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2025 Number Theory A

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. Find the sum of all positive integers $n < 2025$ such that $\gcd(n^2+1, 2n+1) > 1$.

   Proposed by Rohit Agarwal

   Solution: Note that $\gcd(a,b) = \gcd(ra+sb, b)$ for any integers r, s. Thus,

   $$\gcd(n^2+1, 2n+1) = \gcd(4(n^2+1) - (2n-1)(2n+1), 2n+1) = \gcd(5, 2n+1)$$

   Since 5 is prime, we just need $2n+1 \equiv 0 \pmod 5$ or $n \equiv 2 \pmod 5$. Thus, the possible solutions are $2, 7, 12, \ldots, 2022$ with sum $\frac{(2+2022)\cdot 405}{2} = \boxed{409860}$

2. How many composite odd integers $N \leq 10^4$ are there such that $\varphi(N)$ is not a multiple of 4? ($\varphi(N)$ is the number of integers k with $1 \leq k \leq N$ such that $\gcd(k, N) = 1$.)

   Proposed by Pico Gilman

   Solution: Writing out the prime factorization $N = p_1^{k_1} \ldots p_\ell^{k_\ell}$ gives the formula $\varphi(N) = \prod_i (p_i - 1)p_i^{k_i - 1}$. Since N is odd, none of the p_i's are 2. So, the only way to get a multiple of 4 is if we have two distinct primes that are 3 (mod 4) or one odd prime that is 1 (mod 4). Thus, we can only have a perfect power of a prime (with exponent > 1, because N is composite) that is 3 (mod 4). Enumerating such legal primes, we have: $3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83$. Now, notice that $23^3 > 10^4 > 83^2 > 23^2$, so primes 23 to 83 have perfect powers 2, which yields 9. $11^4 > 10^4 > 19^3 > 11^3$, so 11 and 19 go up to a power of 3, adding 4 more. Finally, $7^4 < 10^4 < 7^5$ and $3^8 < 10^4 < 3^9$, yielding $3 + 7 = 10$ more. Thus, in total we have $\boxed{23}$ such N.

3. Compute the sum of all primes $p < 50$ such that there exist linear multinomials P and Q with integer coefficients such that $x^2 - 2xy - y^2 \equiv P(x,y) \cdot Q(x,y) \pmod p$.

   A linear multinomial is an expression of the form $ax + by + c$.

   Proposed by Rohit Agarwal

   Solution: This only works if you can factorize $x^2 - 2xy + y^2 - 2y^2 = (x-y)^2 - 2y^2$. You can only factorize this is 2 is a quadratic residue mod p, which only happens if $p = 2$ or $p \equiv \pm 1 \pmod 8$. Adding such primes that are less than 50 yields: $2+7+17+23+31+41+47 = \boxed{168}$.

4. For p prime and $k \geq 1$, let $S_k^{(p)}$ denote the number of positive integers $x < p^k$ where $x^{2^k} \equiv 1 \pmod{p^k}$. Find

   $$\sum_{k=1}^{4} \sum_{\substack{p<25 \\ p \text{ prime}}} S_k^{(p)}.$$

   Proposed by Rohit Agarwal

   Solution: The total number of coprime x's is $\phi(p^k) = p^{k-1}(p-1)$. Then, $S_k^{(p)} = \gcd(p^{k-1}(p-1), 2^k)$. If $p = 2$, then this is exactly 2^{k-1}. So the $p = 2$ term is exactly $2^4 - 1 = 15$. Otherwise, we have $2^{\min(\nu_2(p-1), k)}$. So it suffices to compute for each prime $p < 25$ the largest factor of 2 that divides $p - 1$. For $p = 3, 7, 11, 19, 23$, we have $\nu_2(p-1) = 1$, so we get $5 \cdot 4 \cdot 2^1 = 40$ contribution. For $p = 5, 13$ we have $\nu_2(p-1) = 2$, so we get $2 \cdot 14 = 28$. Finally for $p = 17$, we have $\nu_2(p-1) = 4$, so we get $2^5 - 1 - 1 = 30$. Thus the total is $40 + 28 + 30 + 15 = \boxed{113}$.

