PUMaC 2024 Power Round:
Measures and Fractals

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Rules and Reminders

Read **ALL** of the following rules. Failure to follow proper rules may result in point deductions.

1. Your solutions should be turned in by **5pm Thursday the 21st of November, EST**. You will submit the solutions through Gradescope. The instructions describing how to log into Gradescope will be sent to the coaches. The deadline for submission is clearly visible on the Gradescope site once you enroll in the course.

   Please make sure you submit your work in time. **No late submissions will be accepted.** Please do not submit your work using email or in any other way. If you have questions about Gradescope, please post them on Piazza.

   You may either typeset the solutions in LaTeX or write them by hand. We strongly encourage you to typeset the solutions. This way, the proofs end up being more clear and the chances are you will not lose points there. Moreover, you might want to use some of the LaTeX resources listed in point 2.

   In case your solutions are handwritten, the cover sheet (the last page of this document) should be the first page of your submission. In case you typeset your solutions, please take a look at the Solutions Template we posted and make sure to make the cover sheet the first page of your submission.

   Each page should have on it the **team number** (not team name) and **problem number**. This number can be found by logging in to the coach portal and selecting the corresponding team. Solutions to problems may span multiple pages. Please put them in order when submitting your solutions.

2. you may resubmit several times before the due date, but **only your final submission will be graded** (moreover, you may not submit any work after the deadline). The last version of the power round solutions that we receive from your team will be graded. Moreover, **you must submit a PDF**. No other file type will be graded. For those new and interested in LaTeX, check out Overleaf as well as its online guides. If you do not know the specific command for a math symbol, check out Detexify or TeX.StackExchange.

3. Do not include identifying information aside from your team number in your solutions.

4. When submitting to Gradescope, assign the solutions to the correct problems on the Gradescope submission outline. Failure to do this WILL result in a point deduction, as it creates a ton of extra work for us on the back-end.

5. On any problem, you may use without proof any result that is stated earlier in the test, as well as any problem from earlier in the test, even if it is a problem that your team has not solved. These are the **only** results you may use. In particular, to solve a problem, you may not cite the subsequent ones. You may not cite parts of your proof of other problems: if you wish to use a lemma in multiple problems, please reproduce it in each one.

6. When a problem asks you to "find", "find with proof," "show," "prove," "demonstrate," or "ascertain" a result, a formal proof is expected, in which you justify each step you take, either by using a method from earlier or by *proving* that *everything* you do is correct. When a problem instead uses the word "explain," an informal explanation suffices. When a problem instead uses the word "sketch" or "draw" a clearly marked diagram is expected.

7. All problems are numbered as "Problem x.y.z" where x.y is the subsection number and z is the the number of the problem within the subsection. Each problem's point distribution can be found in the cover sheet.

8. **You may NOT use any references, such as books or electronic resources, unless otherwise specified. You may NOT use computer programs, calculators, or any other computational aids.**

9. Teams whose members use English as a foreign language may use dictionaries for reference.

10. **Communication with humans outside your team of 8 students about the content of these problems is prohibited.**

11. There are two places where you may ask questions about the test. The first is Piazza. Please ask your coach for instructions to access our Piazza forum. On Piazza, you may ask any question **so long as it does not give away any part of your solution to any problem**. If you ask a question on Piazza, all other teams will be able to see it. If such a question reveals all or part of your solution to a power round question, your team's power round score will be penalized severely. For any questions you have that might reveal part of your solution, or if you are not sure if your question is appropriate for Piazza, please email us at pumac@math.princeton.edu. We will email coaches with important clarifications that are posted on Piazza.

Introduction and Advice

In this power round, we formally investigate fractals, exploring how to make the idea of self-similarity and non-integer dimensions rigorous. We hope that this introduction provides not only an interesting perspective on something often seen as pop-math, but also an introduction to measure theory, a fascinating branch of mathematics.

A large part of the difficulty in this power round will arise from the many different perspectives that one needs to understand the material and tackle the problems. For example, understanding the geometry of fractals is essential to proving facts about them, but grasping set theory and topology is essential to making many of these intuitions formal.

