



Combinatorics A

1. There are 2017 turtles in a room. Every second, two turtles are chosen uniformly at random and combined to form one super-turtle. (Super-turtles are still turtles.) The probability that after 2015 seconds (meaning when there are only two turtles remaining) there is some turtle that has never been combined with another turtle can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.
2. Call a number *unremarkable* if, when written in base 10, no two adjacent digits are equal. For example, 123 is unremarkable, but 122 is not. Find the sum of all unremarkable 3-digit numbers. (Note that 012 and 007 are not 3-digit numbers.)
3. There is a box containing 100 balls, each of which is either orange or black. The box is equally likely to contain any number of black balls between 0 and 100, inclusive. A random black ball rolls out of the box. The probability that the next ball to roll out of the box is also black can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.
4. The four faces of a tetrahedral die are labelled 0, 1, 2, and 3, and the die has the property that, when it is rolled, the die promptly vanishes, and a number of copies of itself appear equal to the number on the face the die landed on. For example, if it lands on the face labelled 0, it disappears. If it lands on the face labelled 1, nothing happens. If it lands on the face labelled 2 or 3, there will then be 2 or 3 copies of the die, respectively (including the original). Suppose the die and all its copies are continually rolled, and let p be the probability that they will all eventually disappear. Find $\left\lfloor \frac{10}{p} \right\rfloor$.
5. *Greedy Algorithms, Inc.* offers the following string-processing service. Each string submitted for processing has a starting price of 1 dollar. The customer can then ask for any two adjacent characters in the string to be swapped. This may be done an arbitrary number of times, but each swap doubles the price for processing the string. Then the company returns the modified string and charges the customer 2^S dollars, where S is the number of swaps executed. If a customer submits **all** permutations of the string PUMAC for processing and wants all of the strings to be identical after processing, what is the lowest price, in dollars, she could pay?
6. Jackson begins at 1 on the number line. At each step, he remains in place with probability 85% and increases his position on the number line by 1 with probability 15%. Let d_n be his position on the number line after n steps, and let E_n be the expected value of $\frac{1}{d_n}$. Find the least n such that $\frac{1}{E_n} > 2017$.
7. If N is the number of ways to place 16 *jumping rooks* on an 8×8 chessboard such that each rook attacks exactly two other rooks, find the remainder when N is divided by 1000. (A jumping rook is said to *attack* a square if the square is in the same row or in the same column as the rook.)
8. Bob chooses a 4-digit binary string uniformly at random, and examines an infinite sequence of uniformly and independently random binary bits. If N is the least number of bits Bob has to examine in order to find his chosen string, then find the expected value of N . For example, if Bob's string is 0000 and the stream of bits begins 101000001..., then $N = 7$.