two multiplications are placed next to each other, and there is a 1 in between them, then one 1 and two 0s will be reduced to a 0. So this time there will be three 1s remaining, which means that the result of the expression will be odd.

Returning to the original setting of the problem, we see that the only way the expression evaluates to an odd number is if Amber places the two multiplication symbols on either side of the same odd number. This is possible in two ways, with 3 and 5. After the multiplication symbols are placed, there are $\frac{4!}{2\times 2} = 6$ ways to place the 4 remaining symbols. Thus, there are 12 ways for Amber to fill in the question marks such that the expression evaluates to an odd number.

4. A regular tetrahedron has a total surface area S_1. Let the surface area of its circumscribed sphere be S_2. It is known that

$$\frac{S_2}{S_1} = \frac{\sqrt{a}\pi}{b},$$

where a and b are positive integers with no common factors other than 1. Find $a + b$.

Proposed by: Peter Wang

Answer: $\boxed{5}$

Solution: Let the edge length of the tetrahedron be a. Then the total surface area is

$$S_1 = 4 \times \frac{\sqrt{3}}{4}a^2 = \sqrt{3}a^2.$$

Let AH be the height from vertex A to the opposite face BCD. By geometry,

$$AH = \sqrt{a^2 - \left(\frac{1}{\sqrt{3}}a\right)^2} = \frac{\sqrt{6}}{3}a.$$

Let the center of the circumscribed sphere be O on segment AH, and let $AO = r$. Since $OB^2 = \left(\frac{\sqrt{6}}{3}a - r\right)^2 + \left(\frac{1}{\sqrt{3}}a\right)^2 = r^2$, solving gives

$$r = \frac{\sqrt{6}}{4}a.$$

Thus, the surface area of the sphere is

$$S_2 = 4\pi r^2 = 4\pi\left(\frac{\sqrt{6}}{4}a\right)^2 = \frac{3}{2}\pi a^2.$$

Therefore,

$$\frac{S_2}{S_1} = \frac{\frac{3}{2}\pi a^2}{\sqrt{3}a^2} = \frac{\sqrt{3}\pi}{2}.$$

Hence $a = 3$, $b = 2$, and

$$\boxed{a + b = 5.}$$

5. A megacorporation's vault performs a "format-flip" on a positive integer N: it writes N in base-3, then interprets that same string of digits as a base-4 numeral and converts it back to base-10. Denote the result by $F(N)$.

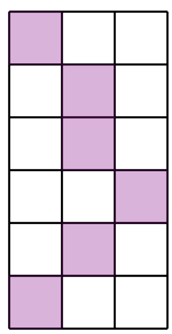
   If $F(N) - N = 2002$ in base-10, the vault unlocks. Among all possible code, the runner uses the smallest possible N to reduce detection time. Find N, with your answer written in base-10.

   Proposed by: Jaeho Lee

   Answer: $\boxed{726}$

   Solution: Let N have base-3 representation $(a_k \cdots a_1 a_0) * 3$ with each $a_i \in 0, 1, 2$. Interpreting the same digits in base 4 gives $F(N) = (a_k \cdots a_1 a_0) * 4$, so $F(N) - N = \sum_{i=0}^{k} a_i(4^i - 3^i) = 2002$. Since $4^0 - 3^0 = 0$, the digit a_0 does not affect the difference and will be chosen at the end to minimize N. Note $4^6 - 3^6 = 4096 - 729 = 3367 > 2002$, so $a_i = 0$ for all $i \geq 6$. Compute $4^1 - 3^1 = 1$, $4^2 - 3^2 = 7$, $4^3 - 3^3 = 37$, $4^4 - 3^4 = 175$, $4^5 - 3^5 = 781$, whose sum is $1+7+37+175+781 = 1001$. Because each $a_i \leq 2$, the maximum possible value of $\sum_{i=1}^{5} a_i(4^i - 3^i)$ is $2 \cdot 1001 = 2002$, exactly the target; hence we must have $a_1 = \cdots = a_5 = 2$ and $a_i = 0$ for $i \geq 6$, otherwise the sum would be strictly smaller than 2002 with no way to compensate. To minimize N, take $a_0 = 0$. Then $N = \sum_{i=1}^{5} 2 \cdot 3^i = 2(3 + 9 + 27 + 81 + 243) = 2 \cdot 363 = 726$. Checking, $F(N) = \sum_{i=1}^{5} 2 \cdot 4^i = 2(4 + 16 + 64 + 256 + 1024) = 2 \cdot 1364 = 2728$, so indeed $F(N) - N = 2728 - 726 = 2002$. Therefore the desired N is $\boxed{726}$.