Here is some further advice with regard to the Power Round:

- **Read the text of every problem!** Many important ideas are included in problems and may be referenced later on. In addition, some of the theorems you are asked to prove are useful or even necessary for later problems.

- **Make sure you understand the definitions**. A lot of the definitions are not easy to grasp; don't worry if it takes you a while to fully understand them. If you don't, then you will not be able to do the problems. Feel free to ask clarifying questions about the definitions on Piazza (or email us).

- **Don't make stuff up**: on problems that ask for proofs, you will receive more points if you demonstrate legitimate and correct intuition than if you fabricate something that *looks* rigorous just for the sake of having "rigor."

- **Check Piazza often!** Clarifications will be posted there, and if you have a question it is possible that it has already been asked and answered in a Piazza thread (and if not, you can ask it, assuming it does not reveal any part of your solution to a question). **If in doubt about whether a question is appropriate for Piazza, please email us at pumac@math.princeton.edu.**

- **Don't cheat**: as stated in Rules and Reminders, you may **NOT** use any references such as books or electronic resources. If you do cheat, you will be disqualified and banned from PUMaC, your school may be disqualified, and relevant external institutions may be notified of any misconduct.

Good luck, and have fun!

– Colby Riley

We would like to acknowledge and thank many individuals and organizations for their support; without their help, this Power Round (and the entire competition) could not exist. Please refer to the solutions of the power round for full acknowledgments and references.

Contents

Notation and Basic Concepts

- $\{x \in S : C(x)\}$: the set of all x in the set S satisfying the condition $C(x)$. *Ex.: $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}\}$ is the set of perfect squares.*
- $A \times B$: cartesian product. It is the set consisting of ordered pairs $\{(a, b) : a \in A, b \in B\}$. *Ex.:* $\mathbb{R}^n = \mathbb{R} \times ... \times \mathbb{R}$
- $A \subset B$: subset. *Ex.:* $\{1, 2\} \subset \{1, 2, 3\}$*, and* $\{1, 2\} \subset \{1, 2\}$
- $f(C)$: for a function $f : A \to B$ and subset $C \subseteq A$, the set of elements of the form $f(c)$, for $c \in C$.
- $\mathbb{N}$: the natural numbers, $\{1, 2, 3, \dots\}$.
- $\mathbb{Z}$: the integers.
- $\mathbb{Q}$: the rational numbers.
- $\mathbb{R}$: the real numbers.
- for $a < b$, $[a, b] = \{x : a \leq x \leq b\}$, $(a, b) = \{x : a < x < b\}$, and $(a, b]$, $[a, b)$ are defined similarly.
- $\sup X$: the supremum. For $X \subset \mathbb{R}$, The supremum is the smallest value s such that for all $x \in X$, $x \leq s$. Always exists, but may be $\pm\infty$. *Ex.:* $\sup\{1, 2, 3\} = 3$. $\sup \mathbb{N} = \infty$.
- $\sup_{x \in X} f(x)$: the supremum applied to the set $f(X)$.
- $\inf X$: the infimum, the largest value s such that for all $x \in X$, $x \geq s$. Always exists, but may be $\pm\infty$.
- for a decreasing sequence a_n, $\lim_{n\to\infty} a_n$ is defined to be $\inf_{n \in \mathbb{N}} a_n$.
- for an increasing function $f : (0, a) \to \mathbb{R}$ defined near 0, $\lim_{\delta \to 0} f(\delta)$ is defined to be $\inf_{\delta > 0} f(\delta)$. *Ex.:* $\lim_{\delta \to 0}(\delta^2 + 5) = 5$. For a decreasing function, use sup.
- log with no base refers to the natural logarithm.
- a function $f : A \to B$ is **injective** if for all $x, y \in A$, $f(x) = f(y) \implies x = y$. It is **surjective** if for all $b \in B$, there is some $a \in A$ such that $f(a) = b$. It is **bijective** if it is injective and surjective.
- a set is **countable** if there exists a bijection from it to $\mathbb{N}$ or a finite set.

