P U M . : . C

2025 Geometry A

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. Two circles ω_1 and ω_2 have radii 3 and 2, respectively, and intersect at two distinct points X and Y. The line through X perpendicular to $\overline{XY}$ intersects ω_1 at $A \neq X$ and ω_2 at $B \neq X$. Given that $AB = 4$, find the area of triangle ABY.

   Proposed by Ana Boiangiu and Zongshu Wu

   Solution: Observe that $\angle AXY = 90°$, so AY is a diameter of ω_1, and hence $AY = 6$. Similarly, BY is a diameter of ω_2, so $BY = 4$. Now let $x = XY$. We have

   $$4 = AB = AX \pm BX,$$

   so by the Pythagorean theorem,

   $$\sqrt{6^2 - x^2} \pm \sqrt{4^2 - x^2} = 4.$$

   Squaring both sides and rearranging gives

   $$\pm\sqrt{(36 - x^2)(16 - x^2)} = x^2 - 18.$$

   Squaring both sides again and simplifying yields $x^2 = \frac{63}{4}$, so $XY = \frac{3\sqrt{7}}{2}$. Therefore, the area of $\triangle ABY$ is

   $$\frac{1}{2} \cdot AB \cdot XY = \boxed{3\sqrt{7}}.$$

2. Consider a right triangular prism with bases $\triangle ABC$ and $\triangle A'B'C'$, where $\triangle ABC \cong \triangle A'B'C'$ and $AC = 10$. Point M lies on $\overline{AB}$ such that $AM = 5$ and $MB = 4$. Let N lie on $\overline{A'C'}$ such that $\overline{BN}$ is parallel to the plane of $\triangle A'CM$. Find $A'N$.

   Proposed by Alexander Divoux

   Solution: Let P lie on $A'B'$ such that the plane of $\triangle BNP$ is parallel to the plane of $\triangle A'CM$. In particular $BP \parallel A'M$ and $NP \parallel CM$. Observe that $A'P = CM$ and $\triangle ACM \simeq \triangle A'NP$. Consequently

   $$A'N = \frac{AC \cdot A'P}{AM} = \boxed{8}.$$

3. Let $ABCD$ be an isosceles trapezoid with $\overline{AB} \parallel \overline{CD}$, $AB = 20$, and $CD = 22$. Let M and N be the midpoints of sides AD and BC, respectively. Ray NM meets the circumcircle of $ABCD$ at K. If the lengths of KM and AM are positive integers, find the sum of all possible values of KM.

   Proposed by Ana Boiangiu

   Solution: First, we have

   $$MN = \frac{AB + CD}{2} = 21.$$

   Let line MN meet the circumcircle of $ABCD$ a second time at L. Then by symmetry, we may denote $KM = LN = x$. By power of a point

   $$MK \cdot ML = MA \cdot MB,$$

or equivalently

$$x(x+21) = MA^2.$$

Now MA is integral if and only if $x(x+21)$ is a perfect square. However

$$x^2 < x(x+21) < (x+11)^2,$$

so $x(x+21) = (x+k)^2$ for some integer $1 \le k \le 10$. Handchecking gives $x \in \{4, 7, 27, 100\}$ as the only solutions, for a final answer of $\boxed{138}$.

4. Let ABC be a triangle with $\angle BAC = 45°$, $AB = 2\sqrt{2}$ and $AC = 3$. Let X be the reflection of B over $\overline{AC}$ and let Y be the reflection of C over $\overline{AB}$. Lines XY and BC intersect at S. Let H be the orthocenter of $\triangle ABC$ and let O be the circumcenter of $\triangle XHY$. Find SO.

Proposed by Ana Boiangiu

Solution: Let O' be the circumcenter of $\triangle ABC$. The synthetic solution relies on two main claims.

Claim 1. O is the reflection of O' across A.

Proof. Define O as the reflection of O' across A. We will show that O is the circumcenter of $\triangle SXY$, which suffices. First, since

$$\angle AHB = 180° - \angle ACB = 180° - \angle AYB,$$

it follows that H lies on the circumcircle of AYB. Let K be the reflection of O' across AB and let M be the midpoint of AB. By symmetry, K is the circumcenter of $\triangle AYB$. Since AM is a midline in $\triangle AKO$, avem $OK \parallel AB$, it follows that $OK \perp HY$. But since K lies on the perpendicular bisector of HY, line OK is the perpendicular bisector of HY. Similarly, O lies on the perpendicular bisector of HX, so O is the circumcenter of triangle XHY, as claimed.

