1. Bruno is teleported into a virtual world and forgets what day of the week it is. He meets a mercenary and a hacker. The mercenary lies on Mondays, Tuesdays, and Wednesdays, and on the other days, they tell the truth. The hacker lies on Thursdays, Fridays, and Saturdays, and on the other days, they tell the truth. "Yesterday, I was lying," the mercenary tells Bruno. "So was I," follows the hacker. What day of the week is it? 1 corresponds to Monday, 2 corresponds to Tuesday, and so on, with 7 corresponding to Sunday.

   Proposed by: Angelina Wang

   Answer: $\boxed{4}$

   Solution: We can break this down case by case.

   On Monday, the mercenary lies and the hacker tells the truth. Yesterday (Sunday), both the mercenary and the hacker were telling the truth. It cannot be Monday because the hacker has to tell the truth on Mondays, but they would be lying if they said they were lying yesterday.

   On Tuesday, the mercenary lies and hacker tells the truth. Yesterday (Monday), the hacker was telling the truth. It cannot be Tuesday because the hacker must tell the truth on Tuesdays, but they would be lying if they said they were lying yesterday.

   On Wednesday, the mercenary lies and hacker tells the truth. Yesterday (Tuesday), the hacker was telling the truth. It cannot be Wednesday because the hacker must tell the truth on Wednesdays, but they would be lying if they said they were lying yesterday.

   On Thursday, the hacker lies and mercenary tells the truth. Yesterday (Wednesday), the mercenary was lying, while the hacker was telling the truth. Everything works out because the mercenary was lying yesterday (so they told the truth today) and the hacker was telling the truth yesterday (so they lied today). Therefore, today could be Thursday.

   On Friday, the hacker lies and mercenary tells the truth. Yesterday (Thursday), the mercenary was telling the truth. It cannot be Friday because the mercenary said they were lying yesterday, which is a lie.

   On Saturday, the hacker lies and mercenary tells the truth. Yesterday (Friday), the mercenary was telling the truth. It cannot be Saturday because the mercenary said they were lying yesterday, which is a lie.

   On Sunday, both the mercenary and the hacker tell the truth. Yesterday (Saturday), the mercenary was telling the truth. It cannot be Sunday because the mercenary said they were lying yesterday, which is a lie.

   Thus, the only possible day is Thursday corresponding to 4.

2. Three numbers are gossiping about a fourth number. One says, "it's prime!" Another says, "it's one more than a square number!" A third says, "it's divisible by four!"

   Given that exactly one of these fellows is lying, and the number lies between 30 and 50, what is the value of the fourth number?

   Proposed by: Jaeho Lee

   Answer: $\boxed{37}$

   Solution: By inspection, the only one who could possibly be lying (if only one of them is lying) is the third one. Prime numbers can't be divisible by four, and (square + one) numbers can't be divisible by four. Then, you find that the only number that is both prime and one more than a square between 30 and 50 is 37.

3. Each row, column, and bolded 2×2 subgrid of the following Sudoku contains the numbers 1, 2, 3, and 4 exactly once. However, exactly one of the given squares has been corrupted and contains an incorrect number. What is the number currently contained in the corrupted square?

1	3		4
4			2
2	4		3

Proposed by: Ethan Brady

Answer: $\boxed{2}$

Solution:

If we finish the top left box by writing a 2, then we have an inconsistency with the 2nd row. In addition, if we finish the bottom row and the right column by writing 1's, then we have an inconsistency in the bottom right box. One approach is to deduce that the incorrect square must be contained in the overlap of these two inconsistencies, namely the digit 2 in the top right square. The completed Sudoku is presented below.

1	3	2	4
4	2	3	**1**
3	1	4	2
2	4	1	3

4. Among 5 people, the tallest is 140 cm and the shortest is 80 cm. If the median height is equal to the average height, what is the smallest possible sum of their heights?

Proposed by: An Cao

Answer: $\boxed{500}$

Solution: Let the heights of the middle 3 children be $80 \leq a \leq b \leq c \leq 140$. Then b is the median height and we have the equality:

$$b = \frac{80 + a + b + c + 140}{5} = \frac{220 + a + b + c}{5}$$

Since $a \geq 80$ and $c \geq b$, we have

$$b \geq \frac{220 + 80 + b + b}{5} = \frac{300 + 2b}{5}$$
$$\Rightarrow b \geq 100$$

Then the sum of their heights, which is equal to 5 times the average b, must be at least 500. This can happen when their heights are $80, 80, 100, 100, 140$.

