



Individual Finals B

1. The Frontier Lands have 50 towns, some pairs of which are directly connected by Morton's railroad tracks (which are bidirectional and may pass over each other), and it is possible to travel from any town to any other town via these tracks, possibly stopping at other towns on the way. Morton decides that he wants some tracks destroyed so that each town is directly connected to an odd number of other towns. (After Morton destroys the tracks, it might no longer be possible to travel from any town to any other town.) Prove that this is possible.
2. Let $a_1(x)$, $a_2(x)$, and $a_3(x)$ be three polynomials with integer coefficients such that every polynomial with integer coefficients can be written in the form

$$p_1(x)a_1(x) + p_2(x)a_2(x) + p_3(x)a_3(x)$$

for some polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$ with integer coefficients. Show that every polynomial is of the form

$$p_1(x)a_1(x)^2 + p_2(x)a_2(x)^2 + p_3(x)a_3(x)^2$$

for some polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$ with integer coefficients.

3. Let $\mathcal{X} = \{1, 2, \dots, 2017\}$. Let k be a positive integer. Given any r such that $1 \leq r \leq k$, there exist k subsets of $\mathcal{X}$ such that the union of any r of them is equal to $\mathcal{X}$, but the union of any fewer than r of them is not equal to $\mathcal{X}$. Find, with proof, the greatest possible value for k .