Remark. This claim holds without the assumption that $\angle BAC = 45°$.

Claim 2. SA is tangent to (ABC).

Proof. By a well-known lemma, it suffices to show that $SB/SC = AB^2/AC^2$. By Menelaus' theorem in triangle BHC we have

$$\frac{SB}{SC} \cdot \frac{YC}{YH} \cdot \frac{XH}{XB} = 1,$$

or

$$\frac{SB}{SC} = \frac{YH}{YC} \cdot \frac{XB}{XH}.$$

Let D be the foot of the B-altitude in $\triangle ABC$. Since $\triangle ADB$ is a right-angled isosceles triangle, we have $BX = 2BD = AB\sqrt{2}$. By symmetry, triangle ADX is also isosceles and right-angled. Since as is HDC, it follows that $CHAX$ is an isosceles trapezoid and so $HX = AC$. Therefore

$$\frac{XB}{XH} = \sqrt{2} \cdot \frac{AB}{AC}.$$

We similarly find that

$$\frac{YC}{YH} = \sqrt{2} \cdot \frac{AC}{AB},$$

whence $SB/SC = AB^2/AC^2$, as needed. Since $SA \perp OO'$ and A is the midpoint of OO', we have $SO = SO'$. By the Pythagorean theorem and power of a point we have

$$SO' = \sqrt{SA^2 - OA^2} = \sqrt{SB \cdot SC - OA^2}.$$

To compute OA, first note that by the law of cosines

$$BC = \sqrt{AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos 45^\circ} = \sqrt{5}.$$

Now the law of sines gives that

$$\frac{BC}{2OA} = \sin 45^\circ,$$

whence $OA = \sqrt{5/2}$. To compute $SB \cdot SC$, we use the same lemma as before to find that $SB/SC = AB^2/AC^2 = 8/9$. It follows that $SB/BC = 8$ and so $SB = 8\sqrt{5}$, while $SC = 9/\sqrt{5}$. The final answer is therefore $SO = \boxed{\sqrt{139/2}}$.

5. Let ABC be a triangle with $\angle BAC = 60^\circ$, $AB = 10$, $AC = 13$, and orthocenter H. Let ω_1 be the circle through H tangent to $\overline{BC}$ at B, and let ω_2 be the circle through H tangent to $\overline{BC}$ at C. Let $\overline{CH}$ intersect ω_1 again at U and let $\overline{BH}$ intersect ω_2 again at V. Let N be the point on the same side of $\overline{BC}$ as A such that triangle BNC is equilateral. Find $\frac{NU}{NV}$.

Proposed by Ana Boiangiu

Solution: We start with an important claim.

Claim. U, N and V are collinear.

Proof. We first show that if (BHU) and (CHV) meet a second time at R, then R lies on (ABC). We have

$$\begin{aligned}\angle BRC &= \angle BHR + \angle CHR \\ &= \angle BUH + \angle CVH \\ &= \angle HBC + \angle HCB \\ &= \angle 180^\circ - \angle BHC \\ &= 60^\circ,\end{aligned}$$

so $BRAC$ is cyclic indeed. Now RN is the external bisector of $\angle BRC$, so it suffices to show that U and V lie on it. Indeed

$$\begin{aligned}\angle BRU &= \angle UHB \\ &= 180^\circ - \angle BHC \\ &= 60^\circ \\ &= 90^\circ - \frac{1}{2}\angle BRC,\end{aligned}$$

so RU is the external bisector of $\angle BRC$. Similarly, RV is the external bisector of $\angle BRC$ and our claim follows.

The main idea is to let UB and VC meet at S. Then

$$\begin{aligned}\angle SBC &= 180^\circ - \angle UBC \\ &= 180^\circ - \angle BHC \\ &= 60^\circ.\end{aligned}$$

Similarly, $\angle SCB = 60^\circ$. It follows that $\triangle BSC$ is equilateral, so $BNCS$ is a rhombus and SN

is the bisector of $\angle BSC$, that is of $\angle USV$. By the angle bisector theorem we have

$$\begin{aligned}\frac{NU}{NV} &= \frac{SU}{SV}\\ &= \frac{SB+BU}{SC+CV}\\ &= \frac{BC+BU}{BC+CV}.\end{aligned}$$

