P U M $\therefore$ C

Number Theory B

1. Let a_n be the least positive integer the sum of whose digits is n. Find $a_1 + a_2 + a_3 + \cdots + a_{20}$.

2. Let $S = \{1, 22, 333, \ldots, 999999999\}$. For how many pairs of integers (a, b) where $a, b \in S$ and $a < b$ is it the case that a divides b?

3. Shaq sees the numbers 1 through 2017 written on a chalkboard. He repeatedly chooses three numbers, erases them, and writes one plus their median. (For instance, if he erased $-2, -1, 0$ he would replace them with 0.) If M is the maximum possible final value remaining on the board, and if m is the minimum, compute $M - m$.

4. The sequence of positive integers $a_1, a_2, \ldots$ has the property that $\gcd(a_m, a_n) > 1$ if and only if $|m - n| = 1$. Find the sum of the four smallest possible values of a_2.

5. Define the *bigness* of a rectangular prism to be the sum of its volume, its surface area, and the lengths of all of its edges. Find the least integer N for which there exists a rectangular prism with integer side lengths and bigness N and another one with integer side lengths and bigness $N + 1$.

6. For any integer $n \geq 2$, let b_n be the least positive integer such that, for any integer N, m divides N whenever m divides the digit sum of N written in base b_n, for $2 \leq m \leq n$. Find the integer nearest to b_{36}/b_{25}.

7. Let $p(n) = n^4 - 6n^2 - 160$. If a_n is the least odd prime dividing $q(n) = |p(n - 30) \cdot p(n + 30)|$, find $\sum_{n=1}^{2017} a_n$. ($a_n = 3$ if $q(n) = 0$.)

8. Find the least positive integer N such that the only values of n for which $1 + N \cdot 2^n$ is prime are multiples of 12.