Copyright © 2018 MathCON National

GRADE 5 SAMPLE ITEMS

Question # 4	Part A - Level 1 (3 points)
	Number Theory

$abcdef$ is the smallest six-digit positive integer whose sum of the digits is 30. What digit is in the hundred thousands place of $abcdef$?

A) 1 B) 2 C) 3 D) 4 E) 6

Solution. $abcdef = 102999$. So it is 1. $\square$

Question # 25	PART B - Level 2 (4 points)
	Algebra

If $A = \frac{21}{19} + \frac{11}{29}$, then which of the following equals $\frac{18}{29} - \frac{2}{19}$?

A) $2 - A$ B) $1 - A$ C) A D) $A + 1$ E) $A + 2$

Solution.

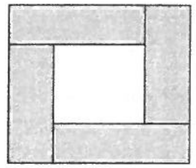
$$\frac{18}{29} - \frac{2}{19} = \left(1 - \frac{11}{29}\right) - \left(\frac{21}{19} - 1\right) = 2 - \left(\frac{11}{29} + \frac{21}{19}\right) = 2 - A$$

$\square$

Question # 31

PART C - Level 3 (5 points)
Geometry

The diagram shows four identical rectangles placed inside a square. The perimeter of each rectangle is 24 cm. What is the perimeter of the large square?



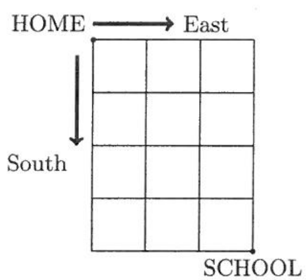
A) 24 cm B) 30 cm C) 36 cm D) 48 cm E) 56 cm

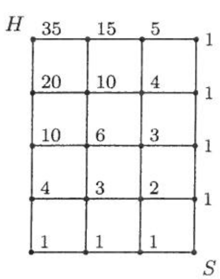
Solution. Let x be the length of short edge of each identical rectangles. Then the length of long edge will be $12 - x$ since the perimeter is 24. Therefore, the length of each side of the square is $x + (12 - x) = 12$ and so the perimeter of the square is 48. □

Question # 45

Part D - Level 4 (6 points)
Combinatorics

Gabby lives in a town whose streets are on a grid system with all streets running east-west or north-south without breaks. Her school, located on a corner, lies four blocks south and three blocks east of her home, also located on a corner. If Gabby only walks south or east on her way to school, how many possible routes can she take to school?





Solution. Give "values" to the each point on the grid as follows: "1" to the closest two points (since there is only one way to school), and, the sum of values of the point on the south and the point on the east. In the end, we have 35 ways to go to the school. □

GRADE 6 SAMPLE ITEMS

Question # 8

PART A - Level 1 (3 points)
Algebra

$x + 4 = y^2 - 1 = z^2 + 2 = t - 3 = m^2 + 12$. Which of the numbers x, y, z, t, and m is the greatest?

A) x B) y C) z D) t E) m

Solution. Let $A = x + 4 = y^2 - 1 = z^2 + 2 = t - 3 = m^2 + 12$. Then $x = A - 4$, $y = \sqrt{A+1}$, $z = \sqrt{A-2}$, $t = A + 3$, $m = \sqrt{A-12}$. Thus the greatest value is t. □

Question # 25

PART B - Level 2 (4 points)
Combinatorics

The first page number of a book is 1. The sum of the number of pages of the book is less than 2020. If there were 1 more page, then the sum of the number of pages of the book would be more than 2020. Find the number of pages of the book.

A) 59 B) 60 C) 61 D) 62 E) 63

Solution. Assume n is the last page of the book. Then we have

$$1+\ldots+n = \frac{n \cdot (n+1)}{2} < 2020 \text{ and } 1+\ldots+(n+1) = \frac{(n+1)\cdot(n+2)}{2} > 2020.$$

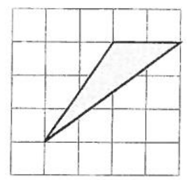
Since n is an integer, we have $n \leq 63$ in the first inequality, and $n+1 \geq 64$ in the second inequality. Thus $n = 63$. □

Question # 35

PART C - Level 3 (5 points)
Geometry

The vertices of a triangle are (1,1), (5,4), and (3,4). Find the area of the triangle.

