P U M ∴ C

2025 Combinatorics A

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. A string of 0s, 1s, and 2s is called *nice* if it doesn't contain "01" or "12", doesn't begin with a "2", and doesn't end with a "0". For example, there are 5 nice strings of length 3, namely "002", "021", "022", "102", and "111". Find the number of nice strings of length 7.

   Proposed by Andrew Li

   Solution: We can set up a recurrence to find this answer. Let a seminice string satisfy all the conditions of a nice string except the not beginning with 2 condition. Let $x_{a,b}$ be the number of ways to make a seminice string of length a with leftmost digit b. This follows the set of recurrences

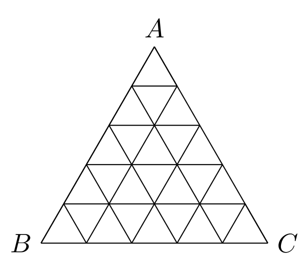
$$x_{a,2} = x_{a-1,0} + x_{a-1,1} + x_{a-1,2}$$
$$x_{a,1} = x_{a-1,0} + x_{a-1,1}$$
$$x_{a,0} = x_{a-1,0} + x_{a-1,2}$$

   The base cases are $x_{1,0} = 0, x_{1,1} = 1, x_{1,2} = 1$. We can crunch the recurrence fairly mechanically.

$$x_{2,2} = 2, x_{2,1} = 1, x_{2,0} = 1$$
$$x_{3,2} = 4, x_{3,1} = 2, x_{3,0} = 3$$
$$x_{4,2} = 9, x_{4,1} = 5, x_{4,0} = 7$$
$$x_{5,2} = 21, x_{5,1} = 12, x_{5,0} = 16$$
$$x_{6,2} = 49, x_{6,1} = 28, x_{6,0} = 37$$
$$x_{7,2} = 114, x_{7,1} = 65, x_{7,0} = 86$$

   So the answer is $x_{7,1} + x_{7,0} = \boxed{151}$.

2. You find yourself standing at vertex A in the triangle grid shown below. Two monsters are standing at vertices B and C, respectively. Each minute, you move along an edge that takes you closer to $\overline{BC}$, choosing an edge randomly out of the two possible edges. In the same way, the monster starting at B moves towards $\overline{AC}$, and the monster starting at C moves towards $\overline{AB}$. If you ever encounter a monster on a vertex not on $\overline{BC}$, you die. Otherwise, if you reach $\overline{BC}$ without dying, you survive (even if you encounter a monster at some vertex on $\overline{BC}$). Compute the probability that you survive.



   Proposed by Andrew Tu

   Solution: Since the grid is a triangle simplex of side length 5, we can use simplex coordinates (i, j, k) with the condition $i + j + k = 5$. Let $A = (5, 0, 0), B = (0, 5, 0), C = (0, 0, 5)$. To move an step along the edge, one's coordinate decreases by 1 and another increases by 1. We can describe the traversal for t minutes as:

$$A(t) = (5 - t, A_t, t - A_t)$$
$$B(t) = (B_t, 5 - t, t - B_t)$$
$$C(t) = (C_t, t - C_t, 5 - t)$$

Where N_t is the number of steps towards the respective side for t moves. We can solve $A(t) = B(t)$ to obtain when A meets B. This forces $A_t = B_t = 5 - t$, so $t \geq 3$. Similarly, $A(t) = C(t)$ forces $A_t = 2t - 5$ which requires $t \geq 3$. Since A reaches $\overline{BC}$ at $t = 5$, only $t = 3, 4$ can kill you. Define $w_t = \binom{t}{5-t}2^{-t}$ as the probability a given monster is in the unique position for t. So, $w_3 = \frac{3}{8}, w_4 = \frac{1}{4}$. The outcome depends on A's first 4 moves (2^4 equally likely prefixes). Let x_3 be the number of A's first 3 moves towards B, and x_4 the number of first 4 moves towards B. We can see that A can meet B at $t = 3$ iff $x_3 = 2$; at $t = 4$ iff $x_4 = 1$ and A can meet C at $t = 3$ iff $x_3 = 1$; at $t = 4$ iff $x_4 = 3$. Thus for each prefix, the survival probability is $(1-\sum_{t\in S_B} w_t)(1-\sum_{t\in S_C} w_t)$ where $\{S_B, S_C \subset 3, 4\}$ are the possible meeting times with (B, C).

