P U M . . . C

2025 Team Round

The Team Round consists of 15 questions and one bonus question. Your team has **50 minutes** to complete the Team Round. Each problem is worth 5 points. Do not discuss or distribute these problems after the test until **January 16th, 2026**.

Bonus: A *Rayo-string* is a string consisting of the symbols (,), =, $\in$, $\neg$, $\wedge$, $\exists$, and x_i for any positive integer i, such that

- $x_i = x_j$ and $x_i \in x_j$ are Rayo-strings for any i, j;
- If φ and ψ are Rayo-strings, then $(\neg\varphi)$ and $(\varphi \wedge \psi)$ and $\exists x_i(\varphi)$ are Rayo-strings;
- No other strings are Rayo-strings.

For example, $\exists x_2((\neg x_1 = x_2))$ is a Rayo-string, but $\exists x_2(\neg x_1 = x_2)$ is not a Rayo-string. Given any Rayo-string, we can interpret it as a mathematical statement: x_i are sets; $x_i = x_j$ means that x_i is equal to x_j; $x_i \in x_j$ means that x_i is an element of x_j; $(\neg\varphi)$ means that φ is false; $(\varphi \wedge \psi)$ means that both φ and ψ are true; $\exists x_i(\varphi)$ means that φ is true for some set x_i.

Let N be the smallest known positive integer for which there exists a Rayo-string of length N which is equivalent to the statement $x_1 = \{\varnothing, \{\varnothing\}\}$. Submit a guess M for N to get

$$\left\lfloor 5e^{-\frac{(M-N)^2}{4N}} \right\rfloor$$

bonus points.

1. Find the sum of all primes p such that p divides $2^p + 4^p + \cdots + 510^p$.

2. A regular octahedron is a polyhedron with 6 vertices, 12 edges, and 8 triangular faces, where each vertex connects to exactly 4 others and there are 3 pairs of opposite vertices. Each vertex is assigned one of 8 colors. Adjacent vertices (connected by an edge) must have different colors, and opposite vertices must have the same color. Additionally, for each triangular face, the three vertices must use at least 3 different colors. Two colorings are considered the same if one can be obtained from the other by rotating the octahedron in 3D space. How many ways can the octahedron be colored?

3. Suppose $x = \sqrt{2 - \sqrt{2 + \sqrt{2 - x}}}$. Find x.

4. There exist positive integers a, b, c such that $a^3 + b^3 + c^3 = 3438$. What is $a + b + c$?

5. Zappy Zack wants to place rectangular chargers on a 6×5 power grid. He has (1×3) chargers and (1×1) chargers. The top leftmost outlet doesn't work, so he cannot plug any chargers into it. Furthermore, any (1×1) charger must be adjacent to a (1×3) charger (diagonals do not count). He also cannot rotate any chargers. Zack places chargers into the slots such that he maximizes the total number of chargers he places down. How many ways could he have done this?

6. Let $\triangle ABC$ be a triangle with AB $= 14$, AC $= 19$, incenter I, and incircle tangent to side $\overline{BC}$ at D. Let H' be the orthocenter of triangle $\triangle BIC$. Given that $\frac{DH'}{DI} = 10$, find the length of BC.

7. For each positive integer n, let $\zeta_{n0}, \zeta_{n1}, \ldots, \zeta_{n(n-1)}$ denote the n distinct complex solutions to $z^n = 1$. Compute

$$\lim_{n\to\infty} \max_{T\subseteq\{0,1,\ldots,n-1\}} \frac{1}{n}\left|\sum_{i\in T}\zeta_{ni}\right|.$$

8. Compute the number of ordered pairs (a, b) of positive integers satisfying $ab \le 100$ and $\gcd(a, b) + \operatorname{lcm}(a, b) = a + b$.

9. For all $(a, b) \in \mathbb{Z}^2$ (excluding the origin), Biao draws the perpendicular bisector of the line segment from $(0, 0)$ to (a, b). An ant starts at $(0,\ 0)$ and walks around on $\mathbb{R}^2$. Let S_n be the set of points that the ant can reach by crossing up to n of Biao's perpendicular bisectors. Thus S_0 is the square whose vertices are $(\pm\frac{1}{2}, \pm\frac{1}{2})$ and S_1 is the square whose vertices are $(0, 1), (1, 0), (0, -1), (-1, 0)$. Find the perimeter of S_3.

10. How many ways can the integers from 0 to 6 be arranged in a cycle such that the difference from one integer to the next is a power of 2 when taken $\pmod 7$?

11. Let $\mathcal{A}$ be a quadrilateral on a sphere, whose sides are all sections of great circles. All the sides of $\mathcal{A}$ are congruent, and all the corners of $\mathcal{A}$ are $120°$. Find the proportion of the sphere's surface area enclosed by the quadrilateral.

12. In a game, each of 6 players get a card on which is printed a nonempty subset of $\{1, 2, 3, 4, 5\}$. Each round, an integer from 1 through 5 is selected at random, and any player who has this number on their card circles it. The player who first circles all numbers on their card wins.

    It is known that every player has a nonzero chance of winning, and there is no possibility of a tie. It is also known that three of the cards are $\{1, 2, 3\}$, $\{2, 3, 4\}$, and $\{3, 4, 5\}$. If each player multiplies the numbers on their card together, what is the sum of these 6 products?

13. Consider the equations

$$\begin{aligned} x + y + z &= A, \\ x^2 + y^2 + z^2 &= B, \\ x^3 + y^3 + z^3 &= C. \end{aligned}$$

    An ordered triple $(A, B, C) \in \mathbb{R}^3$ is called "nice" if the corresponding system above has exactly three real solutions for x. For fixed $(A, B) \in \mathbb{R}^2$, the set of real C such that the system is "nice" is a finite union of finite intervals; denote the sum of their lengths as $\ell(A, B)$. Find $\sum_{A=3}^{5}\sum_{B=3}^{5}\ell(A, B)$.

14. A 24-cell is a region that can be formed by the following procedure: taking convex hull of 8 vertices formed by permuting integer coordinates $(\pm1, 0, 0, 0)$ and 16 vertices at all half-integer coordinates $(\pm1/2, \pm1/2, \pm1/2, \pm1/2)$. If we rotate and scale the 24-cell so that each coordinate has absolute value at most 1 in such a way that the 24-cell has greatest hypervolume, how many vertices (x, y, z, w) of the 24-cell satisfy $x = 1$?

15. Let x be the unique real number which satisfies

$$x = \frac{\sqrt{2}}{\sqrt{\sqrt{x+1}+\sqrt{x}} - \sqrt{\sqrt{x}+\sqrt{x-1}}}.$$

    Find $\lfloor x \rfloor$.