Algebra A

1. Let $a \diamond b = ab - 4(a+b) + 20$. Evaluate
$$1 \diamond (2 \diamond (3 \diamond (\cdots (99 \diamond 100) \cdots))).$$

2. Suppose $z^3 = 2 + 2i$, where $i = \sqrt{-1}$. The product of all possible values of the real part of z can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

3. Let Γ be the maximum possible value of $a + 3b + 9c$ among all triples (a, b, c) of positive real numbers such that
$$\log_{30}(a+b+c) = \log_8(3a) = \log_{27}(3b) = \log_{125}(3c).$$
If $\Gamma = \frac{p}{q}$ where p and q are relatively prime positive integers, then find $p + q$.

4. Let $a_1, a_2, \ldots$ be a sequence of **positive** real numbers such that $a_n = 11a_{n-1} - n$ for all $n > 1$. The smallest possible value of a_1 can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$?

5. Let $f_0(x) = x$, and for each $n \geq 0$, let $f_{n+1}(x) = f_n(x^2(3-2x))$. Find the smallest real number that is at least as large as
$$\sum_{n=0}^{2017} f_n(a) + \sum_{n=0}^{2017} f_n(1-a)$$
for all $a \in [0, 1]$.

6. Together, Kenneth and Ellen pick a real number a. Kenneth subtracts a from every thousandth root of unity (that is, the thousand complex numbers ω for which $\omega^{1000} = 1$) then inverts each, then sums the results. Ellen inverts every thousandth root of unity, then subtracts a from each, and then sums the results. They are surprised to find that they actually got the same answer! How many possible values of a are there?

7. The sum
$$\sum_{k=0}^{\infty} \frac{2^k}{5^{2^k} + 1}$$
can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

8. Let
$$\frac{p}{q} = \cfrac{2017}{2 - \cfrac{1}{3 - \cfrac{2}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{2}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{2}{2 - \cfrac{1}{2 - \cdots}}}}}}}}}}}$$
where p and q are relatively prime positive integers. Find $p + q$.