Counting the 16 prefixes gives: $(1+\frac{3}{4}+3\cdot\frac{15}{32}+3\cdot\frac{5}{8}+3\cdot\frac{15}{32}+\frac{3}{4}+1+3\cdot\frac{5}{8})(\frac{1}{16}) = \frac{161}{16}\cdot\frac{1}{16} = \boxed{\frac{161}{256}}$.

3. A stone is placed on a square of a 5×5 grid. Alice and Bob take turns moving the stone to an orthogonally adjacent square (i.e. up/down/left/right). Alice moves first. Once a square has been visited, it cannot be visited again (and they cannot visit the starting square again). A player loses if they cannot make a legal move. For how many starting squares does Alice have a winning strategy?

Proposed by Andrew Li

Solution: Color the 5×5 grid in a checkerboard pattern, such that there are 13 black squares and 12 white squares.
We first show that Alice has a winning strategy if the starting square is any of the 12 white squares. Alice can tile the entire checkerboard using 1×2 dominos, leaving a single black square untiled. Then, each time Alice makes a move (including her first move), she can move the stone into the square which is tiled by the same domino as the square the stone currently is in. In this way, Alice always has a legal move available and thus cannot lose.
We now show that Bob has a winning strategy if the starting square is any of the 13 black squares. Regardless of what the starting square is, Bob can tile the remainder of the checkerboard using 1×2 dominoes. Then, each time Alice makes a move into a new square, Bob can move the stone into the square which is tiled by the same domino. In this way, Bob always has a legal move available and thus cannot lose.

Thus, the answer is $\boxed{12}$ starting squares.

4. Zongshu is selling socks in packs of 2025. Each sock in each pack is equally likely to be red, green, or yellow. You buy (a positive number of) packs of socks until you can split all of the socks into pairs of the same color. Find the expected number of packs you buy.

Proposed by Rohit Agarwal

Solution: Since you must buy an even number of socks in total and each pack has an odd number of socks, we can let the random variable n denoting the number of packs you buy be written as $2k$. We can describe states by vectors in $\{0, 1\}^3$ where 1 represents a certain color is unpaired, i.e. $(0, 1, 0)$ represents having paired your red and yellow socks and having an unpaired green sock. After buying two packs of socks, the state vector may remain unchanged, or exactly two bits may flip. Let q be the probability that the state vector remains unchanged. By symmetry, the three cases $v \to v \oplus (1,1,0), v \to v \oplus (1,0,1), v \to v \oplus (0,1,1)$ happen with equal probability $\frac{1-q}{3}$, where $\oplus$ (XOR) adds the entries of two vectors modulo 2. So we have a Markov chain on the states $(0,0,0),(1,1,0),(1,0,1),(0,1,1)$ where each state is treated identically: upon buying two packs, there is a q probability of returning to itself, and $\frac{1-q}{3}$ probability of going to each of the other three states. Hence, the expected return time is $q \cdot 1 + (1-q)(1+\mathbb{E}[P])$ where P is the number of $\frac{1-q}{3}$-biased coin flips needed to flip a heads. Miraculously we compute $q + (1-q)(1+\frac{3}{1-q}) = q + (1-q) + 3 = 4$.

Alternatively, one can recall that the expected return time to state i on a Markov chain where it is possible to get from any state to any other state in finite time is $1/p_i$, where p_i is the steady-state probability of being in state i. When buying packs two at a time, each of the four states has equal steady-state probability, so the expected return time is 4.

Hence the expected number of sets of two packs we need is $\mathbb{E}[k] = 4$, hence $\mathbb{E}[n] = \boxed{8}$.

5. Alice wants to send Bob the integer 120, represented as a binary string of length 7. However, her computer is being bombarded by cosmic rays, which can flip 00 to 11 and vice versa an arbitrary amount of times! Bob receives a possibly altered binary string, and writes down the integer in decimal. Find the sum of the possible numbers Bob could have written down (including 120 itself).