5. Find the sum of the smallest five positive integers x such that there exists a positive integer y satisfying $x^{x+y} = y^{y-x}$.

Proposed by Aiden Sharp

Solution: Note that if $y < x$, then the exponent of y is negative so $y = 1$. However, if $x > y$, $x^{x+1} > 1$. Thus, there are no solutions for $y < x$. If $y \geq x$, let $\nu_p(x)$ denote the highest power of prime p that divides x , that is $p^n \mid x$ but $p^{n+1} \nmid x$, so $\nu_p(x) = n$. Then, note that for any prime $p \mid x$, $p \mid y$ so every prime divisor of x is a prime divisor of y . Then, $(x+y)\nu_p(x) = (y-x)\nu_p(y)$ so $\nu_p(y) = \frac{x+y}{y-x}\nu_p(x) > \nu_p(x)$. Thus, $\nu_p(y) \geq \nu_p(x)$ for all $p \mid x$ so $x|y$. Thus, let $y = dx$. Plugging this back in, $x^{x(d+1)} = (dx)^{x(d-1)} \implies x^2 = d^{(d-1)}$. Hence, the only solutions are $x = d^{(d-1)/2}$ and $y = d^{(d+1)/2}$. For this to have integer solutions, d^{d-1} must be a perfect square so either d is odd or a perfect square. Thus, the smallest 5 such x occur when $d = 1, 3, 4, 5, 7$, which gives $x = 1, 3, 8, 25, 343$. Thus, the answer is $\boxed{380}$.

6. Suppose that complex numbers t_j for $0 \leq j \leq 2025$ satisfy

$$\prod_{\substack{0 \leq a < 2025 \\ \gcd(a,2025)=1}} (x - e^{2\pi i a/2025}) = \sum_{j=0}^{2025} t_j x^j.$$

Find the sum of all j such that $t_j \neq 0$.

Proposed by Pico Gilman

Solution: We have that

$$\prod_{\substack{0 \leq a < 2025 \\ \gcd(a,2025)=1}} (x - e^{2\pi i a/2025}) = \frac{\prod_{k_1=0}^{2025}(x - e^{2\pi i k_1/2025}) \prod_{k_2=0}^{135}(x - e^{2\pi i k_1/135})}{\prod_{k_1=0}^{2025}(x - e^{2\pi i k_1/675}) \prod_{k_2=0}^{135}(x - e^{2\pi i k_1/405})}$$

$$= \frac{(x^{2025} - 1)(x^{135} - 1)}{(x^{675} - 1)(x^{405} - 1)}$$

This is because $2025 = 3^4 5^2$, so we first need to remove the cases where k_1 is a multiple of 3 and where it is multiple of 5, which are the $2025/3 = 675$th and $2025/5 = 405$th roots of unity respectively. This double removes roots that are multiples of 15, so we need to multiply by $2025/15 = 135$th roots of unity. Substituting $y = x^{135}$ gives

$$\frac{(y^{15} - 1)(y - 1)}{(y^5 - 1)(y^3 - 1)} = \frac{y^{10} + y^5 + 1}{y^2 + y + 1} = y^8 - y^7 + y^5 - y^4 + y^3 - y + 1$$

and so the sum of all j such that $t_j \neq 0$ is equal to $135(8+7+5+4+3+1+0) = 135 \cdot 28 = \boxed{3780}$.

7. For a positive integer n, let $\text{rad}(n)$ denote the product of the unique prime divisors of n, where $\text{rad}(1) = 1$. For example, $\text{rad}(12) = 2 \cdot 3 = 6$. Find the sum of all integers n satisfying

$$\sum_{d \mid n} \text{rad}(d) = n.$$

(The sum is over all positive integer divisors d of n.)

