P U M $\therefore$ C

Combinatorics A

1. There are five dots arranged in a line from left to right. Each of the dots is colored from one of five colors so that no 3 consecutive dots are all the same color. How many ways are there to color the dots?

2. In an election between A and B, during the counting of the votes, neither candidate was more than 2 votes ahead, and the vote ended in a tie, 6 votes to 6 votes. Two votes for the same candidate are indistinguishable. In how many orders could the votes have been counted? One possibility is AABBABBABABA.

3. Alex starts at the origin O of a hexagonal lattice. Every second, he moves to one of the six vertices adjacent to the vertex he is currently at. If he ends up at X after 2018 moves, then let p be the probability that the shortest walk from O to X (where a valid move is from a vertex to an adjacent vertex) has length 2018. Then p can be expressed as $\frac{a^m-b}{c^n}$, where a, b, and c are positive integers less than 10; a and c are not perfect squares; and m and n are positive integers less than 10000. Find $a+b+c+m+n$.

4. If a and b are selected uniformly from $\{0, 1, ..., 511\}$ with replacement, the expected number of 1's in the binary representation of $a+b$ can be written in simplest form as $\frac{m}{n}$. Compute $m+n$.

5. How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and reflection.

6. Michael is trying to drive a bus from his home, $(0, 0)$ to school, located at $(6, 6)$. There are horizontal and vertical roads at every line $x = 0, 1, \ldots, 6$ and $y = 0, 1, \ldots, 6$. The city has placed 6 roadblocks on lattice point intersections (x, y) with $0 \leq x, y \leq 6$. Michael notes that the only path he can take that only goes up and to the right is directly up from $(0, 0)$ to $(0, 6)$, and then right to $(6, 6)$. How many sets of 6 locations could the city have blocked?

7. Frankie the Frog starts his morning at the origin in $\mathbb{R}^2$. He decides to go on a leisurely stroll, consisting of $3^1 + 3^{10} + 3^{11} + 3^{100} + 3^{101} + 3^{110} + 3^{111} + 3^{1000}$ moves, starting with the 1st move. On the nth move, he hops a distance of
$$\max\{k \in \mathbb{Z} : 3^k | n\} + 1,$$
then turns 90° degrees counterclockwise. What is the square of the distance from his final position to the origin?

8. Let S_5 be the set of permutations of $\{1, 2, 3, 4, 5\}$, and let C be the convex hull of the set
$$\{(\sigma(1), \sigma(2), \ldots, \sigma(5)) \mid \sigma \in S_5\}.$$
Then C is a polyhedron. What is the total number of 2-dimensional faces of C?