Proposed by Zongshu Wu and Atharva Pathak

Solution: Let $n = 7$ and $s = 1111000 = (120)_2$. Let α_i denote the bit in the 2^i of string α. Define a new string t of length n by

$$t_i = \begin{cases} s_i, & \text{if } i \text{ is even,} \\ 1 - s_i, & \text{if } i \text{ is odd.} \end{cases}$$

Explicitly, $t = 1010010$.

Then a cosmic ray transformation on s is equivalent to flipping 01 to 10 and vice-versa in t. It follows that any string with the same number of 1s and 0 as t is attainable from t via comsic ray transformations.

First, letting $k = 3$ be the number of 1s in t, then there are $\binom{n}{k}$ attainable strings.

Now the expected value of any bit in t is $\frac{k}{n}$. It follows that

$$\mathbb{E}\left[s_i\right] = \mathbb{E}\left[t_i\right] = \frac{k}{n}$$

for all even $0 \le i \le n-1$, while

$$\mathbb{E}\left[s_i\right] = \mathbb{E}\left[1 - t_i\right] = 1 - \frac{k}{n}$$

for all odd $0 \le i \le n-1$. Therefore the expected value of s upon transformations is

$$\sum_{i \text{ even}} \frac{k}{n} 2^i + \sum_{i \text{ odd}} \frac{n-k}{k} 2^i,$$

so the sum of all possible values of s is

$$\binom{n}{k} \cdot \left(\sum_{i \text{ even}} \frac{k}{n} 2^i + \sum_{i \text{ odd}} \frac{n-k}{k} 2^i \right) = \binom{7}{3} \cdot \left[\frac{3}{7}\left(2^0 + 2^2 + 2^4 + 2^6\right) + \frac{4}{7} \cdot \left(2^1 + 2^3 + 2^5\right) \right]$$
$$= \boxed{2115}.$$

6. Charles challenges David to a game of Fraction Fixer. In this game, David first picks two positive integers $m \leq n$ that sum to 2025. Next, Charles picks integers a and b at random, so that $0 \leq a < m$ and $0 \leq b < n$, and computes $\frac{a}{m} + \frac{b}{n}$. If the sum is at least 1, David wins. Otherwise, Charles wins. If David chooses m and n optimally, what is the probability that David wins?

   Proposed by David Ji

   Solution: Consider any positive integers m, n that sum to 2025. If Charles picks $a = i$, then the probability that David wins is equal to

$$\frac{1}{n} \sum_{j=0}^{n-1} \left\lfloor \frac{i}{m} + \frac{j}{n} \right\rfloor$$

   Thus, the overall probability that David wins is

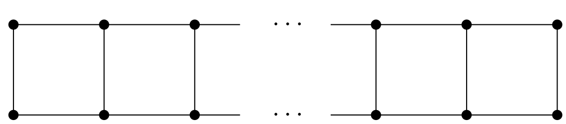
$$\frac{1}{m} \sum_{i=0}^{m-1} \frac{1}{n} \sum_{j=0}^{n-1} \left\lfloor \frac{i}{m} + \frac{j}{n} \right\rfloor = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left\lfloor \frac{i}{m} + \frac{j}{n} \right\rfloor$$

   Applying Hermite's identity, this is equal to

$$\frac{1}{mn} \sum_{i=0}^{m-1} \left\lfloor \frac{n}{m} i \right\rfloor = \frac{(m-1)(n-1) + \gcd(m,n) - 1}{2mn} = \frac{1}{2} - \frac{m + n - \gcd(m,n)}{2mn}$$

   The expression $\frac{m+n-\gcd(m,n)}{2mn}$ is minimized when $m = 675$ and $n = 1350$. Thus, the answer is $\boxed{\frac{337}{675}}$.

7. Consider the graph shown below with 4050 vertices and 6073 edges. Each edge is independently deleted with probability $\frac{1}{2}$. If C is the expected value of the number of connected components of the resulting graph, compute the nearest integer to $49C$.