A) $\frac{7}{2}$ B) 3 C) $\frac{5}{2}$ D) 2 E) $\frac{3}{2}$



Solution. The lengths of the the height and the base of the triangle is $4-1=3$ and $5-3=2$. Thus the area of the triangle is $3.2/2=3$. $\square$

Question # 42

Part D - Level 4 (6 points)
Number Theory

What is the number of ordered pairs (x, y) of positive integers that satisfy the equation?

$$2x+3y=120$$

A) 19 B) 24 C) 29 D) 36 E) None

Solution. We have $x=60-\dfrac{3y}{2}$. Since x is an integer, y should be divisible bu 2, say $y=2k$ for some $k \in \mathbb{Z}^{+}$. Then $x=60-3k$ and $y=2k$ for $k \in \mathbb{Z}^{+}$. Then we have (x, y) solutions for $k=1,2,\ldots,19$, that is, there are 19 solutions. $\square$

GRADE 7 SAMPLE ITEMS

Question # 5

PART A - Level 1 (3 points)
Geometry

The ratio of the corresponding side lengths of two similar triangles is $5:4$, and the perimeter of the greater triangle is 30 cm. What is length the of the shortest side of the smaller triangle, if its side lengths are consecutive even numbers?

A) 10 cm B) 8 cm C) 6 cm D) 4 cm E) 2 cm

Solution. Since the similarity ratio is $5:4$, the perimeter of the smaller triangle is $\frac{30\cdot 4}{5}=24$. Let the length of the shortest side of the smaller triangle be a. Then the other lengths are $a+2$ and $a+4$. Since $a+(a+2)+(a+4)=24$, we have $a=6$ as the solution of this equation. $\square$

Question # 19

PART B - Level 2 (4 points)
Algebra

If a and b are additive inverses (opposites) that are nonzero, then

$$2(a+b-1)^{2017}+3\left(\frac{a}{b}\right)^{2018}+2018$$

equals

A) 1 B) 2017 C) 2018 D) 2019 E) None

Solution. Since a and b are additive inverses, we have $a+b=0$ or $b=-a$. Then $\frac{a}{b}=-1$ and so

$$2(a+b-1)^{2017}+3\left(\frac{a}{b}\right)^{2018}+2018=2(-1)^{2017}+3(-1)^{2018}+2018=2019$$

$\square$

Question # 29	PART C - Level 3 (5 points)
	Combinatorics

4 elements from the set $A = \{1, 2, 3, \ldots, 12\}$ is chosen and arranged along a circle. The oddness or evenness of any number on the circle and the sum of the other numbers on the circle are the same. How many different arrangements are possible?

A) 180 B) 90 C) 75 D) 60 E) None

Solution. All of the chosen numbers should be even or odd. There are 6 even numbers and 6 odd numbers in the set. There is $\binom{6}{4}$ ways to choose and $(4-1)!$ ways to arrange on the circle. Thus our answer is $2 \cdot \binom{6}{4} \cdot (4-1)! = 180$ ways to arrange in total. □

Question # 42	Part D - Level 4 (6 points)
	Number Theory

The sum of three positive numbers is 314. How many zeros, at most, the product of these three numbers end with?

Solution. Let x, y and z be such positive integers, i.e. $x+y+z = 314$. We want to find the maximum number of zeroes where the product xyz to end with. Let $2^{x_1}5^{x_2}|x$, $2^{y_1}5^{y_2}|y$ and $2^{z_1}5^{z_2}|z$. One of x_2, y_2, z_2 must be 0, otherwise $5|314$, which is a contradiction. Say $x_2 = 0$. On the other hand, $y_2, z_2 \leq 3$, otherwise $y, z \geq 625$, which is again a contradiction. So we can have $y = z = 125$ and $x = 64$. Then $xyz = 10^6$ and thus the answer is 6. □

GRADE 8 SAMPLE ITEMS

Question # 3	PART A - Level 1 (3 points)
	Combinatorics

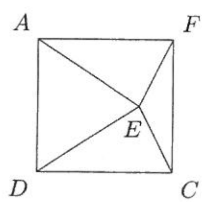
Five companies each send three representatives to a networking event. At the event, each representative must shake hands with all the representatives from companies other than their own. How many handshakes must take place?