Now $\triangle BUC \simeq \triangle HBC$ by tangency, so

$$\frac{BU}{BC} = \frac{HB}{HC} = \frac{\cos B}{\cos C}.$$

Similarly

$$\frac{CV}{BC} = \frac{\cos C}{\cos B},$$

So we compute

$$\frac{NU}{NV} = \frac{1+\frac{\cos B}{\cos C}}{1+\frac{\cos C}{\cos B}} = \frac{\cos B}{\cos C}.$$

Now, by the law of cosines

$$BC = \sqrt{AB^2 + AC^2 - 2\cdot AB\cdot AC\cdot \cos 60^\circ} = \sqrt{139}.$$

Also, by the law of cosines

$$\cos B = \frac{BA^2+BC^2-AC^2}{2\cdot BA\cdot BC} = \frac{7\sqrt{139}}{278},$$

while

$$\cos C = \frac{CA^2+CB^2-AB^2}{2\cdot CA\cdot CB} = \frac{8\sqrt{139}}{139},$$

for a final answer of

$$\frac{NU}{NV} = \boxed{\frac{7}{16}}.$$

6. Let ABC be a triangle with $AB = 7$, $BC = 6$ and $AC = 8$. Let M be the midpoint of $\overline{BC}$ and let S be on segment AM such that $\frac{MS}{AM} = \frac{1}{4}$. The line through A parallel to $\overline{BC}$ meets the circumcircles of triangles ASB and ASC at $D \neq A$ and $E \neq A$, respectively. Find DE.

   Remark. Regrettably, due to an error on behalf of the author, the in-contest problem statement had the ratio written as $\frac{MS}{AM} = 4$ instead of $\frac{MS}{AM} = \frac{1}{4}$.

   Proposed by Ana Boiangiu

   Solution: The next two problems use a relatively obscure result called *Ptolemy's sine lemma.* While problem 6 can be solved without it, the author is not aware of a different solution to problem 7 at the time of writing. You can read about it here.

   Let $AS/AM = k = 3/4$. By Ptolemy's sine lemma, we find that

$$AS\cdot \sin\angle CAE + AE\cdot \sin\angle CAS = AC\cdot \sin\angle SAE.$$

Using the law of sines, this becomes

$$\begin{aligned} AE &= AC \cdot \frac{\sin \angle AMC}{\sin \angle CAM} - k \cdot AM \cdot \frac{\sin \angle ACM}{\sin \angle CAM} \\ &= AC \cdot \frac{AC}{CM} - k \cdot AM \cdot \frac{AM}{CM} \\ &= \frac{2b^2}{a} - k \cdot \frac{2\left(b^2+c^2\right)-a^2}{2a} \\ &= \frac{(4-2k)\cdot b^2 - 2k \cdot c^2 + k \cdot a^2}{2a}. \end{aligned}$$

We similarly compute

$$AD = \frac{(4-2k)\cdot c^2 - 2k \cdot b^2 + k \cdot a^2}{2a}.$$

Consequently

$$DE = AD + AE = \frac{(2-2k)\cdot\left(b^2+c^2\right) + k \cdot a^2}{a} = \boxed{\frac{167}{12}}.$$

7. Let ABC be a triangle with $AC = 33$, $BC = 16$, $AB = 28$ and incenter I. Let U and V be points on sides AB and AC respectively such that $AU = AV = 20$. Let P be the reflection of B over U and let Q be the reflection of C over V. The circumcircles of triangles BIP and CIQ intersect at a point $S \neq I$. Let M be the midpoint of $\overline{BC}$. Find MS.

   Proposed by Ana Boiangiu

   Solution: The main claim is the following.

   Claim. M lies on the radical axis of (BIP) and (CIQ). In particular, M, I and S are collinear.

   Proof. Denote $AU = AV = x$. Let BC meet (BIP) and (CIQ) at X and Y, respectively. We need to prove that

$$MB \cdot MX = MC \cdot MY,$$

   or $MX = MY$, that is $BX = CY$. By Ptolemy's sine lemma we have

$$BX \cdot \sin \angle IBX + BI \cdot \sin \angle PBX = BP \cdot \sin \angle IBX.$$

   Equivalently

$$BX \cdot \sin \frac{B}{2} + BI \cdot \sin B = 2(c-x) \cdot \sin \frac{B}{2},$$

   or

$$BX = 2(c-x) - 2BI \cos \frac{B}{2} = 2(c-x) - 2(s-b) = 2(b+c-s-x).$$

   This is symmetric in b and c, so $BX = CY$, as desired.

