PUM.C

Number Theory B

1. Find the largest prime factor of 8001.

2. Find the number of positive integers $n < 2018$ such that $25^n + 9^n$ is divisible by 13.

3. For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of n. For example, $f(1) = 0$, $f(2) = 1$, and $f(4) = f(6) = 2$. Let $g(n)$ be the number of positive integers $k \leq n$ such that $f(k) \geq f(j)$ for all $j \leq n$. Find $g(1) + g(2) + \ldots + g(100)$.

4. You come across an ancient mathematical manuscript. It reads, "To find out whether a number is divisible by seventeen, take the number formed by the last two digits of the number, subtract the number formed by the third- and fourth-to-last digits of the number, add the number formed by the fifth- and sixth-to-last digits of the number, and so on. The resulting number is divisible by seventeen if and only if the original number is divisible by seventeen." What is the sum of the five smallest bases that the ancient culture might have been using? (Note that "seventeen" is the number represented by 17 in base 10, not 17 in the base that the ancient culture was using. Express your answer in base 10.)

5. What is the largest integer $n < 2018$ such that for all integers $b > 1$, n has at least as many 1's in its base-4 representation as it has in its base-b representation?

6. Let n be a positive integer. Let $f(n)$ be the probability that, if divisors a, b, c of n are selected uniformly at random with replacement, then $\gcd(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(a, \gcd(b, c))$. Let $s(n)$ be the sum of the distinct prime divisors of n. If $f(n) < \frac{1}{2018}$, compute the smallest possible value of $s(n)$.

7. Find the remainder of
$$\prod_{n=2}^{99}(1 - n^2 + n^4)(1 - 2n^2 + n^4)$$
when divided by 101.

8. Find the smallest positive integer G such that there exist distinct positive integers a, b, c with the following properties:
   - $\gcd(a, b, c) = G$.
   - $\operatorname{lcm}(a, b) = \operatorname{lcm}(a, c) = \operatorname{lcm}(b, c)$.
   - $\frac{1}{a} + \frac{1}{b}$, $\frac{1}{a} + \frac{1}{c}$, and $\frac{1}{b} + \frac{1}{c}$ are reciprocals of integers.
   - $\gcd(a, b) + \gcd(a, c) + \gcd(b, c) = 16G$.