Number Theory A

1. [**3**] What is the 22nd positive integer n such that 22^n ends in a 2? (when written in base 10).

2. [**3**] What is the sum of all positive integers n such that $\operatorname{lcm}(2n, n^2) = 14n - 24$?

3. [**4**] What is the largest positive integer n less than $10,000$ such that in base 4, n and $3n$ have the same number of digits; in base 8, n and $7n$ have the same number of digits; and in base 16, n and $15n$ have the same number of digits? Express your answer in base 10.

4. [**4**] What is the smallest positive integer n such that $20 \equiv n^{15} \pmod{29}$?

5. [**5**] Given that there are 24 primes between 3 and 100, inclusive, what is the number of ordered pairs (p, a) with p prime, $3 \le p < 100$, and $1 \le a < p$ such that the sum

$$a + a^2 + a^3 + \cdots + a^{(p-2)!}$$

is not divisible by p?

6. [**6**] For a positive integer n, let $d(n)$ be the number of positive divisors of n. What is the smallest positive integer n such that

$$\sum_{t|n} d(t)^3$$

is divisible by 35?

7. [**7**] Given a positive integer k, let $f(k)$ be the sum of the k-th powers of the primitive roots of 73. For how many positive integers $k < 2015$ is $f(k)$ divisible by 73?

Note: A primitive root r of a prime p is an integer $1 \le r < p$ such that the smallest positive integer k such that $r^k \equiv 1 \pmod{p}$ is $k = p - 1$.

8. [**8**] Let $n = 2^{2015} - 1$. For any integer $1 \le x < n$, let

$$f_n(x) = \sum_p s_p(n - x) + s_p(x) - s_p(n),$$

where $s_q(k)$ denotes the sum of the digits of k when written in base q and the summation is over all primes p. Let N be the number of values of x such that $4 \mid f_n(x)$. What is the remainder when N is divided by 1000?