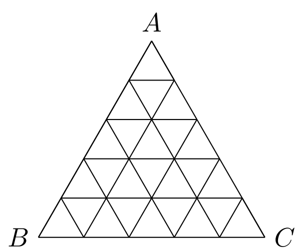
2025 Combinatorics A*

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. A string of 0s, 1s, and 2s is called *nice* if it doesn't contain "01" or "12", doesn't begin with a "2", and doesn't end with a "0". For example, there are 5 nice strings of length 3, namely "002", "021", "022", "102", and "111". Find the number of nice strings of length 7.

2. You find yourself standing at vertex A in the triangle grid shown below. Two monsters are standing at vertices B and C, respectively. Each minute, you move along an edge that takes you closer to $\overline{BC}$, choosing an edge randomly out of the two possible edges. In the same way, the monster starting at B moves towards $\overline{AC}$, and the monster starting at C moves towards $\overline{AB}$. If you ever encounter a monster on a vertex not on $\overline{BC}$, you die. Otherwise, if you reach $\overline{BC}$ without dying, you survive (even if you encounter a monster at some vertex on $\overline{BC}$). Compute the probability that you survive.



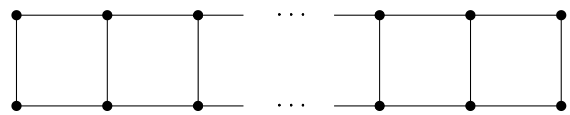
3. A stone is placed on a square of a 5×5 grid. Alice and Bob take turns moving the stone to an orthogonally adjacent square (i.e. up/down/left/right). Alice moves first. Once a square has been visited, it cannot be visited again (and they cannot visit the starting square again). A player loses if they cannot make a legal move. For how many starting squares does Alice have a winning strategy?

4. Zongshu is selling socks in packs of 2025. Each sock in each pack is equally likely to be red, green, or yellow. You buy (a positive number of) packs of socks until you can split all of the socks into pairs of the same color. Find the expected number of packs you buy.

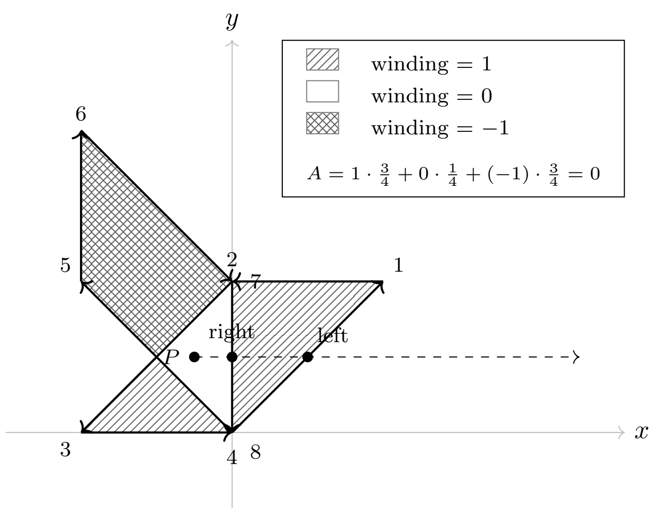
5. Alice wants to send Bob the integer 120, represented as a binary string of length 7. However, her computer is being bombarded by cosmic rays, which can flip 00 to 11 and vice versa an arbitrary amount of times! Bob receives a possibly altered binary string, and writes down the integer in decimal. Find the sum of the possible numbers Bob could have written down (including 120 itself).

6. Charles challenges David to a game of Fraction Fixer. In this game, David first picks two positive integers $m \le n$ that sum to 2025. Next, Charles picks integers a and b at random, so that $0 \le a < m$ and $0 \le b < n$, and computes $\frac{a}{m} + \frac{b}{n}$. If the sum is at least 1, David wins. Otherwise, Charles wins. If David chooses m and n optimally, what is the probability that David wins?

7. Consider the graph shown below with 4050 vertices and 6073 edges. Each edge is independently deleted with probability $\frac{1}{2}$. If C is the expected value of the number of connected components of the resulting graph, compute the nearest integer to $49C$.



8. Trevor the traveller starts at the origin of $\mathbb{R}^2$. He then takes a series of 8 steps that change his position by a vector in the set $\{(a, b) \in \mathbb{Z}^2 : |a|, |b| \leq 1 \text{ and } (a, b) \neq (0, 0)\}$. He takes each of the 8 possible steps exactly once, and takes the steps in uniformly random order, so that he returns to the origin. The *winding number* of each point P in $\mathbb{R}^2$, except those exactly on the path, is defined as the number of times any ray starting at P going out to infinity crosses a left-pointing portion of Trevor's path minus the number of times the ray crosses a right-pointing portion of the path, where left and right are defined relative to the ray. Let A be the area in $\mathbb{R}^2$ weighted by winding number. An example is shown below. Find the expected value of A^2.



Weighted areas for steps, in order,
$(1, 1), (-1, 0), (-1, -1), (1, 0), (-1, 1), (0, 1), (1, -1), (0, -1)$