Proposed by Azalea Li

Solution: Let $f(n) = \sum_{d \mid n} \text{rad}(d)$. We claim that $f(n)$ is multiplicative. For a prime power p^e, the divisors are $1, p, p^2, \ldots, p^e$. The radical of 1 is 1, and the radical of p^k for any $k \geq 1$ is

p. Thus, we can evaluate $f(p^e)$ as:

$$f(p^e) = 1 + \underbrace{p + p + \cdots + p}_{e \text{ times}} = 1 + ep$$

Now, let $n = Np^a$ where $p \nmid N$ and $a \geq 1$. Then

$$f(n) = \sum_{d|N} \sum_{i=0}^{a} \text{rad}(dp^i) = \sum_{d|N} \text{rad}(d) \sum_{i=0}^{a} \text{rad}(p^i) = \sum_{d|N} \text{rad}(d)(1 + ap) = f(N)f(p^a),$$

so f is multiplicative by induction over unique prime factors of n.

For any integer $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, the condition $f(n) = n$ translates to:

$$\prod_{i=1}^{k} (1 + e_i p_i) = \prod_{i=1}^{k} p_i^{e_i}$$

First, consider $n = 1$. The only divisor of 1 is 1, and $\text{rad}(1) = 1$. Thus, $\sum_{d|1} \text{rad}(d) = 1$, so $n = 1$ is a valid solution. Assume $n > 1$. We can rewrite our equation as a product of ratios that must equal 1:

$$\prod_{i=1}^{k} \frac{1 + e_i p_i}{p_i^{e_i}} = 1$$

Suppose n is odd. This implies $p_i \geq 3$ for all i. For the product $\prod(1 + e_i p_i)$ to be odd, every $1 + e_i p_i$ must be odd, which means every e_i must be even (and thus $e_i \geq 2$). However, for any prime $p \geq 3$ and integer $e \geq 2$, we have:

$$\frac{1 + ep}{p^e} \leq \frac{1 + 2p}{p^2} \leq \frac{7}{9} < 1$$

Since every factor in the product would be strictly less than 1, the product cannot equal 1. Thus, no odd integer $n > 1$ can be a solution. Therefore, n must be even. Let $p_1 = 2$. Since $1 + 2e_1$ is always odd, all the factors of 2 in the numerator product must come from the terms $(1 + e_i p_i)$ where $i \geq 2$. Specifically, the number of factors of 2 must exactly match e_1. We can systematically check small values for e_1:

- **Case $e_1 = 1$:** The numerator has a factor of $1 + 2(1) = 3$. This forces $3 \mid n$, so let $p_2 = 3$. To get exactly one factor of 2 among the $1 + e_i p_i$ to cancel the $2^{e_1} = 2$ in the denominator, we need exactly one e_i for $i \geq 2$ to be odd. If $e_2 = 1$, the numerator term is $1 + 3(1) = 4 = 2^2$, which gives two factors of 2. If $e_2 \geq 3$, then the product over $i = 1, 2$ is $\leq \frac{5}{9} < 1$, so at least some other i has $\frac{1+e_i p_i}{p_i} > 1$, which requires that $e_i = 1$, but this introduces more factors of 2 in the numerator. If $e_2 = 2$, then the product over $i = 1, 2$ is $\frac{7}{6}$, so $7 \mid n$. Let $p_3 = 7$. Then $e_3 \geq 2$, since $e_3 = 1$ would give excess factors of 2. But then $\frac{1+e_3 p_3}{p_3^{e_3}} \leq \frac{15}{49}$, so the product over $i = 1, 2, 3$ is $\leq \frac{5}{14}$, so for another i we must have $e_i = 1$, but again this introduces excess factors of 2. So we have ruled out the case $e_1 = 1$.
- **Case $e_1 = 2$:** The numerator has $1 + 2(2) = 5$, forcing $5 \mid n$. Let $p_2 = 5$. If $e_2 = 1$, $1 + 5(1) = 6 = 2 \cdot 3$. This gives one factor of 2 and forces $3 \mid n$. Let $p_3 = 3$. If $e_3 \geq 3$, then the product over $i = 1, 2, 3$ is $\leq \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{7}{9} = \frac{10}{27}$. Hence, the contribution from the remaining primes $i \geq 4$ must be at most $\frac{27}{10}$. However, among all those, we only have one factor of 2 left to use. To compensate and bring the total product up to 1, we must introduce factors where $\frac{1+e_i p_i}{p_i^{e_i}} > 1$, which only occurs when $e_i = 1$, which has $\frac{1+p_i}{p_i} \leq \frac{6}{5} < \frac{27}{10}$. Hence this case is not possible.

