P U M $\therefore$ C

Algebra B

1. If x is a positive number such that $x^{x^{x^x}} = ((x^x)^x)^x$, find $(x^x)^{(x^x)}$.

2. Find the coefficient of x^7y^6 in $(xy + x + 3y + 3)^8$.

3. Let $a \diamond b = ab - 4(a + b) + 20$. Evaluate
$$1 \diamond (2 \diamond (3 \diamond (\cdots (99 \diamond 100) \cdots))).$$

4. Suppose $z^3 = 2 + 2i$, where $i = \sqrt{-1}$. The product of all possible values of the real part of z can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

5. Let Γ be the maximum possible value of $a + 3b + 9c$ among all triples (a, b, c) of positive real numbers such that
$$\log_{30}(a + b + c) = \log_8(3a) = \log_{27}(3b) = \log_{125}(3c).$$
If $\Gamma = \frac{p}{q}$ where p and q are relatively prime positive integers, then find $p + q$.

6. Let the sequence $a_1, a_2, \ldots$ be defined recursively as follows: $a_n = 11a_{n-1} - n$. If all terms of the sequence are positive, the smallest possible value of a_1 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

7. Let $f_0(x) = x$, and for each $n \geq 0$, let $f_{n+1}(x) = f_n(x^2(3-2x))$. Find the smallest real number that is at least as large as
$$\sum_{n=0}^{2017} f_n(a) + \sum_{n=0}^{2017} f_n(1 - a)$$
for all $a \in [0, 1]$.

8. Together, Kenneth and Ellen pick a real number a. Kenneth subtracts a from every thousandth root of unity (that is, the thousand complex numbers ω for which $\omega^{1000} = 1$) then inverts each, then sums the results. Ellen inverts every thousandth root of unity, then subtracts a from each, and then sums the results. They are surprised to find that they actually got the same answer! How many possible values of a are there?