Team Round

The Team Round consists of 15 questions. Your team has **50 minutes** to complete the Team Round. Each problem is worth 5 points.

Let s be the number of correct answers your team submits on this team round as an integer between 0 and 15. Let $c = -0.75 + 0.01 \cdot (s+1)i$. Let $f_a(z) = z^2 + a$. Let N be the least positive integer n for which $|f_c^n(0)| > 2024$. (You may recognize that $|f_a^n(0)|$ remaining bounded is the condition for a to be in the Mandelbrot set. The particular c we have chosen in this problem is outside the Mandelbrot set, nestled in between the main cardioid and circle.) Submit an integer guess M for N. The number of bonus points you will receive is

$$\left\lfloor 8\frac{M^2 e^{(2-\frac{2M}{N})}}{N^2} \right\rfloor.$$

We have constructed this scoring function so that it is maximized when $M = N$ and minimized when $M = 0$ or $M \to \infty$.

1. Justin chooses a number n uniformly at random from the set of integers between 90 and 99, inclusive. He then chooses a positive divisor d of n uniformly at random. Justin notices that d and n/d are relatively prime. If the probability that $n = 90$ can be expressed as a/b for relatively prime positive integers, find $a+b$.

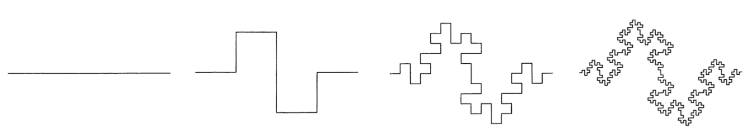
2. Let real number sequences a_k and x_k be defined for $1 \le k \le 7$ and suppose that $a_1 = 1$ and $a_{k+1} = a_k + x_k$ for $1 \le k \le 7$. Let x_k be chosen such that the quantity $S = \sum_{k=1}^{7}(a_k^2 + x_k^2)$ is minimized. Then $S = \frac{m}{n}$ for coprime positive integers m and n. Find $m+n$.

3. Let $f(x) = x^2 - 3x + 1$, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the 4 roots of $f(f(x)) = x$. Evaluate $\lfloor 10\alpha_1 \rfloor + \lfloor 10\alpha_2 \rfloor + \lfloor 10\alpha_3 \rfloor + \lfloor 10\alpha_4 \rfloor$

4. Consider the 100×100 grid of points with integer coordinates $S = \{(x,y) \in \mathbb{Z}^2 | 1 \le x \le 100, 1 \le y \le 100\}$. A set C is formed by selecting each point $p \in S$ with probability $\frac{1}{2}$ uniformly at random. The *expansion* of C is defined as the set of points $q \in S$ such that $\min_{p \in C} d(q,p) \le 1$, where $d(q,p)$ denotes the Euclidean distance between q, p. If the expected size of the expansion of C can be written as $\frac{A}{B}$ for relatively prime positive integers A, B, find $A+B$.

5. The *Minkowski sausage* is constructed as follows. M_0 is the line segment from $(0,0)$ to $(1,0)$. M_{i+1} is constructed by replacing each segment in M_i with eight segments, each of length $1/4^{i+1}$ (see figure below, where we have provided M_0 through M_3).

   Let M_∞ be the limiting shape of $M_0, M_1, \ldots$. The area of the smallest convex polygon which encloses M_∞ can be written as $\frac{a}{b}$ for relatively prime positive integers a and b. Find $a+b$.



6. Ben has a square of side length 2. He wants to put a circle and equilateral triangle inside the square such that the circle and equilateral triangle do not overlap. The maximum possible sum of the areas of the circle and triangle is $\frac{a\pi + b\sqrt{c} + d\sqrt{e}}{f}$, where a, c, e, f are positive integers, b and d are integers, c and e are square-free, and $\gcd(a,b,d,f) = 1$. Find $a+b+c+d+e+f$.

7. Consider a regular 24-gon $\mathcal{P}$. A quadrilateral is said to be inscribed in $\mathcal{P}$ if its vertices are among those of $\mathcal{P}$. We consider two inscribed quadrilaterals equivalent if one can be obtained from the other via a rotation about the center of $\mathcal{P}$. How many distinct (i.e. not equivalent) quadrilaterals can be inscribed in $\mathcal{P}$?

8. Let $\triangle ABC$ be a triangle. Let points D and E be on segment BC in the order B, D, E, C such that $\angle BAD = \angle DAE = \angle EAC$. Suppose also that $BD = F_{2024}, DE = F_{2025}, EC = F_{2027}$, where F_k is the kth Fibonacci number where $F_1 = F_2 = 1$. To the nearest degree, $\angle BAC$ is n°. Find n.

9. Define a sequence called the 2020-nacci sequence. It is defined as followed: If $n \leq 2020$ then $S_n = 1$, if $n > 2020$ then $S_n = \sum_{i=n-2020}^{n-1} S_i$. Find the last 2 digits of S_{4040}.

10. Suppose that A is a set of real numbers between 3 and 2024 inclusive such that for any $x, y \in A$ with $x \neq y$, we have $|x - y| > \frac{xy}{2+xy}$. What is the largest possible size of A?

11. Austen has a regular icosahedron (20-sided polyhedron with all triangular faces). He randomly chooses 3 distinct points among the vertices and constructs the circle through these three points. The expected value of the total number of the icosahedron's vertices that lie on this circle can be written as m/n for relatively prime positive integers m and n. Find $m + n$.

12. Find the number of positive integers $10 \leq n \leq 99$ with last digit at most 5 such that the last two digits of n^n are the same as n.

13. Consider the square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. The line segments from $(t, 0)$ to $(0, 1 - t)$ are drawn for $0 \leq t \leq 1$. The set of points inside the square but not on one of these line segments has area $\frac{m}{n}$ for coprime positive integers m and n. Find $m + n$.

14. What is the largest value for m for which I can find nonnegative integers $a_1, a_2, \ldots, a_m < 2024$ such that for all indices $i > j$, 17 divides $\binom{a_i}{a_j}$?

    Note: This should say "... nonnegative integers $a_1 < a_2 < \cdots < a_m < 2024$ ..."

15. There are 10 teams, named T_1 through T_{10}, participating in a draft in which there are 20 players available, named P_1 through P_{20}. Suppose each team independent of the others has uniform random preference on the 20 players. Team T_1 will draft their favorite player, and then each subsequent team $T_2, \ldots, T_{10}$ draft their favorite player among the ones not already drafted. Each team drafts exactly one player. Given that P_1 is among the 10 favorite players for each team, the probability that P_1 is drafted can be written as $\frac{m}{n}$ where m and n are coprime positive integers. Find $m + n$.

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8

Q9	Q10	Q11	Q12	Q13	Q14	Q15	M