P U M $\therefore$ C

Individual Finals A

1. Let ABC be a triangle. Construct three circles k_1, k_2 and k_3 with the same radius such that they intersect each other at a common point O inside the triangle ABC and $k_1 \cap k_2 = \{A, O\}, k_2 \cap k_3 = \{B, O\}, k_3 \cap k_1 = \{C, O\}$. Let t_a be a common tangent of circles k_1 and k_2 such that A is closer to t_a than O. Define t_b and t_c similarly. Those three tangents determine a triangle MNP such that triangle ABC is inside the triangle MNP. Prove that the area of MNP is at least 9 times the area of ABC.

2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$ it holds that

$$f\left(xy\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{x+y}\right)\right)=f\left(xy\left(\frac{1}{x}+\frac{1}{y}\right)\right)+f\left(x\right)f\left(\frac{y}{x+y}\right).$$

3. We say that the prime numbers $p_1, \ldots, p_n$ construct the graph G if we can assign to each vertex of G a natural number whose prime divisors are among $p_1, \ldots, p_n$ and there is an edge between two vertices in G if and only if the numbers assigned to the two vertices have a common divisor greater than 1. What is the minimal n such that there exist prime numbers $p_1, \ldots, p_n$ which construct any graph G with N vertices.