PUMaC

2025 Individual Finals B

You have **90 minutes** for this examination. To get full credit, give a rigorous mathematical proof of your answer. Do not discuss or distribute these problems until **January 16th, 2026**.

1. In an atom of PUMaCium, there are $2N$ energy levels that an electron can occupy, indexed $\{1, 2, \ldots, 2N\}$. A given electron can only travel from a lower-indexed energy level to a higher-indexed one. Whenever it travels from i to j, it emits light of energy E_{ij}. Supposing that all the E_{ij} are distinct, show that the minimum number of electrons you need to use to get every possible energy of light to be emitted is N^2. You are allowed to reuse an electron to get multiple light emissions, as long as it does not go from a higher-indexed energy level to a lower one.

2. Find all integers $x, y, z \geq 1$ such that $3^x + 4^y = z^2$.

3. Let λ^* be the maximum value of λ such that for all cubic polynomials $P(x) = x^3 + ax^2 + bx + c$ whose zeros are all real and nonnegative, then $P(x) \geq \lambda(x - a)^3$ for all $x \geq 0$. Prove that $\lambda^* = -\frac{1}{27}$.