P U M ∴ C

Individual Finals A

1. Let $\mathcal{X} = \{1, 2, \ldots, 2017\}$. Let k be a positive integer. Given any r such that $1 \leq r \leq k$, there exist k subsets of $\mathcal{X}$ such that the union of any r of them is equal to $\mathcal{X}$, but the union of any fewer than r of them is not equal to $\mathcal{X}$. Find, with proof, the greatest possible value for k.

2. Let $a_1, a_2, a_3, \ldots$ be a monotonically decreasing sequence of positive real numbers converging to zero. Suppose that $\sum_{i=1}^{\infty} \frac{a_i}{i}$ diverges. Show that $\sum_{i=1}^{\infty} a_i^{2^{2017}}$ also diverges. You may assume in your proof that $\sum_{i=1}^{\infty} \frac{1}{i^p}$ converges for all real numbers $p > 1$. (A sum $\sum_{i=1}^{\infty} b_i$ of positive real numbers b_i *diverges* if for each real number N there is a positive integer k such that $b_1 + b_2 + \cdots + b_k > N$.)

3. Triangle ABC has incenter I. The line through I perpendicular to AI meets the circumcircle of ABC at points P and Q, where P and B are on the same side of AI. Let X be the point such that $PX \parallel CI$ and $QX \parallel BI$. Show that PB, QC, and IX intersect at a common point.