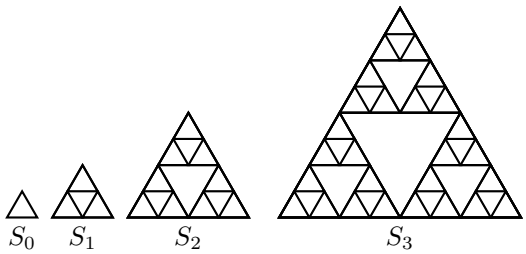
P U M ∴ C

Combinatorics A

1. Consider the Sierpinski triangle iterations drawn below. S_0 is a single triangle, and S_{n+1} consists of three copies of S_n. Let a *maximal line segment* be line segment in the drawing of S_k which cannot be extended any further while remaining in S_k. For example, S_0 has three maximal line segments and S_1 has 6 maximal line segments. How many maximal line segments are there in S_5?



2. Arnie draws 20 real numbers independently and uniformly at random from the interval $[0, 1]$. Given that the largest number that Arnie draws equals $\frac{19}{20}$, the expected value of the average of the 20 numbers can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

3. Joseph chooses a permutation of the numbers $1, 2, 3, 4, 5, 6$ uniformly at random. Then, he goes through his permutation, and deletes the numbers which are not the maximum among each of the preceding numbers. For example, if he chooses the permutation $3, 2, 4, 5, 1, 6$, then he deletes 2 and 1, leaving him with $3, 4, 5, 6$. The expected number of numbers remaining can be expressed as m/n for relatively prime positive integers m and n. Find $m + n$.

4. Michael and Steven are playing the card game War with a deck of 4 cards numbered 1 through 4. The deck is shuffled randomly and Michael gets a stack of 1 card and Steven gets a stack of 3 cards. In each round, the players reveal the top card from their stack, and the player whose card was higher collects both cards to the bottom of their stack in random order. A player wins when they get all four cards in their hand. The probability that Michael wins after exactly 5 rounds is $\frac{m}{n}$ for coprime positive integers m and n. Find $m + n$.

5. It is election year in PUMACland, and for the presidential election there are 27 people voting for either Vraj Patel or Vedant Shah. Each voter selects a candidate uniformly at random, and their ballots are labeled 1 through 27.

   The election takes place as a series of rounds. In each round, the surviving ballots are sorted by label and separated into consecutive groups of three. From each group, the person with the most votes wins, and exactly one of the ballots bearing the winner's name is allowed to proceed to the next round. This procedure continues until a single ballot remains, and the person whose name is on the ballot wins.

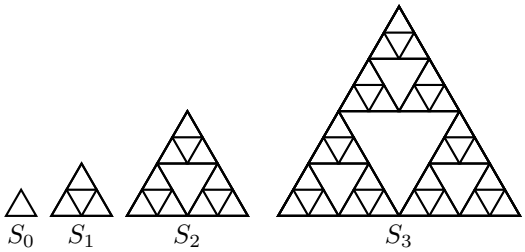
   Alice, Bob, and Carol submitted ballots numbered 1, 15, and 27, respectively. Suppose that Alice, Bob, and Carol had all flipped their votes. If the probability that the outcome of the election would have changed is $\frac{a}{b}$ for relatively prime positive integers a, b, find $a + b$.

6. Ezzie is walking around the perimeter of a regular hexagon. Each vertex of the hexagon has an instruction telling him to move clockwise or counterclockwise around the hexagon. However, when he leaves a vertex the instruction switches from clockwise to counterclockwise on that vertex, or vice versa. We say that a configuration C of Ezzie's position and the instructions

on the vertices is *irrepeatable* if, when starting from configuration C, configuration C only appears finitely many more times. Find the number of irrepeatable configurations.

7. Let $\omega = e^{2\pi i/20}$ and let S be the set $\{1, \omega, \ldots, \omega^{19}\}$. How many subsets of S sum to 0? Include both S itself and the empty set in your count.

8. Let the Sierpinski triangle graph S_n be defined as follows. S_0 consists of three vertices and three edges in a triangle. S_1 consists of 6 vertices and 9 edges. To make S_{n+1}, we take three copies of S_n and merge vertices at the corners, where the bottom-left corner of the top copy merges with the top corner of the bottom-left copy, etc. Then the number of cycles on S_4, which visit each vertex exactly once and traverse each edge at most once, can be expressed as $p_1^{e_1} p_2^{e_2}$ for some primes p_1, p_2 and positive integers e_1, e_2. Find $p_1 + p_2 + e_1 + e_2$.



Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8