P U M . : . C

Number Theory B

1. Let $f(n)$ be the sum of the factors of $2^n \cdot 31$. Find $\sum_{n=0}^{4} f(n)$.

2. Find the remainder when
$$\sum_{x=1}^{2024}\sum_{y=1}^{2024}(xy)$$
is divided by 31.

3. A quadratic polynomial $f(x) = Ax^2 + Bx + C$ is *small* if A, B, C are single-digit positive integers. It is *full* if there are only finitely many positive integers that cannot be expressed as $f(x) + 3y$ for some positive integers x and y. Find the number of quadratic polynomials that are both small and full.

4. A quadratic polynomial with positive integer coefficients and rational roots can be written as $196x^2 + Bx + 135$ for some integer B. What is the sum of all possible values of B such that $\gcd(B, 196 \cdot 135) = 1$?

5. Let σ be a permutation of the set $S \coloneqq \{1, 2, \ldots, 100\}$, such that $\sigma(a + b) \equiv \sigma(a) + \sigma(b) \pmod{100}$ whenever $a, b, a + b \in S$. Denote by $f(s)$ the sum of the distinct values $\sigma(s)$ can take over all possible σs satisfying the given condition. What is the nonnegative difference between the maximum and minimum value f takes on when ranging over all $s \in S$?

6. Let $r(m)$ be the number of positive integers a less than or equal to m where $\gcd(a, m)$ is prime. Find the sum of all positive integers $m < 300$ such that $r(m) = \varphi(m)$, where $\varphi(m)$ denotes the number of positive integers a less than m where $\gcd(a, m) = 1$.

7. Call a positive integer *nice* if the sum of its even proper divisors is larger than the sum of its odd proper divisors. What is the smallest nice number that is congruent to 2 mod 4?

8. Let Pascal's triangle be constructed where each $\binom{n}{i}$ is written inside its own cell in row n. Colby colors the cells red for $1 \le n \le 63$ when $\binom{n}{i}$ is divisible by 4. How many cells does he color red?

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8