6. A rectangular grid has 6 rows and 3 columns. Two squares in the grid are said to be connected if they meet at a corner or an edge. If one square in each row is selected randomly, what is the probability that the 6 selected squares form a connected path?

   Proposed by: Jake Rosenberg

   Answer: $\boxed{\dfrac{239}{729}}$

   Solution: Let $m(n)$ be defined as the number of connected paths of length n that have the square selected in the bottom row, and let $s(n)$ be defined as the number of connected paths of length n that have an edge square selected in the bottom row (in each case, we're only allowing one square per row). We see that for $m > 1$, $m(n) = m(n-1) + s(n-1)$, since the bottom middle square will connect to any of the three possible squares above it. Similarly, $s(n) = s(n-1) + 2m(n-1)$, as the bottom left and bottom right squares each connect to the square directly above them and both connect to the middle square above them. Clearly $m(1) = 1$, and $s(1) = 2$, so we can used this recursion to calculate $m(6)$ and $s(6)$.

$m(1) = 1, s(1) = 2.$
$m(2) = 3, s(2) = 4.$
$m(3) = 7, s(2) = 10.$
$m(4) = 17, s(4) = 24.$
$m(5) = 41, s(5) = 58.$
$m(6) = 99, s(6) = 140.$

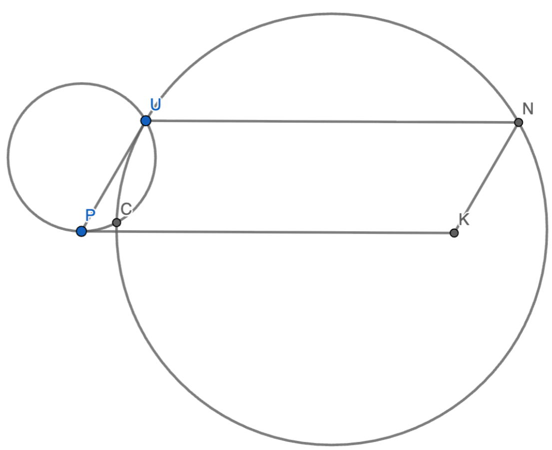
This means that there are a total of $99 + 140 = 239$ possible ways for the selected squares to form a connected path. Since each row has three squares, there are $3^6 = 729$ ways to choose the squares. Thus the probability of the selected squares forming a connected path is $\frac{239}{729}$.

7. Let $PUNK$ be a parallelogram with $PU = 24, PK = 70$, and $\angle UPK = 60°$. Let Ω_1 be the circle passing through P and U tangent to $\overline{PK}$, and let Ω_2 be the circle passing through U and N tangent to $\overline{PU}$. Let Ω_1 and Ω_2 intersect at $C \neq U$. The length CP can be expressed as $\frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by a square greater than 1. Compute $m + n$.

   Proposed by: Max Liu

   Answer: $\boxed{2077}$

   Solution:



We show that C lies on $\overline{PN}$. From inscribed arcs on Ω_1, we can see that $\angle KPC = \angle PUC$ since they both subtend arc PC. Similarly, we can see that $\angle PUC = \angle UNC$ since they both subtend arc UC in Ω_2. Therefore, $\angle KPC = \angle UNC$.

Let $\overline{NC}$ intersect $\overline{PK}$ at P'. Then $\angle UNC$ and $\angle KP'N$ are alternate interior angles, so they are equal. This implies that $P = P'$, so C lies on $\overline{PN}$.

By Power of a Point, we have $PC \cdot PN = PU^2$. Using the Law of Cosines, we can compute $PN^2 = 24^2 + 70^2 - 2 \cdot 24 \cdot 70 \cdot \cos(120°) = 7156$, so

$$PC = \frac{PU^2}{PN} = \frac{576}{\sqrt{7156}} = \frac{288}{\sqrt{1789}}.$$

Therefore, $m + n = 2077$.