   Proposed by Atharva Pathak

   Solution: We can use induction to compute the answer. Think of the graph as 2025 pairs of vertices from left to right as $(A_{2025}, B_{2025}), (A_{2024}, B_{2024}), \ldots$. Let $C(n)$ be the expected number of connected components in the induced subgraph from pair n onwards. Let $D(n)$ be the expected number of connected components from pair n onwards as well, with an additional undeletable multiedge between A_n and B_n, to represent that there is some path from top to bottom from some leftward edges that were undeleted. We can now make a system of recurrences with $C(n)$ and $D(n)$.

   First for $C(n)$. Then we do casework on how many of the edges $(A_n, B_n), (A_n, A_{n-1}), (B_n, B_{n-1})$ were cut.

   - No edges cut. Then we get $D(n-1)$.

- Exactly one edge cut. Then we get $C(n-1)$ because no new connected components have formed, but there is no extra multi-edge.
- Exactly two edges cut. Then we form exactly one new connected component and no extra multi-edge, giving $1 + C(n-1)$.
- All three edges cut. Similarly, $2 + C(n-1)$.

This yields the equation

$$C(n) = \frac{1}{8}D(n-1) + \frac{7}{8}C(n-1) + \frac{5}{8}$$

Now we can proceed similarly with $D(n)$.

- No edges cut. Then we get $D(n-1)$.
- Only Top or bottom edge cut. Then the path from top to bottom is broken, so we get $C(n-1)$.
- Only Left edge cut. Then the path from top to bottom is not broken, so this is $D(n-1)$.
- Top and bottom edge cut. Then we get $1 + C(n-1)$.
- Left and one other edge cut. Then the connection from top to bottom is broken but we still don't create any new connected components, so this is $C(n-1)$.
- All three cut. This creates one new connected component, yielding $1 + C(n-1)$.

Now this yields recurrence

$$D(n) = \frac{1}{4}D(n-1) + \frac{3}{4}C(n-1) + \frac{1}{4}$$

Now we can combine the recurrences. In particular, consider $C(n) - D(n) = -\frac{1}{8}D(n-1) + \frac{1}{8}C(n-1) + \frac{3}{8} = \frac{1}{8}(C(n-1) - D(n-1)) + \frac{3}{8}$. Now defining $E(n) = C(n) - D(n)$ and noting that $C(1) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 2 = \frac{3}{2}$ and $D(1) = 1$ whence $E(1) = \frac{1}{2}$, we can solve this recurrence as $E(n) = \frac{1}{14}\left(\frac{1}{8}\right)^{n-1} + \frac{3}{7}$ using standard techniques. Now we can use this to solve for C. Substituting $D(n-1) = C(n-1) - E(n-1)$, and simplifying, we get that

$$C(n) = C(n-1) + \frac{4}{7} - \frac{1}{14}\left(\frac{1}{8}\right)^{n-1}$$

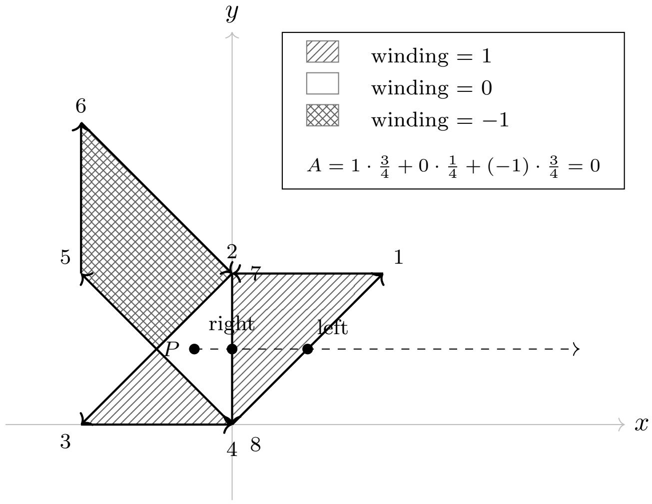
Unrolling the recurrence yields

$$C(n) = C(1) + \frac{4}{7}(n-1) - \frac{1}{14}\sum_{k=2}^{n}\left(\frac{1}{8}\right)^{k-1} = \frac{3}{2} + \frac{4}{7}(2024) - \frac{1}{98}\left(1 - \left(\frac{1}{8}\right)^{n-1}\right)$$

Finally evaluating $C(2025)$ yields (apart from the exponentially small term, which we disregard) $\frac{56745}{49}$, giving an answer of $\boxed{56745}$.