A) 72 B) 90 C) 108 D) 120 E) 156

Solution. There are 15 representatives in total. So there are $\binom{15}{2}$ handshake without any restriction. However we want each representative to shake hands with all the representatives from companies other than their own, so we need to exclude $\binom{3}{2}$ handshakes per company. Thus our answer is $\binom{15}{2} - 5 \cdot \binom{3}{2} = 90$. □

Question # 11	PART B - Level 2 (4 points)
	Geometry

Equilateral $\triangle ADE$ shares a side with square $ADCF$. What is the sum of $\angle FEC$ and $\angle DEC$?



A) 195° B) 200° C) 215° D) 225° E) 245°

Solution. Since $\triangle ADE$ is equilateral and $ADCF$ is square, we have $|AF| = |AE|$ and $|DE| = |DC|$. Then these are 30-75-75 triangles. So $m(\angle FEC) = 360 - (60 + 75 + 75) = 150$. Thus

$$m(\angle FEC) + \angle DEC) = 150 + 75 = 225.$$

□

Question # 33

PART C - Level 3 (5 points)
Algebra

If n is positive numbers that satisfies the equation

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{16}$$

What's the value of n?

A) 4 B) 8 C) 9 D) 12 E) 16

Solution.

$$\begin{aligned}\frac{n+1}{16} &= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\\ &= \frac{(2^2-1)}{2^2}\frac{(3^2-1)}{3^2}\cdots\frac{(n^2-1)}{n^2}\\ &= \frac{[(2-1)(3-1)\ldots(n-1)]\,[(2+1)(3+1)\ldots(n+1)]}{2^2 3^2\ldots n^2}\\ &= \frac{n+1}{2n}\end{aligned}$$

Thus $n=8$ by simplifying both sides. □

Question # 44

Part D - Level 4 (6 points)
Number Theory

If a, b and c are distinct prime numbers with $a-c=8300$ and $a+b+c=9074$, then what is the value of $a+2b$?

Solution. One of a, b, c is even prime, that is 2, otherwise $a+b+c$ will be odd. Since $a-c$ is even, both a and c are odd and so $b=2$. Thus the equations $a-c=8300$ and $a+c==9702$ gives $a=9001$ and $c=701$. So $a+2b=9005$. □

GRADE 9 SAMPLE ITEMS

Question # 5

PART A - Level 1 (3 points)
Number Theory

Which of the following is the units digit of the number 2018^{1776}?

A) 1 B) 3 C) 6 D) 7 E) None

Solution. Note that $2018^{1776} \equiv 8^{1776} \mod 10$. We can deduce that $8^{4k} \equiv 6 \mod 10$, $8^{4k+1} \equiv 8 \mod 10$, $8^{4k+2} \equiv 4 \mod 10$ and $8^{4k+3} \equiv 2 \mod 10$ for any integer k. Since 1776 is divisible by 4, we have $2018^{1776} \equiv 6 \mod 10$. □

Question # 16

PART B - Level 2 (4 points)
Algebra

Find the sum of all real solutions of the equation $(x-3)^{x^2-9}=1$?

A) 1 B) 3 C) 4 D) 7 E) 8

Solution. There are three cases: i) $(x-3)=1$, ii) $(x-3)=-1$ but x^2-9 is odd, iii) $x^2-9=0$ but $x-3\neq 0$. Then the possible solutions are 4 (in the Case i) and $x=4$ in the Case iii. Thus the answer is 1. □

Question # 32

PART C - Level 3 (5 points)
Combinatorics

In a regular decagon, all diagonals are drawn. If a diagonal is chosen at random, what is the probability it is neither one of the shortest nor one of the longest?

