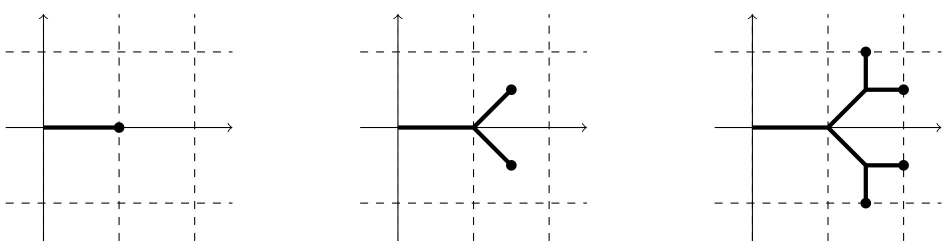
Algebra A

1. Consider polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are nonnegative integers satisfying $ab + bc + cd + ad = 20$. Find the sum of all distinct possible values of $f(1)$.

2. Let f and g be two polynomials such that $f(g(x)) = g(f(x))$. If $g(x)$ is linear but not identically equal to x, and $f(x) = x^3 + 60x^2 + 1000x + c$ for some c, find the value of c.

   Note: This problem was scrapped from the contest. It should state: "... If $g(x)$ is linear, **non-constant**, and not identically equal to x, ..."

3. T_1 consists of a single branch from 0 to 1 in the complex plane. This branch then splits into two branches at the endpoint which each form a $135°$ angle with the branch in T_1, and each branch has length $\frac{1}{\sqrt{2}}$. This process is repeated, so that from each terminal branch in T_n, we form two more branches at angles $135°$ with $\frac{1}{\sqrt{2}}$ the length. Let L_n be the collection of the 2^{n-1} endpoints of the tree T_n. If multiple terminal branches end at the same point, then that point is counted multiple times in L_n. Shown below is T_k for $k = 1, 2, 3$ with dots at the points in L_k. Find the sum of ℓ^2 over all points ℓ in L_{10}.



(a) T_1 and L_1 (b) T_2 and L_2 (c) T_3 and L_3

4. Compute the number of solutions to $1 + \cos(\theta) + \cos(2\theta) + \cos(3\theta) + ... + \cos(2024\theta) = \frac{1}{2}$ for $\theta \in [0, 2\pi]$.

5. Real numbers a, b, c satisfy $\frac{1}{ab} = b + 2c$, $\frac{1}{bc} = 2c + 3a$, $\frac{1}{ca} = 3a + b$. Then $(a + b + c)^3$ can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

6. Let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by the recurrence relation $a_{n+3} = 2a_{n+2} - 23a_{n+1} + 3a_n$ for all $n \geq 0$, with initial conditions $a_0 = 20$, $a_1 = 0$, and $a_2 = 23$. Let $b_n = a_n^3$ for all $n \geq 0$. Then there exists a unique positive integer k and constants $c_0, \ldots, c_{k-1}$ with $c_0 \neq 0$ and $c_{k-1} \neq 0$ such that for all sufficiently large n, we have the recurrence relation $b_{n+k} = \sum_{t=0}^{k-1} c_t b_{n+t}$. Find $k + \sqrt{|c_{k-1}|} + \sqrt{|c_0|}$.

7. Let $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Then

$$S = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{F_n}\right) \arctan\left(\frac{1}{F_{n+1}}\right)$$

   Find $\lfloor 80S \rfloor$.

   (Hint: it may be useful to note that $\arctan\left(\frac{1}{1}\right) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$.)

8. Let $[n]$ denote the set of integers $0, 1, \ldots, n-1$. Let $\omega_n = e^{2\pi i/n}$. Let

$$f(n) = \prod_{\substack{i \in [n] \\ \gcd(i,n)=1}} \prod_{\substack{j \in [n] \\ \gcd(j,n) \neq 1}} (\omega_n^i - \omega_n^j).$$

Then, $f(2024) = 2^{e_1} \cdot 11^{e_2} \cdot 23^{e_3}$ for positive integers e_1, e_2, e_3. Find $e_1 + e_2 + e_3$.

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8