   Now by power of a point we have

$$\begin{aligned} MI \cdot MS &= MB \cdot MX \\ &= MB \cdot (MB + BX) \\ &= \frac{a^2 + 4a(b+c-s-x)}{4} \\ &= 104. \end{aligned}$$

It remains to compute MI. By well-known formulas we have

$$\begin{aligned} MI^2 &= \frac{2(IB^2 + IC^2) - BC^2}{4} \\ &= \frac{ac(s-b)}{2s} + \frac{ab(s-c)}{2s} - \frac{a^2}{4} \\ &= 40, \end{aligned}$$

so $MI = 2\sqrt{10}$. Finally, we compute

$$MS = \frac{104}{MI} = \frac{52}{\sqrt{10}} = \boxed{\frac{26\sqrt{10}}{5}}.$$

8. Connor is standing on an infinite, flat ground plane at $z = 0$. The ground is perfectly solid everywhere except for a circular hole of radius 1 centered at the origin, i.e., the set $x^2 + y^2 \leq 1$ is missing from the plane. He holds a cube of side length 1000 starting entirely above the ground in the half-space $z > 0$. He may translate and rotate the cube in any way he likes, subject to the constraint that no part of the cube ever passes through the solid portion of the ground plane; it may only pass through the circular hole.

Let S be the set of all points with $z < 0$ that can coincide with a vertex of the cube during such motions. Find the volume of S.

Proposed by Atharva Pathak

Solution: Let O be $(0, 0, 0)$. Let V be the vertex of the cube going below $z = 0$. We lower the cube down to the deepest possible depth directly below any point P (P for projection) inside the unit circle, and must find a formula for this depth d as a function of the distance r from P to O.

When lowered fully, the three edges of the cube sit tangent to the unit circle. Say these three points are A, B, C. By the orthogonality of the edges of the cube, we have $\overrightarrow{VA} \cdot \overrightarrow{VB} = 0$ and $\overrightarrow{VA} \cdot \overrightarrow{VC} = 0$, so subtracting we get $\overrightarrow{VA} \cdot \overrightarrow{BC} = 0$. Since $\overrightarrow{VA} = \overrightarrow{VP} + \overrightarrow{PA}$ and $\overrightarrow{VP}$ is perpendicular to the xy plane, we get $\overrightarrow{PA} \cdot \overrightarrow{BC} = 0$. Combining this with the analogous cyclic results, we find that $P = H$ is the orthocenter of $\triangle ABC$.

We also have

$$0 = \overrightarrow{VA} \cdot \overrightarrow{VB} = (\overrightarrow{VP} + \overrightarrow{PA}) \cdot (\overrightarrow{VP} + \overrightarrow{PB}) = d^2 + \overrightarrow{PA} \cdot \overrightarrow{PB},$$

so with the analogous cyclic results we find $\overrightarrow{PA} \cdot \overrightarrow{PB} = \overrightarrow{PB} \cdot \overrightarrow{PC} = \overrightarrow{PC} \cdot \overrightarrow{PA} = -d^2$. Using the orthocenter fact $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, we have

$$\begin{aligned} -3d^2 &= \overrightarrow{PA} \cdot \overrightarrow{PB} + \overrightarrow{PB} \cdot \overrightarrow{PC} + \overrightarrow{PC} \cdot \overrightarrow{PA} \\ &= (-(\overrightarrow{OB} + \overrightarrow{OC})) \cdot (-(\overrightarrow{OA} + \overrightarrow{OC})) + \text{cyc} \\ &= (|\overrightarrow{OC}|^2 + \overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OB} \cdot \overrightarrow{OC}) + \text{cyc} \\ &= 3 + 3(\overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OB} \cdot \overrightarrow{OC}), \end{aligned}$$

so $\overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OB} \cdot \overrightarrow{OC} = -d^2 - 1$, and

$$\begin{aligned} r^2 = |\overrightarrow{OP}|^2 &= (|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 + |\overrightarrow{OC}|^2) + 2(\overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OB} \cdot \overrightarrow{OC}) \\ &= 3 + 2(-d^2 - 1), \end{aligned}$$

so $d = \sqrt{\frac{1-r^2}{2}}$. So the region below the xy plane that the vertex can access is a hemisphere of radius 1 squished by a factor of $\sqrt{2}$, whose volume is $\boxed{\dfrac{\pi\sqrt{2}}{3}}$.