A) $\frac{2}{7}$ B) $\frac{3}{7}$ C) $\frac{12}{35}$ D) $\frac{4}{9}$ E) $\frac{4}{7}$

Solution. There are $\binom{10}{2} = 35$ diagonals in total. Label the vertices as $0, 1, \ldots, 9$. Then the shortest diagonals are in the form $(x-2, x+2)$. Then the set of such diagonals are $\{02, 08, 13, 19, 24, 35, 46, 57, 68, 79\}$. On the other hand, the longest diagonals are in the form $(x, x+5)$. Then the set of such diagonals are $\{05, 16, 27, 38, 49\}$. Thus the probability that the chosen diagonal is neither one of the shortest nor one of the longest is

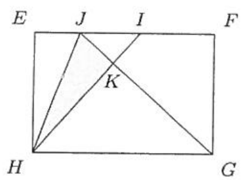
$$1 - \frac{10+5}{35} = \frac{4}{7}$$

. □

Question # 43

Part D - Level 4 (6 points)
Geometry

$EFGH$ is a rectangle with $|FG| = 4$ and $|EF| = 6$. If the area of the shaded triangle is 3 square units, then what is the length of JI?



Solution. Let $|JI| = x$. Note that the area of $\triangle JGH = \frac{4 \cdot 6}{2} = 12$ and the area of $\triangle JIH = \frac{4 \cdot x}{2} = 2x$. Then the area of $\triangle KGH = 12 - 3 = 9$ and the area of $\triangle JIK = 2x - 3$. Using the similarity ratio on the areas of the triangles JIK and KGH we have

$$\frac{2x-3}{9} = \left(\frac{x}{6}\right)^2.$$

Finally we have $x = 2$ by solving this equation. □

GRADE 10 SAMPLE ITEMS

Question # 7

PART A - Level 1 (3 points)
Algebra

If $a^2 + 4a = 1$ then find the value of $a^2 + \frac{1}{a^2}$.

A) 12 B) 14 C) 16 D) 18 E) 20

Solution. It's clear that $a \neq 0$. Then

$$a^2 + 4a = 1 \Rightarrow a + 4 = \frac{1}{a} \Rightarrow 16 = (-4)^2 = \left(\frac{1}{a} - a\right)^2 = a^2 + \frac{1}{a^2} - 2$$

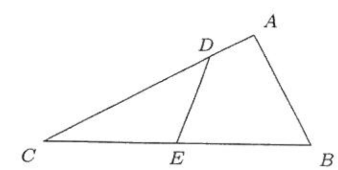
$$\Rightarrow a^2 + \frac{1}{a^2} = 18.$$

□

Question # 21

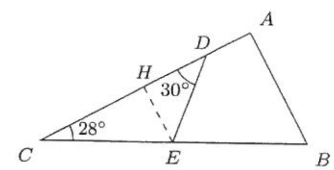
PART B - Level 2 (4 points)
Geometry

ABC is a right triangle with $AB \perp AC$. If $|CE| = |EB|$, $|DE| = |AB|$ and $m(\angle ACB) = 28°$, then find $m(\angle DEC)$.



A) 90 B) 108 C) 122 D) 132 E) None

Solution. Let H be a point on AC such that $AC \perp HE$. Then $|HE| = |AB|/2$ by similarity. Since, $|AB| = |DE|$, so DEH is a 30-60-90 triangle. Thus $m(\angle DEC) = 180 - (30 + 28) = 122$. □



Question # 32

PART C - Level 3 (5 points)
Combinatorics

If x, y, and z are three numbers picked randomly and with replacement from the set $\{1,2,3,4,5\}$ then what is the probability that $xz+y$ is even number?

A) $\frac{2}{5}$ B) $\frac{23}{25}$ C) $\frac{39}{125}$ D) $\frac{64}{125}$ E) $\frac{59}{125}$

Solution. Total number of outcomes is $5^3 = 125$. There are two cases: 1) Both xz and y are even. Then there are $(25-9)\cdot 2 = 32$ triples. 2) Both xz and y are odd. Then there are $9 \cdot 3 = 27$ triples. The answer is $\frac{32+27}{125} = \frac{59}{125}$. □

Question # 42

Part D - Level 4 (6 points)
Number Theory

If $(y+x)(y-x) = 111+6y$ then how many (x,y) integer pair solution does the equation have?