5. A cryptarithm is a mathematical equation where each letter represents a unique digit and leading zeros are not allowed. What is the sum of all possible values for the letter A in the following equation?

$$\begin{array}{cccccc} & A & P & P & L & E \\ + & A & P & P & L & E \\ \hline O & R & A & N & G & E \end{array}$$

Proposed by: Annabel Qin

Answer: $\boxed{13}$

Solution: In the third and fourth columns, $P + P = A$ and $P + P = N$, respectively. Since A and N can't be the same digit, this discrepancy must result from a carried 1 in either the fourth or fifth column. Thus, A is either odd, with $A = N + 1$ and $P > 5$, or even, with $N = A + 1$ and $L > 5$. Further note that $A > 5$, O must be 1, and E must be 0. Solving for all valid combinations leads to the two possible solutions for the cryptarithm:

- APPLE = 63390, ORANGE = 126780, where $A = 6$.
- APPLE = 78820, ORANGE = 157640, where $A = 7$.

The possible values of A are 6 and 7. Thus, the sum of all possible values of A is $\boxed{13}$.

6. Bruno the cow grazes outside, tied to the corner of a rectangular shed measuring 3 feet by 8 feet with a 10-foot-long rope. What is the total area of ground, in square feet, that Bruno can graze on (not including the shed)? Leave π in your answer.

Proposed by: Annabel Qin

Answer: $\boxed{\frac{353}{4}\pi}$

Solution: Bruno can move anywhere in the 270° sector that does not include the shed, so this contributes 75π to the total area. Bruno can also bend the rope around either of the two adjacent corners of the shed. Going around the long side yields a quarter circle with radius 2, which contributes π. Going around the short side yields a quarter circle with radius 7, which contributes $\frac{49}{4}\pi$. The total area is $(75 + 1 + \frac{49}{4})\pi = \frac{353}{4}\pi$.

7. In the year 2052, our friend Mruno ends up working at the megacorporation Metapple. His job is to turn a special dial that only points to (possibly negative) integers. Then, he calculates the stability score using the formula $S(x) = x + \frac{12}{x+1}$, where x is the number his dial is pointing to.

One day, he moves the dial from n to $n + 1$. But oddly enough, he finds that the stability score doesn't change. What is the sum of all possible integer values of n?

Proposed by: Jaeho Lee

Answer: $\boxed{-3}$

Solution: We are given the stability score $S(x) = x + \frac{12}{x+1}$. The problem states that the score at n is the same as the score at $n+1$, so we must have $S(n) = S(n+1)$. We can write this as the equation

$$n + \frac{12}{n+1} = (n+1) + \frac{12}{(n+1)+1},$$

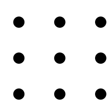
which simplifies to $n+\frac{12}{n+1} = n+1+\frac{12}{n+2}$. We can subtract n from both sides to get $\frac{12}{n+1} = 1+\frac{12}{n+2}$. Next, let's group the fractions together by subtracting $\frac{12}{n+2}$ from both sides, which gives us $\frac{12}{n+1} - \frac{12}{n+2} = 1$. To combine the fractions on the left, we find a common denominator, which is $(n+1)(n+2)$. This gives $\frac{12(n+2)-12(n+1)}{(n+1)(n+2)} = 1$. When we expand the numerator (the top part), we get $\frac{12n+24-12n-12}{(n+1)(n+2)} = 1$, which simplifies to $\frac{12}{(n+1)(n+2)} = 1$. This means that $12 = (n+1)(n+2)$.

We are looking for an integer n such that $n+1$ and $n+2$ are two consecutive integers that multiply to 12. There are only two pairs of consecutive integers that do this: $3 \times 4 = 12$ and $-4 \times -3 = 12$. In the first case, $n+1 = 3$, which means $n = 2$. In the second case, $n+1 = -4$, which means $n = -5$. The two possible integer values for n are 2 and -5. The question asks for their sum, which is $2 + (-5) = -3$.

8. There are 9 lattice points (x, y) in the coordinate plane such that $|x| \leq 1$ and $|y| \leq 1$. How many quadratic equations

$$f(x) = ax^2 + bx + c$$

exist such that $f(x)$ goes through 3 of the points and $a > 0$?



Proposed by: Joshua Kou

Answer: $\boxed{11}$

Solution: By the vertical line test, f cannot pass through two points with the same x-coordinate, so for f to pass through 3 points, it must pass through one point in each column.

One approach is to do case work on which point in the middle column f passes through. If f passes through the bottom point $(0, -1)$, then any choice of points in the left and right column yields a concave up function except $(-1, -1), (0, -1), (1, -1)$. Similarly by inspection there are 3 concave up functions going through the center point $(0, 0)$, and no concave up functions going through $(0, 1)$, yielding a combined sum of $8 + 3 = \boxed{11}$.

