P U M $\therefore$ C

2025 Individual Finals A

You have **90 minutes** for this examination. To get full credit, give a rigorous mathematical proof of your answer. Do not discuss or distribute these problems until **January 16th, 2026**.

1. Let λ^* be the maximum value of λ such that for all cubic polynomials $P(x) = x^3 + ax^2 + bx + c$ whose zeros are all real and nonnegative, then $P(x) \geq \lambda(x-a)^3$ for all $x \geq 0$. Prove that $\lambda^* = -\frac{1}{27}$.

   Proposed by Aiden Sharp

   Solution: Let P have zeros r, s, t. WLOG, assume that $0 \leq r \leq s \leq t$. For $0 \leq x \leq r$, we have
   $$-P(x) = (r-x)(s-x)(t-x) \leq \frac{1}{27}(-a-3x)^3 \leq \frac{1}{27}(x-a)^3$$
   where the first inequality holds by AM-GM and the second holds because $x \geq 0$.

   Thus, for $P(x) \geq -\frac{1}{27}(x-a)^3$ for all $0 \leq x \leq r$. There is equality if $r = s = t$ and $x = 0$, so $-\frac{1}{27}$ is maximal.

   When $s \leq x \leq t$, using the same argument,
   $$-P(x) = (x-r)(x-s)(t-x) \leq \frac{1}{27}(x-(r+s-t))^3 \leq \frac{1}{27}(x-(-r-s-t))^3 = \frac{1}{27}(x-a)^3$$
   so $P(x) \geq -\frac{1}{27}(x-a)^3$.

   Finally, if $r < x < s$ or $t < x$, then
   $$P(x) > 0 \geq -\frac{1}{27}(x-a)^3.$$

   Thus, $\lambda^* = \boxed{-\frac{1}{27}}$.

2. Prove that there exists a constant $c > 0$ such that for any set of integers $a_1 > \cdots > a_n$ there exist indices $1 \leq i < j \leq n$ such that $a_i - a_j$ has $\geq 2^{c \log n / \log\log n}$ divisors.

   Proposed by Arpon Basu

   Solution: Let $m < n$ be such that $\tau(m) \geq \tau(k)$ for all $1 \leq k < n$ where τ counts the number of divisors. By pigeonhole principle, two numbers amongst $a_1, \ldots, a_n$ will have the same remainder modulo m. Consequently, $\max_{i \neq j} \tau(a_i - a_j) \geq \tau(m)$. Now, let $m := \prod_{1 \leq k \leq r} p_k$ be the largest number $< n$ which is the product of first r primes. Then $\tau(m) = 2^r$. Furthermore, let $\pi(x)$ be the number of primes less than x. We have that, by the Prime Number Theorem,
   $$\log m = \sum_{p \leq p_r,\, p \text{ prime}} \log p < \pi(p_r) \log p_r < \left(c_1 \frac{p_r}{\log p_r}\right) \log p_r = c_1 p_r$$
   for some absolute constant $c_1 > 0$. Hence to have $\log m < \log n$, it suffices to have $p_r < \frac{\log n}{c_1}$. Thus, applying the prime number theorem again, we must be able to take, for any $\varepsilon \in (0, 1)$, constant c_3 that only depends on ε, and absolute constant c_2, that
   $$r = \pi(p_r) \geq c_2 \frac{(1-\varepsilon)}{c_1} \frac{\log n}{\log\left(\frac{(1-\varepsilon)}{c_1} \log n\right)} \geq \frac{c_2(1-\varepsilon)}{c_1} \frac{\log n}{\log(1-\varepsilon) - \log c_1 + \log\log n} > \frac{c_2(1-\varepsilon)}{c_1 c_3} \frac{\log n}{\log\log n}$$
   to ensure $p_r \leq \frac{(1-\varepsilon)\log n}{c_1}$.

3. Let ABC be an acute scalene triangle with orthocenter H, circumcenter O and circumcircle Ω. It is given that $OH \parallel BC$. Line OH meets side AB at U and side AC at V. Let R be on Ω such that $\angle ARH = 90°$. Let the circumcircle of triangle ROH meet Ω a second time at S. Show that lines SO and RH meet on the circumcircle of triangle RUV.

   Proposed by Ana Boiangiu

   Solution: The main observation is the following claim.

   Claim. $AS \parallel BC$.

   Proof. Since $OS = OR$ and R, H, O and S are concyclic, HO is the external bisector of $\angle RHS$ and so $\angle SHO = \angle A'HO$.

   Because $OS = OA'$, it follows that OH is the perpendicular bisector of SA'. In particular $OH \perp SA'$, so $SA' \perp BC$. Thus

   $$\angle SA'C = 90° - \angle A'CB = \angle ACB.$$

   It follows that $AB = SC$, so $BASC$ is an isosceles trapezoid and $AS \parallel BC$.

   Now $HA \perp BC$ gives that $\angle HAS = 90°$, so AH and SO meet at H', the antipode of S in (ABC) and the reflection of H across BC.

   Since $OH \parallel BC$, we also have that $OH \perp AH'$, so OH is the perpendicular bisector of AH' and H is the midpoint of AH'. It follows that OH is a midline in $\triangle AH'A'$, so $OH \parallel H'A'$. Finally

   $$\angle SH'A' = \angle SRA' = \angle A'HO,$$

   so X lies on (SOH), as desired.