Solution.

$$y^2 - x^2 = (y+x)(y-x) = 111 + 6y \Rightarrow y^2 - x^2 - 6y + 9 = 120$$

$$\Rightarrow (y-3)^2 - x^2 = 120 \Rightarrow (y-3+x)(y-3-x) = 120$$

Since 120 and $(y-3+x)-(y-3-x) = 2x$ are even, both $(y-3+x)$ and $(y-3-x)$ should be even. Say $y-3-x = 2k$, then $y-3+x = 2k+2x$. By simplifying,

$$k(k+x) = 30$$

For each divisor of 30 for k we get a integer pair (x,y), thus the number of pairs is 16. □

GRADE 11 SAMPLE ITEMS

Question # 8

PART A - Level 1 (3 points)
Algebra

Find the number of integer solutions of the following equation $(x^2 - 3x + 1)^{x+1} = 1$.

A) 0 B) 1 C) 2 D) 3 E) 4

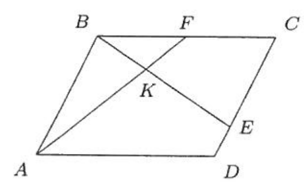
Solution. There are three cases: i) $x^2 - 3x + 1 = 1$. So $x = 3$ or $x = 0$. ii) $x^2 - 3x + 1 = -1$ but $x+1$ is even. So $x = 1$. iii) $x + 1 = 0$ but $x^2 - 3x + 1 \neq 0$. So $x = -1$. Then there are 4 solutions. □

Question # 17

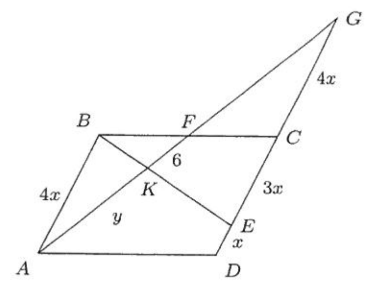
PART B - Level 2 (4 points)
Geometry

$ABCD$ is a parallelogram. $|CE| = 3|ED|$, $|BF| = |FC|$ and $|KF| = 6$. Find $|AK|$.

A) 10 B) 12 C) 14 D) 15 E) 16



Solution. Let $|ED| = x$ and $|AK| = y$. Then $|CE| = 3x$ and $|AB| = 4x$. Since $|BF| = |FC|$, we have $|GC| = 4x$ and $|GF| = y + 6$ by similarity on ABF and GCF. We have another similarity on ABK and GEK, so $\frac{4x}{7x} = \frac{y}{y+12}$. Solving this equation, we have $y = 16$. □



Question # 35

PART C - Level 3 (5 points)
Combinatorics

3 of 101 coins are fake. Weights of the fake coins are equivalent and lighter than the right size. In how many weights, at least, can we find 25 right size coins by using an equal arm balance?

A) 2 B) 3 C) 4 D) 5 E) None

Solution. Partition 101 coins into 3 groups A, B, C of sizes 50, 50 and 1 respectively, then compare the weights of A and B. If their weights are equal, each of A and B contains a fake coin (while the other fake coin is in C). Partition A into two subgroups A_1 and A_2 of sizes 25 and 25 and compare their weights. Their weights can not be equal, moreover, the group with heavy weight contains no fake coin. In total we used equal arm balance twice.
On the other hand, if the weights of A and B are not equal, then the group heavier than the other contains one fake coin (while the other contains two fake coins). Partition the heavy group, which has one fake coin, into two subgroups of sizes 25 and 25. Again their weights can not be equal, moreover, the group with heavy weight contains no fake coin. In total we used equal arm balance twice again. □

Question # 43

Part D - Level 4 (6 points)
Number Theory

The sum of two real numbers is n, and the sum of their squares is $n + 19$, for a positive integer n. What is the possible maximum value of n?

