P U M ∴ C

Number Theory A

1. Find the number of positive integers $n < 2018$ such that $25^n + 9^n$ is divisible by 13.

2. For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of n. For example, $f(1) = 0$, $f(2) = 1$, and $f(4) = f(6) = 2$. Let $g(n)$ be the number of positive integers $k \leq n$ such that $f(k) \geq f(j)$ for all $j \leq n$. Find $g(1) + g(2) + \ldots + g(100)$.

3. What is the largest integer $n < 2018$ such that for all integers $b > 1$, n has at least as many 1's in its base-4 representation as it has in its base-b representation?

4. Let n be a positive integer. Let $f(n)$ be the probability that, if divisors a, b, c of n are selected uniformly at random with replacement, then $\gcd(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(a, \gcd(b, c))$. Let $s(n)$ be the sum of the distinct prime divisors of n. If $f(n) < \frac{1}{2018}$, compute the smallest possible value of $s(n)$.

5. Find the remainder when
$$\prod_{i=1}^{1903} (2^i + 5)$$
is divided by 1000.

6. Find the remainder of
$$\prod_{n=2}^{99} (1 - n^2 + n^4)(1 - 2n^2 + n^4)$$
when divided by 101.

7. Find the smallest positive integer G such that there exist distinct positive integers a, b, c with the following properties:
   - $\gcd(a, b, c) = G$.
   - $\operatorname{lcm}(a, b) = \operatorname{lcm}(a, c) = \operatorname{lcm}(b, c)$.
   - $\frac{1}{a} + \frac{1}{b}$, $\frac{1}{a} + \frac{1}{c}$, and $\frac{1}{b} + \frac{1}{c}$ are reciprocals of integers.
   - $\gcd(a, b) + \gcd(a, c) + \gcd(b, c) = 16G$.

8. Let p be a prime. Let $f(x)$ be the number of ordered pairs (a, b) of positive integers less than p, such that $a^b \equiv x \pmod{p}$. Suppose that there do not exist positive integers x and y, both less than p, such that $f(x) = 2f(y)$, and that the maximum value of f is greater than 2018. Find the smallest possible value of p.