Notes on the use of calculus in the Power Round: we here at PuMaC do not expect anyone in this competition to necessarily know calculus. Thus, we have tried to minimize the use of calc in this power round. Accordingly, for proofs which involve limits, infimums, and supremums, we do not expect formal, $\epsilon - \delta$ type proofs, nor do we want you to be citing continuity results willy nilly. Thus, the limit parts of your proofs will be graded to a lesser standard than other parts of your proofs, as a little treat. Still try to be as rigorous as possible, and we will keep it in mind during grading. Thank you!

1 Topology in $\mathbb{R}^n$

This section is a crash course in the parts of topology relevant for this power round. There are no problems, but the concepts introduced here are important technical details that we need for fractals. Without them, fractals would be too hard to study and understand.

$\mathbb{R}^n$ is the set of n-tuples $(x_1, ..., x_n)$ with that every $x_i \in \mathbb{R}$.[1] Two points of $\mathbb{R}^n$ can be added term-wise and multiplied by a scalar $c \in \mathbb{R}$:

$$(x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$$

$$cx = c(x_1, ..., x_n) = (cx_1, ...cx_n)$$

Denote the special element $(0, ..., 0)$ as simply 0. Given an element $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we define the **Euclidean norm** to be

$$|x| = \sqrt{\sum_{i=0}^{n} x_i^2}$$

For example, in $\mathbb{R}^2$, this is the distance formula

$$|x| = \sqrt{x_1^2 + x_2^2}$$

The **Euclidean metric** is how we measure distance. We define $\text{dist}(x, y) = |x - y|$. This function satisfies 3 very important properties:

1. $\text{dist}(x, y) \geq 0$ for all x, y and equality is achieved if and only if $x = y$.
2. $\text{dist}(x, y) = \text{dist}(y, x)$.
3. (triangle inequality) $\text{dist}(x, t) \leq \text{dist}(x, z) + \text{dist}(z, y)$.

We will usually avoid the notation "$\text{dist}(x, y)$" and just use $|x - y|$.

For a point $x_0 \in \mathbb{R}^n$, we define an **open ball at** x_0 with radius $r \geq 0$ as

$$B_{x_0}(r) = \{x : |x - x_0| < r\}$$

That is, it is the set of all points less than r away from x_0. Notice the "$<$" - it is a strict inequality.

Similarly, the **closed ball** is the set of points

$$\overline{B_{x_0}(r)} = \{x : |x - x_0| \leq r\}$$

An **open set** is a subset of $\mathbb{R}^n$ which is a(n arbitrarily large, even infinite) union of open balls. Clearly, any open ball in $\mathbb{R}^n$ is also an open set.

Here are three crucial properties of open sets:

[1] for $n = 2$, n-tuple notation conflicts with the open interval notation (a, b). Context should make the notation clear.

1. $\emptyset, \mathbb{R}^n$ are both open sets.

2. If $\{U_\alpha\}_\alpha$ is an arbitrary collection of open sets (indexed by α), then $\bigcup_\alpha U_\alpha$ is an open set. This is immediate from the definition.

3. (Nontrivial) If $\{U_i\}_i$ is a *finite* collection of open sets, then $\bigcap_i U_i$ is an open set. This is not true for arbitrary intersections.

Example. The unit open cube, consisting of the points $[0,1]^n = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $0 < x_i < 1$ for all x_i, is an open set.

Proof. For any point $x \in [0,1]^n$, there exists some $r_0(x)$ depending on x such that $B_x(r_0(x)) \subset [0,1]^n$ (convince yourself that this is true!). Then

$$[0,1]^n = \bigcup_{x \in [0,1]^n} B_x(r_0(x))$$

Notice that this union is over an uncountably infinite number of elements, one ball for each x in the cube. □

A set is a **closed set** if it is a complement of an open set. For example, given an r and an x_0, the set of points $\{x : |x - x_0| \geq r\}$ is a closed set. The closed ball,

$$\overline{B_{x_0}(r)} = \{x : |x - x_0| \leq r\}$$

is a closed set. Again, convince yourself this is true. Here are the properties of closed sets:

1. $\emptyset, \mathbb{R}^n$ are both closed sets.

2. If $\{C_\alpha\}_\alpha$ is an arbitrary collection of closed sets, then $\bigcap_\alpha C_\alpha$ is a closed set.

3. If $\{C_i\}_i$ is a *finite* collection of closed sets, then $\bigcup_i U_i$ is a closed set. This is not true for arbitrary intersections.