8. Trevor the traveller starts at the origin of $\mathbb{R}^2$. He then takes a series of 8 steps that change his position by a vector in the set $\{(a,b) \in \mathbb{Z}^2 : |a|, |b| \leq 1 \text{ and } (a,b) \neq (0,0)\}$. He takes each of the 8 possible steps exactly once, and takes the steps in uniformly random order, so that he returns to the origin. The *winding number* of each point P in $\mathbb{R}^2$, except those exactly on the path, is defined as the number of times any ray starting at P going out to infinity crosses a left-pointing portion of Trevor's path minus the number of times the ray crosses a right-pointing portion of

the path, where left and right are defined relative to the ray. Let A be the area in $\mathbb{R}^2$ weighted by winding number. An example is shown below. Find the expected value of A^2.



Weighted areas for steps, in order,
$(1,1), (-1,0), (-1,-1), (1,0), (-1,1), (0,1), (1,-1), (0,-1)$

Proposed by Atharva Pathak

Solution: Let $v_1, \ldots, v_8$ be the vectors in $\{(a,b) \in \mathbb{Z}^2 : |a|, |b| \leq 1 \text{ and } (a,b) \neq (0,0)\}$ in some arbitrary order. Let π be a permutation on $1, \ldots, 8$ taken uniformly at random. Let $S_0 = (0,0)$ and $S_k = \sum_{t=1}^{k} v_{\pi(t)}$ for $k = 1, \ldots, 8$. Note for our set of possible vectors that S_8 is always $(0,0)$. For $X = (X_1, X_2)$ and $Z = (Z_1, Z_2)$, define $\det(X, Z) = X_1 Z_2 - X_2 Z_1$. By the shoelace formula, the area weighted by winding number contained inside the path is $A = \frac{1}{2} \sum_{k=0}^{7} \det(S_k, S_{k+1})$, and we seek $\mathbb{E}_\pi[A^2]$.

We can write A in terms of the v_is as $A = \frac{1}{2} \sum_{1 \leq r < s \leq 8} \det(v_{\pi(r)}, v_{\pi(s)})$. Let σ_{ij} be $+1$ if i appears before j in π and -1 otherwise. Then the contribution of the unordered pair $\{i, j\}$ to $\sum_{1 \leq r < s \leq 8} \det(v_{\pi(r)}, v_{\pi(s)})$ is $\det(v_{\pi(i)}, v_{\pi(j)})\sigma_{ij}$. Let $c_{ij} = \det(v_i, v_j)$. So $A = \frac{1}{2} \sum_{1 \leq i < j \leq 8} c_{ij}\sigma_{ij}$, so

$$\mathbb{E}_\pi[A^2] = \frac{1}{4}\mathbb{E}_\pi\left[\left(\sum_{i<j} c_{ij}\sigma_{ij}\right)\right] = \frac{1}{4}\sum_{i<j,k<\ell} c_{ij}c_{k\ell}\mathbb{E}_\pi[\sigma_{ij}\sigma_{k\ell}]$$

