P U M $\therefore$ C

2025 Individual Finals A

You have **90 minutes** for this examination. To get full credit, give a rigorous mathematical proof of your answer. Do not discuss or distribute these problems until **January 16th, 2026**.

1. Let λ^* be the maximum value of λ such that for all cubic polynomials $P(x) = x^3 + ax^2 + bx + c$ whose zeros are all real and nonnegative, then $P(x) \geq \lambda(x - a)^3$ for all $x \geq 0$. Prove that $\lambda^* = -\frac{1}{27}$.

2. Prove that there exists a constant $c > 0$ such that for any set of integers $a_1 > \cdots > a_n$ there exist indices $1 \leq i < j \leq n$ such that $a_i - a_j$ has $\geq 2^{c \log n / \log \log n}$ divisors.

3. Let ABC be an acute scalene triangle with orthocenter H, circumcenter O and circumcircle Ω. It is given that $OH \parallel BC$. Line OH meets side AB at U and side AC at V. Let R be on Ω such that $\angle ARH = 90^\circ$. Let the circumcircle of triangle ROH meet Ω a second time at S. Show that lines SO and RH meet on the circumcircle of triangle RUV.