Solution. Let x and y be real numbers with $x + y = n$ and $x^2 + y^2 = n + 19$. When we substitute $y = n - x$, we have $x^2 + (n - x)^2 = n + 19$, or, $2x^2 - 2nx + (n^2 - n - 19) = 0$. This can be seen as an quadratic equation in terms of x. To have a real number solution for x, the discriminant of the equation, that is $\Delta = (2n)^2 - 4 \times 2 \times (n^2 - n - 19) = -4n^2 + 8n + 152 = -4(n-1)^2 + 156$, can not be less than 0, in other words, $-4(n-1)^2 + 156 \geq 0$. Then by solving this inequality we have $n = 7$ as the maximum value for n. □

GRADE 12 SAMPLE ITEMS

Question # 10

PART A - Level 1 (3 points)
Algebra

If x and y are real numbers so that $4x^2 + y^4 - 20x + 29 = 0$, then what does $y^2 + 2x$ equal?

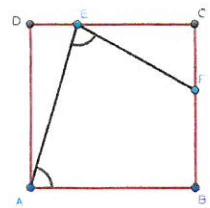
A) 7 B) 8 C) 10 D) 12 E) None

Solution. Note that $4x^2 + y^4 - 4y^2 - 20x + 29 = 0$ can be re-expressed as $(y^2 - 2)^2 + (2x - 5)^2 = 0$. Thus $y^2 = 2$ and $2x = 5$. Hence $y^2 + 2x = 7$. □

Question # 18

PART B - Level 2 (4 points)
Geometry

$ABCD$ is a square. $|FB| = 2|CF|$ and $m(\angle EAB) = m(\angle FEA)$. Find $\tan(\angle DAE)$.



A) $\frac{1}{5}$ B) $\frac{2}{5}$ C) $\frac{3}{5}$ D) 1 E) $\frac{3}{2}$

Solution. Let $|DE| = 1$, $|AD| = p$ and $m(\angle EAB) = \alpha$. Then $|BC| = |CD| = p$, $|CF| = p/3$, $|EC| = p - 1$ and $m(\angle FEA) = \alpha$. Note that $\tan\alpha = p$ and $\frac{p/3}{p-1} = \tan(\angle FEC) = \tan(180 - 2\alpha) = -tan(2\alpha)$. Also note that $tan(2\alpha) = \frac{2tan(\alpha)}{1-tan^2(\alpha)} = \frac{2p}{1-p^2}$. Thus, by solving $\frac{p/3}{p-1} = -\frac{2p}{1-p^2}$, we have $p = 5$. Since $\tan(\angle DAE) = \frac{1}{\tan\alpha}$, the answer is $1/5$. □

Question # 27

PART C - Level 3 (5 points)
Combinatorics

5 soccer teams participate in a tournament. Each team plays all the other teams exactly once. If a team wins a game, 3 points are awarded. If a team loses a game, no points are awarded. If the game is tie, both teams get 1 point. At the end of all the games, 4 of the 5 teams get 1, 2, 5, and 8 points in total. What is the total score of the fifth team?

A) 10 B) 12 C) 14 D) 16 E) None

Solution. Let W, L and T represent a win, a lose and a tie respectively. Let A, B, C and D be the four teams who gets 1, 2, 5 and 8 points in total, respectively. Let E be the fifth team.
Neither A nor B has a win, since both of them have less than 3 points, i.e. no wins. So A has $\{T, L, L, L\}$ and B has $\{T, T, L, L\}$. Moreover, the match between A and B must be a tie. Thus A lost all other matches against C, D, E.
Since C has a win against A and 5 points in total, C has $\{W, T, T, L\}$. Then the match between B and C must be a tie, since neither B or C has no other wins. Thus B losses its matches against D and E.
Since D has two wins against A and B, and 8 points in total, D has $\{W, W, T, T\}$, i.e. its other matches against C or E are tie.
Finally, E has a win against A, B and C, while it has a tie against D. So its total point is $3 \cdot 2 + 1 = 10$. □

Question # 43

Part D - Level 4 (6 points)
Number Theory

$T(n)$ is the digit sum of positive integer n. (e.g. $T(5081) = 5 + 0 + 8 + 1 = 14$) Find the number of three-digit numbers that satisfy $T(n) + 3n = 2020$.

Solution. Since $1 \leq T(n) \leq 27$ for any three-digit number n and $T(n) = 2020 - 3n$, we have $1 \leq 2020 - 3n \leq 27$, or, $665 \leq n \leq 673$ after we solve the inequality. The only integer in this interval, satisfying the equation $T(n) + 3n = 2020$, is 667. □