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2025 Number Theory B

You have **50 minutes** for this examination.
Do not discuss or distribute these problems until **January 16th, 2026**.

1. A rectangle with integer side lengths has area 2025. Find the sum of the possible perimeters of the rectangle.

2. For a prime p, let $\nu_p(n)$ denote the largest nonnegative integer k such that p^k divides n. For example, $\nu_3(9) = 2$, $\nu_2(12) = 2$, and $\nu_5(12) = 0$. Find the number of odd primes p less than 100 such that $\nu_p(2^p - 1) = \nu_p(2^{2p} - 1) - \nu_p(2)$.

3. Find the sum of all positive integers $n < 2025$ such that $\gcd(n^2 + 1, 2n + 1) > 1$.

4. How many composite odd integers $N \leq 10^4$ are there such that $\varphi(N)$ is not a multiple of 4? ($\varphi(N)$ is the number of integers k with $1 \leq k \leq N$ such that $\gcd(k, N) = 1$.)

5. Compute the sum of all primes $p < 50$ such that there exist linear multinomials P and Q with integer coefficients such that $x^2 - 2xy - y^2 \equiv P(x, y) \cdot Q(x, y) \pmod{p}$.

   A linear multinomial is an expression of the form $ax + by + c$.

6. For p prime and $k \geq 1$, let $S_k^{(p)}$ denote the number of positive integers $x < p^k$ where $x^{2^k} \equiv 1 \pmod{p^k}$. Find
$$\sum_{k=1}^{4} \sum_{\substack{p<25 \\ p \text{ prime}}} S_k^{(p)}.$$

7. Find the sum of the smallest five positive integers x such that there exists a positive integer y satisfying $x^{x+y} = y^{y-x}$.

8. Suppose that complex numbers t_j for $0 \leq j \leq 2025$ satisfy
$$\prod_{\substack{0 \leq a < 2025 \\ \gcd(a, 2025) = 1}} (x - e^{2\pi i a/2025}) = \sum_{j=0}^{2025} t_j x^j.$$

   Find the sum of all j such that $t_j \neq 0$.