8. Calculate:
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{6(mn+1)^2-(mn+n+m+2)^2}{2^n3^m}$$

Proposed by: Ethan Bove

Answer: $\boxed{15}$

Solution: Consider the function
$$f(n,m)=\frac{(mn+1)^2}{2^n3^m}$$

We can rewrite our summation as:
$$6\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f(n,m)-f(n+1,m+1)$$

For any fixed n,m, we note that $f(n+i,m+i)\to 0$, so
$$\sum_{i=0}^{\infty}f(n+i,m+i)-f(n+i+1,m+i+1)=f(n,m)$$

We can break the sum up into:
$$\sum_{i=0}^{\infty}f(i,i)+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}f(i,m+i)+\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}f(n+i,i)$$

Applying our result about diagonal sums, this becomes:
$$f(0,0)+\sum_{m=0}^{\infty}f(0,m)+\sum_{n=0}^{\infty}f(n,0)$$
$$=1+\sum_{m=0}^{\infty}2^m+\sum_{n=0}^{\infty}3^n=1+1+\frac{1}{2}.$$

Multiplying through by 6 again, we get $6(1+1+\frac{1}{2})=15$.

9. Let $f_1,f_2\dots f_{100}$ be real-valued functions such that for all positive integers k such that $k\le 100$, $f_k(x)$ is either $\lfloor x\rfloor$ or $\lceil x\rceil$. Define $f(x)=f_{100}(x\cdot f_{99}(x\cdot f_{98}(\dots x\cdot f_1(x)\dots)))$. Suppose that, for a real number y, there are exactly 111 different possible values of $f(y)$. What is the greatest possible rational value of y? Express your answer as a fraction $\frac{a}{b}$, where a and b are positive integers with no common factors other than 1.

Proposed by: Mason Yu

Answer: $\boxed{\frac{91}{90}}$

Solution: Note that $y>1$, since if $y\le 1$ then the only possible values for $f(y)$ are 0 and 1.

Then, suppose $1<y<2$. Consider $g_k(x)=f_k(x\cdot f_{k-1}(\dots x\cdot f_1(x)\dots))$. Then $g_k(x)=f_k(x\cdot g_{k-1}(x))$. Let G_k be the set of all possible values for $g_1(y)$.

$G_1 = \{1, 2\}$. Then, $G_2 = \{\lfloor y \rfloor, \lceil y \rceil, \lfloor 2y \rfloor, \lceil 2y \rceil\}$. Since $4 > 2y$, $\lfloor 2y \rfloor = 3$ or $\lceil 2y \rceil = 3$. Therefore, we have all consecutive positive integers in G_2. Because we only have differences of less than 2 between integers after multiplying by y, every set G_k contains all positive integers $\{1, 2, \cdots \lceil y \cdot \max_{x \in G_{k-1}}(x) \rceil\}$. Note that because $y > 1$, $\lceil y \cdot \max_{x \in G_{k-1}}(x) \rceil > \max_{x \in G_{k-1}}(x)$. Also note that if $\lceil y \cdot \max_{x \in G_{k-1}}(x) \rceil > 1 + \max_{x \in G_{k-1}}(x)$, then for all positive integers $N > \max_{x \in G_{k-1}}(x)$, $\lceil y \cdot N \rceil > N + 1$.

Note that this implies that $|G_k| \geq |G_{k-1}| + 1$ for all $k = 2, 3 \ldots 100$. So $|G_{100}| \geq 101$. To get $|G_{100}| = 111$, we consider the following cases:

Case 1: $|G_k|$ increases by 1 up to $k = 90$, then $|G_{91}| = 93, |G_{92}| = 95, \ldots |G_{100}| = 111$. Then, $y \cdot 91 > 92$. So $y > \frac{92}{91}$, and we also know that $y \leq \frac{91}{90}$ because otherwise $|G_{90}|$ would also increase by 2 which we don't want. Checking up to $G_{99} = 109$, we can verify that multiplying the maximum by $\frac{91}{90}$ doesn't change G_k by more than two up to G_{100}, which is what we want.