Alternatively, there are $3 \times 3 \times 3 = 27$ choices of left, middle, and center point, each yielding a function. 5 of these are straight lines with the coefficient of x^2 being 0. Then there are 22 functions with nonzero x^2 coefficient, half of which will be concave up and half of which will be concave down. Hence our answer is $\boxed{11}$.

9. The Salomon Auditorium contains a rectangular grid of chairs, with an unknown number of rows and columns. There are exactly 15 math students in each row and 9 computer science students in each column. If there are 7 empty seats, what is the smallest possible number of seats in the auditorium? Assume that no student studies both math and computer science.

Proposed by: Amber Bajaj

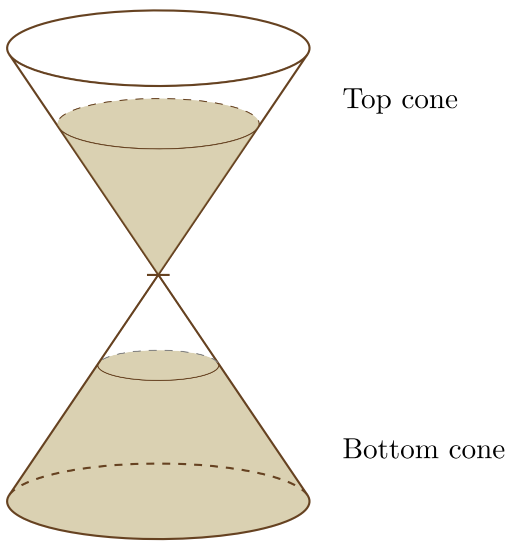
Answer: $\boxed{946}$

Solution: Let r represent the number of rows, and c represent the number of columns, and use MS and CS as shorthands for "math students" and "computer science students". Since there are 15 MS in each row, there are $15r$ MS total. Since there are 9 CS in each column, there are $9c$ CS total. Since there are 7 empty seats, the total number of seats is $15r + 9c + 7$. The seats are in a rectangular grid, so $rc = 15r + 9c + 7$. $c \geq 15$ since there are 15 MS in each row. Similarly, $r \geq 9$, since there are 9 CS in each column.

We must solve the equation $rc = 15r + 9c + 7$. Re-arranging, we find that $(r - 9)(c - 15) = 142$. The factors of 142 are 1, 2, 71, 142. We thus get the following table, which shows that the fewest number of seats is 946.

$c-15$	$r-9$	c	r	rc
1	142	16	151	2416
2	71	17	80	1360
71	2	86	11	946
142	1	157	10	1570

10. An hourglass consists of two identical cones with tips vertically touching. Sand flows at a constant rate from the top cone to the bottom cone. The top cone starts with sand 12 cm deep, and a bit later, the level of sand in the top cone has decreased by 2 cm while that of the bottom cone has increased by 8 cm. Given that the top cone starts with 64 cm^3 of sand, how much sand is remaining in the top cone (in cm^3) when the bottom cone has filled up?



Proposed by: Ryan Lee

Answer: $\boxed{37}$

Solution: By similarity, given the top cone started with 64cm^3 of sand, once its level decreased by 2cm it has $\left(\frac{10}{12}\right)^3 \cdot 64\text{cm}^3$. Moreover, if the top cone had sand 1cm deep, then it has $\left(\frac{1}{12}\right)^3 \cdot 64\text{cm}^3$ of sand.

The amount of sand not filled in the bottom cone is also a cone similar to the amount of sand filled in the top cone. Thus, if the level of sand not filled in the bottom cone at first is h cm, then the volume of this unfilled area starts as $h^3 \cdot \left(\frac{1}{12}\right)^3 \cdot 64$ and goes to $(h-8)^3 \cdot \left(\frac{1}{12}\right)^3 \cdot 64$ at time t. This volume change is equal to the volume change in the top cone, so

$$[h^3 - (h-8)^3]\left(\frac{1}{12}\right)^3 \cdot 64 = (12^3 - 10^3)\left(\frac{1}{12}\right)^3 \cdot 64$$

.

We can cancel $\left(\frac{1}{12}\right)^3 \cdot 64$ from both sides, so $24h^2 - 192h + 512 = 728$, so $h^2 - 8h - 9 = 0$, so $h = 9$. Thus, the bottom cone has 9cm of sand left to fill at the beginning, which means the top cone loses $\left(\frac{9}{12}\right)^3 \cdot 64\text{cm}^3$ of sand. So it is left with $(1 - (\frac{9}{12})^3) \cdot 64 = \boxed{37\text{cm}^3}$.