The last important property is that of compactness: a set $K \subset \mathbb{R}^n$ is **compact** if and only if it is both closed and bounded; that is, its complement is open, and there exists some $B_{(0,...,0)}(r)$ such that $K \subset B_{(0,...,0)}(r)$.

Theorem 1.0.1. *Let*

$$K_0 \supset K_1 \supset K_2 \supset ...$$

be a countably infinite sequence of compact sets in $\mathbb{R}^n$, each one of which containing the rest of them. Suppose that they are all non-empty. Then $\bigcap_i K_i$ is also non-empty.

Note that $\cap_i K_i$ is also compact since it is bounded and closed, being an intersection of closed sets.

2 Measures

A measure is any way which we use to describe "size" to sets, especially when seen as subsets of a larger space. They are designed to generalize and formalize the intuitive notions of length, area, volume, and probability that we know and love.

Let X be a set, and $\Sigma \subset \mathcal{P}(X)$ a subset of the power set of X.

Definition 2.0.1. Σ is called a σ*-algebra* on X if it is closed under complements, countable unions, and countable intersections. That is, if $\{E_i\}_{i=1}^{\infty} \subset \Sigma \subset \mathcal{P}(X)$, then

1. $\emptyset \in \Sigma$
2. $X \setminus E_i \in \Sigma$
3. $\bigcup_{i=1}^{\infty} E_i \in \Sigma$
4. $\bigcap_{i=1}^{\infty} E_i \in \Sigma$

These include finite intersections and unions.

In short, a σ-algebra is a set of subsets which play very nicely with set-theoretic operations. Also notice that first two conditions immediately imply that $X \in \Sigma$.

Problem 2.0.1. Which of the following sets are σ-algebras over $X = \{0, 1, 2, 3, 4\}$?

1. $\Sigma = \{\emptyset, X\}$
2. $\Sigma = \{\emptyset, 0, 1, 2, 3, 4, X\}$
3. $\Sigma = \{\emptyset, \{0, 1\}, \{2, 3, 4\}, \{0, 1, 2, 3, 4\}\}$
4. $\Sigma = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{1, 2\}, \{1, 2, 3, 4\}, \{0, 2, 3, 4\}, X\}$

It turns out that when doing work with fractals, it is imposible to attempt to deal with arbitrary subsets of $\mathbb{R}^n$, which is why we will restrict ourselves to the following σ-algebra:

The **Borel algebra** over $\mathbb{R}^n$ is defined to be the smallest σ-algebra on $\mathbb{R}^n$ which contains all open sets. By the definition of σ-algebras, this means that it also contains all countable unions, intersections, and complements of open sets in $\mathbb{R}^n$. Elements of the the borel sigma algebra are called **Borel sets**.

Now, let $\mu : \Sigma \to \mathbb{R} \cup \{\infty\}$ be a function to the **extended real line**. Here we take the extra element infinity to be such that if $a \in \mathbb{R}$, then $a < \infty$. We may sometimes denote it as $\overline{\mathbb{R}}$.

Definition 2.0.2. μ is a *measure* if:

1. $\mu(\emptyset) = 0$
2. $\mu(E) \geq 0$ for all $E \in \Sigma$.

3. Let $\{E_i\}_{i=1}^{\infty} \subset \Sigma$ be a collection of elements of Σ which are pairwise disjoint: that is, for $i \neq j$, $E_i \cap E_j = \emptyset$. Then [2]

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Condition 1 is mostly a technicality, whereas conditions 2 and 3 are important to what it means to be a measure. Condition 2 represents the idea that something may not have a "negative area." Condition 3 tells us that for a measure, ant set is the sum of its parts. This should be familiar with one's intuition for the regular area or volume of a shape.

We call the triplet (X, Σ, μ) a **measure space** if Σ is a σ-algebra on X and μ is a measure for Σ.

Problem 2.0.2. Let X be any set and $\Sigma = \mathcal{P}(X)$. Let $\mu : \Sigma \to \overline{\mathbb{R}}$ be defined as $\mu(A) = \#A$, the cardinality of A, if $A \subset X$ is finite, and $\mu(A) = \infty$ if A is infinite.