Case 2: $|G_k|$ increases by 1 up to $k > 90$, then increase by 2 up to $m < 100$, then increase by 3 up to $|G_{100}| = 111$. This case is impossible because $y < \frac{92}{91}$ would mean that the function increases slower than in Case 1, but we want it to increase by 3 up to $|G_{100}|$ which is impossible.

So the maximum is $y = \frac{91}{90}$.

Now, if $y > 2$ (and is not integer) then each of the $|G_k|$ would increase by more than 1 each time, since consecutive elements in a set G_k always map to four different elements in set G_{k+1}, as $\lceil ny \rceil < \lfloor (n+1)y \rfloor$ for positive integer n and $y > 2$. Therefore on the upper end of each G_k, the number of elements is growing by at least 2 each iteration. So indeed the highest y that can work is $\frac{91}{90}$.

10. How many pairs of positive integers (n, k), both of which are not greater than 1000, are there such that
$$\sum_{i=0}^{16} k^{n+i} \equiv \sum_{i=0}^{16} (n+i)^k \pmod{17}?$$

Proposed by: Mason Yu

Answer: $\boxed{118515}$

Solution: Simplifying the LHS:

If $k \equiv 0, 1 \pmod{17}$ then the sum is 0. Otherwise, it is k^n by Fermat's Little Theorem.

Simplifying the RHS, if 16 divides k, then by Fermat's Little Theorem, $x^k = 1$ and therefore we have that the RHS is $-1 \pmod{17}$. If 16 does not divide k then the RHS is $0 \pmod{17}$.

The first case is where both sides are $0 \pmod{17}$. This means that $k \equiv 0, 1 \pmod{17}$ and $k \not\equiv 0 \pmod{16}$. There are
$$\left\lfloor \frac{1000}{17} \right\rfloor + 1 + \left\lfloor \frac{1000}{17} \right\rfloor = 117$$
cases for the first condition. The second condition invalidates
$$\left\lfloor \frac{55}{16} \right\rfloor + \left\lfloor \frac{56}{16} \right\rfloor = 6$$
of them, so there are 111 cases for k. Then, since n can be any number, that means that this case has 111000 values.

Now we check the second case, where 16 divides k and both sides are $-1 \pmod{17}$. In this case, $k^n \equiv -1 \pmod{17}$, and k could be one of $\lfloor \frac{1000}{16} \rfloor = 62$ multiples of 16.

Each of these multiples of 16 can be expressed as $16m$. For $m \equiv 1, 2, 3, \ldots, 11 \pmod{17}$, there are 4 values of m between 1 and 62, and for $m \equiv 12, \ldots, 16, 0 \pmod{17}$, there are 3 values. Since $k \equiv -m \pmod{17}$, there are 4 cases for $k = 6, 7, \ldots, 16$, and 3 cases for $0, 1, 2, \ldots, 5$.

Let $a = k \pmod{17}$, $1 \le a \le 16$. We do casework on a given that $a^n \equiv -1 \pmod{17}$.

If $a = 1$, the equality is never satisfied.

If $a = 3, 5, 6, 7, 10, 11, 12$, or 14, then n must be $8 \pmod{16}$. Thus there are 63 possible values of n, and there are $3 \cdot 2 + 4 \cdot 6 = 30$ cases for k. Thus there $30 \cdot 63 = 1890$ cases here.

If $a = 2, 8, 9$, or 15, then n must be $4 \pmod{8}$. Thus there are 125 possible values of n, and there are $3 + 4 \cdot 3 = 15$ cases for k. Thus there are $15 \cdot 125 = 1875$ cases here.

If $a = 13$ or 4, then n must be $2 \pmod{4}$. Then there are 250 possible of n, and there are $3 + 4 = 7$ cases for k. Then there are $7 \cdot 250 = 1750$ cases here.

Finally, if $a = 8$ then n must be odd, so there are 500 possible values of n, and there are 4 possible values of k. Thus there are $4 \cdot 500 = 2000$ cases here.

The total of all these two cases is $111000 + 1890 + 1875 + 1750 + 2000 = \boxed{118515}$ cases.