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2025 Number Theory A

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. Find the sum of all positive integers $n < 2025$ such that $\gcd(n^2 + 1, 2n + 1) > 1$.

2. How many composite odd integers $N \leq 10^4$ are there such that $\varphi(N)$ is not a multiple of 4? ($\varphi(N)$ is the number of integers k with $1 \leq k \leq N$ such that $\gcd(k, N) = 1$.)

3. Compute the sum of all primes $p < 50$ such that there exist linear multinomials P and Q with integer coefficients such that $x^2 - 2xy - y^2 \equiv P(x, y) \cdot Q(x, y) \pmod{p}$.

   A linear multinomial is an expression of the form $ax + by + c$.

4. For p prime and $k \geq 1$, let $S_k^{(p)}$ denote the number of positive integers $x < p^k$ where $x^{2^k} \equiv 1 \pmod{p^k}$. Find
$$\sum_{k=1}^{4} \sum_{\substack{p<25 \\ p \text{ prime}}} S_k^{(p)}.$$

5. Find the sum of the smallest five positive integers x such that there exists a positive integer y satisfying $x^{x+y} = y^{y-x}$.

6. Suppose that complex numbers t_j for $0 \leq j \leq 2025$ satisfy
$$\prod_{\substack{0 \leq a < 2025 \\ \gcd(a, 2025) = 1}} (x - e^{2\pi i a/2025}) = \sum_{j=0}^{2025} t_j x^j.$$

   Find the sum of all j such that $t_j \neq 0$.

7. For a positive integer n, let $\operatorname{rad}(n)$ denote the product of the unique prime divisors of n, where $\operatorname{rad}(1) = 1$. For example, $\operatorname{rad}(12) = 2 \cdot 3 = 6$. Find the sum of all integers n satisfying
$$\sum_{d \mid n} \operatorname{rad}(d) = n.$$

   (The sum is over all positive integer divisors d of n.)

8. Find the number of integer solutions (x, y), where $0 \leq x, y \leq 46$ and $y \neq 0$, to the equation
$$y^2 \equiv (x^2 - 4x + 1)(x^3 - 6x^2 + 16) \pmod{47}.$$