1. Verify that Σ forms a σ-algebra over X.
2. Verify that μ forms a measure over (X, Σ).

This measure is called the **counting measure**. It is the simplest non-trivial measure.

Problem 2.0.3. Verify that in a measure space (X, Σ, μ) with $A, B \in \Sigma$, if $A \subset B$ then $\mu(A) \leq \mu(B)$.

Problem 2.0.4 (Countable Subadditivity). Let (X, Σ, μ) a measure space; show that for any countable collection $\{E_i\}_{i=1}^{\infty} \subset \Sigma$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

[Hint: think about what you would do for the finite case first.]

Problem 2.0.5. Let X be any set and $\Sigma = \mathcal{P}(X)$. Let $x_0 \in X$ be a designated point in X. Now, define $\mu : \Sigma \to \overline{\mathbb{R}}$ so that $\mu(A) = 1$ if $x_0 \in A$, and $\mu(A) = 0$ otherwise. Is (X, Σ, μ) a measure space?

[2] Since we are working with non-negative numbers, the infinite sum, if it converges, is well-defined and absolute. If it diverges or has an ∞ term, then the sum is defined to equal ∞. When working with infinite sums of positive numbers, you may assume standard facts that are true for finite sums, such as the distributive law, commutativity, associativity, etc. Also, this definition includes the case of a finite sum / union

2.1 Hausdorff measure

The Hausdorff measure is the measure we actually care about in this Power Round. Actually, there is no one Hausdorff measure; there is a Hausdorff measure for every non-negative real number s. If s is an integer, then the Hausdorff meaure is just the regular length, area, volume, etc that we are accustomed to! Thus, it is a generalization of the regular concept of area.

Let $S \subset \mathbb{R}^n$. Define the **diameter** of S as

$$|S| = \sup_{x,y \in S} |x - y|$$

Notice that S is a set but $x - y$ is a point, so the notation does not clash.

> **Problem 2.1.1.** For any set E and any $\epsilon > 0$, show that there exists a convex, open set $U \supset E$ such that $|U| \leq |E| + \epsilon$. (for U to be convex means that if $x, y \in U$, then for any $t \in [0, 1]$, $tx + (1 - t)y \in U$). [hint: first find a set satisfying just convexity]

Now, let $F \subset \mathbb{R}^n$. Then, fixing some $\delta > 0$, let $\{U_i\}_i$ be any finite or countably infinite number of sets, such that two properties are satisfied:

1. $F \subset \bigcup_i U_i$
2. $|U_i| < \delta$

Then $\{U_i\}_i$ is called a δ-cover of F.

> **Problem 2.1.2.** let $[0, 1] \subset \mathbb{R}$. Which of the following are δ-covers of $[0, 1]$, for $\delta = \frac{1}{2}$?
>
> 1. $\{(-\frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{4}{3})\}$
> 2. $\{[0, \frac{1}{n}] : n \in \mathbb{N}\}$
> 3. $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$

We define the **outer s-dimensional Hausdorff measure** as follows:

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}$$

Where the infimum is taken over *all possible* δ-coverings of F. By problem 2.1.1, it suffices to consider δ-coverings where every element of the covering is open and convex.

> **Problem 2.1.3.** Let $0 < \delta_1 < \delta_2$; show that
>
> $$\mathcal{H}^s_{\delta_1}(F) \geq \mathcal{H}^s_{\delta_2}(F)$$

Because of the previous exercise, there is a well defined limit

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

(where $\mathcal{H}^s(F)$ could be infinity). This is the **s-dimensional Hausdorff measure** of F. We need to justify that it actually is a measure:

Problem 2.1.4. Prove the following:

1. $\mathcal{H}^s(E) \geq 0$ for all $E \subset \mathbb{R}^n$.

2. $\mathcal{H}^s(\emptyset) = 0$

3. Prove countable subadditivity for the s-dimensional Hausdorff measure (see problem 2.0.4).

Full additivity for the measure is true over Borel sets (but harder to prove), and thus

Theorem 2.1.1. *For each $s \geq 0$, the s-dimensional Hausdorff measure is a measure on $\mathbb{R}^n$ over the Borel σ-algebra.*

From here on, we may assume that every set mentioned is a Borel set and thus has a Hausdorff measure (this includes when you are solving problems).