Similarly, for the case when $e_2 \geq 2$, then $\frac{1+5e_2}{5^{e_2}} \leq \frac{11}{25}$, so the remaining primes must contribute at least $\frac{20}{11}$. However, the total 2-adic valuation available is 2, so the remaining contribution is $(6/5)^2 = 36/25 < 20/11$. This rules out $e_1 = 2$.

- **Case $e_1 = 3$:** The numerator has $1 + 2(3) = 7$, forcing $7 \mid n$. Let $p_2 = 7$. To get exactly three factors of 2, we check $e_2 = 1$. This gives $1 + 7(1) = 8 = 2^3$. This perfectly matches our required 2^3 without introducing any new prime factors! Checking this subset, we get $(1 + 2(3))(1 + 7(1)) = 7 \cdot 8 = 56$, and $2^3 \cdot 7^1 = 56$. Thus, $n = 56$ is a solution.

For $e_1 \geq 4$, the ratio $\frac{1+2e_1}{2^{e_1}} \leq \frac{9}{16}$. To compensate and bring the total product up to 1, we must introduce factors where $\frac{1+e_i p_i}{p_i^{e_i}} > 1$, which only occurs when $e_i = 1$. However, adding primes with an exponent of 1 introduces terms of the form $1 + p_i$, which are even. The 2-adic valuation of these combined terms grows much faster than e_1, forcing contradictions in the parity and matching of prime factors. Thus, no solutions exist for $e_1 \geq 4$. The only integers satisfying the condition are 1 and 56. The sum of all such integers is $1 + 56 = \boxed{57}$.

8. Find the number of integer solutions (x, y), where $0 \leq x, y \leq 46$ and $y \neq 0$, to the equation

$$y^2 \equiv (x^2 - 4x + 1)(x^3 - 6x^2 + 16) \pmod{47}.$$

Proposed by Kaivalya Kulkarni

Solution: Work in $\mathbb{F}_{47}$ (that is, modulo 47). Substitute $x = x + 2$ (the number of solutions remains the same). The given magically factors as

$$\begin{aligned} y^2 &= \left[(x+2)^2 - 4(x+2) + 1\right] \cdot \left[(x+2)^3 - 6(x+2)^2 + 16\right] \\ &= x\left(x^2 - 3\right)\left(x^2 - 12\right). \end{aligned}$$

We now discard the $y = 0$ cases. If $y = 0$ we have:

- $x = 0$ or
- $x^2 - 3 = 0$, giving $x = \pm 12$,
- $x^2 = 12$, giving $x = \pm 23$.

Alternatively, one can show that 3 is a quadratic residue modulo 47. It follows that $12 = 2^2 \cdot 3$ is also a quadratic residue, so we discard 5 solutions in all. Now note that for each of the $47 - 5 = 42$ remaining x, exactly one of x and $-x$ gives a solution (because $x \to -x$ changes the sign of the right-hand side and -1 is not a quadratic residue modulo 47). Furthermore, for each good x, there are exactly two values of y. This gives a final answer of $2 \cdot (42/2) = \boxed{42}$.