Next, we split the sum into cases based on i, j, k, ℓ. First, if i, j, k, ℓ are all distinct, then $\mathbb{E}_\pi[\sigma_{ij}\sigma_{k\ell}] = 0$ because σ_{ij} and $\sigma_{k\ell}$ are independent of each other. For terms with $(i,j) = (k,\ell)$, we have $\sigma_{ij} = \sigma_{k\ell}$ so we can simplify into $\sum_{i<j} c_{ij}^2$. In the last case, (i,j) and (k,ℓ) share exactly one label. Say the three labels are $a < b < c$. Then $(i,j), (k,\ell)$ may be (a,b), (a,c), (a,b), (b,c), (a,c), (b,c), or the pairs obtained by swapping (i,j) and (k,ℓ), which we account for with a factor of 2. By listing out 6 cases, we can find $\mathbb{E}_\pi[\sigma_{ab}\sigma_{ac}] = +\frac{1}{3}$, $\mathbb{E}_\pi[\sigma_{ab}\sigma_{bc}] = -\frac{1}{3}$, and $\mathbb{E}_\pi[\sigma_{ac}\sigma_{bc}] = +\frac{1}{3}$. So the contribution from this case to the summation is $2 \cdot \frac{1}{3} \cdot \sum_{i<j<k}(c_{ij}c_{ik} - c_{ij}c_{jk} + c_{ik}c_{jk})$, so now we must compute

$$\sum_{i<j} c_{ij}^2 + \frac{2}{3}\sum_{i<j<k}(c_{ij}c_{ik} - c_{ij}c_{jk} + c_{ik}c_{jk}).$$

Let $S_2 = \sum_{i<j} c_{ij}^2$ and $S_3 = \sum_{i<j<k}(c_{ij}c_{ik} - c_{ij}c_{jk} + c_{ik}c_{jk})$.

Let $L_j = \sum_{i<j} c_{ij}$. We now use the fact that $\sum_{t=1}^{8} v_t = (0,0)$ to simplify S_3. We have

$$0 = \det(v_j, \sum_{j\neq m} v_m) = \sum_{m\neq j} \det(v_j, v_m) = \sum_{k>j} c_{jk} - \sum_{i<j} c_{ij},$$

so $\sum_{k>j} c_{jk} = L_j$ as well. Now we split up S_3 into three sums based on the common index in the two cs. We have

$$\begin{aligned}
\sum_{i<j<k} c_{ij}c_{ik} &= \frac{1}{2}\sum_i \left(\left(\sum_{t>i} c_{it}\right)^2 - \sum_{t>i} c_{it}^2\right) = \frac{1}{2}\sum_i \left(L_i^2 - \sum_{t>i} c_{it}^2\right), \\
\sum_{i<j<k} c_{ij}c_{jk} &= \sum_j \left(\sum_{i<j} c_{ij}\right)\left(\sum_{k>j} c_{jk}\right) = \sum_j L_j^2, \\
\text{and } \sum_{i<j<k} c_{ik}c_{jk} &= \frac{1}{2}\sum_k \left(\left(\sum_{t<k} c_{tk}\right)^2 - \sum_{t<k} c_{tk}^2\right) = \frac{1}{2}\sum_k \left(L_k^2 - \sum_{t<k} c_{tk}^2\right), \\
\text{so } S_3 &= -\frac{1}{2}\left(\sum_i \sum_{t>i} c_{it}^2 + \sum_k \sum_{t<k} c_{tk}^2\right) = -\frac{1}{2}\sum_{i<j} c_{ij}^2 = -S_2.
\end{aligned}$$

So we are left to compute $\mathbb{E}_\pi[A^2] = \frac{1}{4}(S_2 + \frac{2}{3}(-S_2)) = \frac{1}{12}S_2$. We do the calculation for $S_2 = \sum_{1\le i<j\le 8} \det(v_i, v_j)^2$ by casework on which pairs of vectors we consider. Say the vectors $(1,0),(0,1),(-1,0),(0,-1)$ are the axis vectors and $(1,1),(-1,1),(-1,-1),(1,-1)$ are the diagonal vectors. Among the 6 axis-axis pairs, just the four orthogonal pairs give $c_{ij}^2 = 1$ each, so the contribution in this case is 4. Among the 6 diagonal-diagonal pairs, the four orthogonal pairs each gives $(\sqrt{2}\cdot\sqrt{2})^2 = 4$, so the contribution in this case is 16. Among the 16 axis-diagonal pairs, each has $|\det(v_i, v_j)| = 1$, so the contribution in this case is 16. So $S_2 = 36$, so our final answer is $\mathbb{E}_\pi[A^2] = \boxed{3}$.