Problem 2.1.5. (**homogeneity**) Let $E \subset \mathbb{R}^n$ and $0 < c \in \mathbb{R}$ Then let E' be E scaled up by c, that is, $E' = \{cx : x \in E\}$. Show that for all s,

$$\mathcal{H}^s(E') = c^s H^s(E)$$

[Hint: cover E using covers of E' and vice-versa]

Problem 2.1.6. (**translation invariance**) Let $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Let $E' = E+x = \{e + x : e \in E\}$. Show that for all s, $\mathcal{H}^s(E) = \mathcal{H}^s(E')$.

Problem 2.1.7. For $0 < r < s$ and $E \subset \mathbb{R}^n$, prove the following:

1. $\mathcal{H}^r(E) \geq \mathcal{H}^s(E)$.

2. If $\mathcal{H}^r(E) < \infty$, then $\mathcal{H}^s(E) = 0$.

[Hint: for both of these, fix $0 < \delta < 1$ and use coverings where $|U_i| < \delta$]

This problem demonstrates that for any E, there is a critical value s where the Hausdorff measure jumps from 0 to ∞, and this value s is the only possible value where the Hausdorff measure *might* be both non-zero and non-infinite.

Definition 2.1.2. This critical value of s is the **Hausdorff dimension** of E.

Note: just because the dimension of a set E is s does not mean that $\mathcal{H}^s(E) > 0$. It could very well be that the measure drops from infinity directly to zero. That said, the dimension is still be defined.

Problem 2.1.8. Prove that if $E \subset \mathbb{R}^n$, then the Hausdorff dimension of E is $\leq n$.

Problem 2.1.9. Prove that $\mathcal{H}^0$ is simply the counting measure.

Because of this, we will freely assume $s > 0$.

3 Interlude: some fractal constructions

There are no problems in this section: instead, we will provide several constructions of fractals which we will base problems off of in later sections.

3.1 Cantor Set

Let C_0 be the closed unit interval $C_0 = [0,1] \subset \mathbb{R}$. Then let $C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Notice that C_1 is the disjoint union of two closed intervals. We will construct C_{n+1} from C_n as follows: by induction, each C_n is a finite disjoint union of closed intervals $C_n = \bigcup_j [a_j, b_j]$, and each interval is length 3^{-n}. Thus, let

$$C_{n+1} = \bigcup_j \left([a_j, b_j] \setminus (a_j + 3^{-n-1}, b_j - 3^{-n-1})\right)$$

This process removes the middle third of each interval. Since C_{n+1} is clearly also the finite union of disjoint closed intervals, it follows by induction that C_i exists for every i, and that $C_{i+1} \subset C_i$.

Then, define the Cantor Set C as

$$C = \cap_{i=0}^{\infty} C_i$$

This intersection is non-empty and compact since it is the intersection of a decreasing sequence of compact sets.

3.2 Sierpinski Carpet

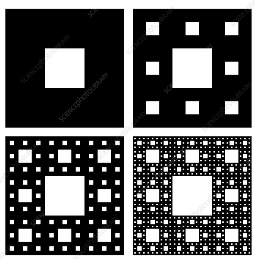


Figure 1: The construction of the Sierpinski Carpet.

Start with the closed unit square $\square_0 = [0,1] \times [0,1]$. To go from $\square_i$ to $\square_{i+1}$, divide each component square into 9 equal new squares, and remove the central open square, as in figure 1. At stage $\square_k$ there will be 8^k squares, each with sidelength 3^{-k}.

Note that when we remove the center square, we are deleting the open component. For example, $\square_1 = \square_0 \setminus \left((\frac{1}{3}, \frac{2}{3}) \times (\frac{1}{3}, \frac{2}{3})\right)$. Thus, each $\square_i$ is closed and so compact, and $\square_{i+1} \subset \square_i$. Thus,

$$\square = \bigcap_i \square_i$$

Is non-empty by compactness. $\square$ is the Sierpinski Carpet.

3.3 Minkowski Sausage

Let $M_0 = [0, 1]$ the interval. At each stage, replace every interval in M_i with the interval pattern as in figure 2 on the left to create M_{i+1}. Each interval in M_k is length 4^{-k}, and there are 8^k such intervals. Notice that in M_1, for example, we consider there to be 8 intervals each with length $\frac{1}{4}$, *even though* two of the intervals are colinear. There is a well-defined limit of this process which results in a compact fractal denoted as M, called the Minkowski Sausage.

3.4 Koch Curve

Let $K_0 = [0, 1]$ the interval. At each stage, replace every interval in K_i with the interval pattern as in figure 2 on the right to create K_{i+1}. Each interval in K_k is length 3^{-k}, and there are 4^k such intervals. Notice that in K_1, for example, we consider there to be 4 intervals each with length $\frac{1}{3}$. There is a well-defined limit of this process which results in a compact fractal denoted as K, called the Koch Curve.

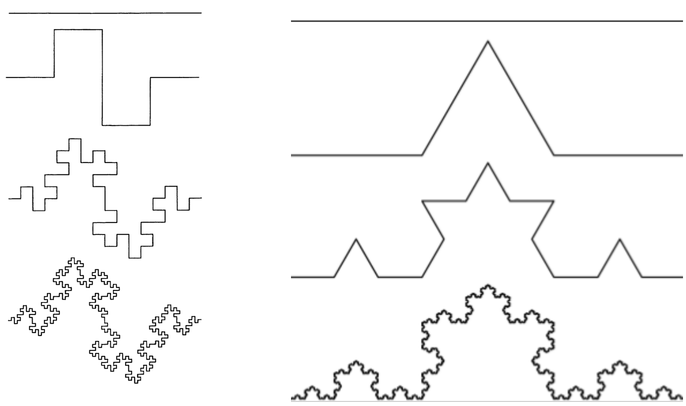


Figure 2: (Left) the construction of the Minkowski Sausage. (Right) the Koch Curve

4 Iterated Function Systems

An iterated function system is a method of generating self-similar fractals. All the fractals in the interlude, as well as many more, may be created using this method.

Definition 4.0.1. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a **contraction** if there exists some constant $0 < c < 1$ such that for every $x, y \in \mathbb{R}^n$, $|f(x) - f(y)| = c|x - y|$. The value c is called the **contraction ratio** for f.

Reworded, a contraction is a function that sends points closer to each-other by some common factor.

An **iterated function system** (**IFS**), then, is just a finite set of contractions $(f_1, ..., f_m)$, $m \geq 2$ (they are assumed to all have the same domain and codomain, but *not* necessarily the same contraction ratios).

Problem 4.0.1. Let f be a contraction in $\mathbb{R}^n$. Denote $f^{(k)}$ to be f composed with itself k times (e.g., $f^{(2)}(x) = f(f(x))$).

1. Show that for any $x, y \in \mathbb{R}^n$ and any $\epsilon > 0$, there exists some k such that $|f^{(k)}(x) - f^{(k)}(y)| < \epsilon$.

2. Show that if one replaces the contraction assumption with the weaker $|f(x) - f(y)| < |x - y|$ (for $x \neq y$), then there exist such functions with no fixed points; that is, no x such that $f(x) = x$.

Problem 4.0.2. If f is a contraction and K is compact, show that $f(K)$ is compact.

A subset $D \subset \mathbb{R}^n$ is said to be **self-similar** if it is the union of contractions of itself; that is, if there exists an IFS $(f_1, ..., f_m)$ such that

$$D = \bigcup_{i=1}^{m} f_i(D)$$

The union is allowed to intersect itself, so that for instance $f_1(D) \cap f_2(D) \neq \emptyset$. In this case, the D is called and **attractor** of the IFS $(f_1, ..., f_m)$.

Theorem 4.0.2. *Every IFS $(f_1, ..., f_m)$ is the attractor of a unique, non-empty, compact set F.*

Proof. Let the IFS be $(f_1, ..., f_n)$: there exists a compact set C such that $f_i(C) \subset C$ for all i: in fact, the closed ball $\overline{B_{(0,...,0)}(r)}$ suffices for r sufficiently large. C exists by the following argument:

Let c_i be the contraction ratio for each f_i; let $r_i = \frac{|f_i(0)|}{1-c_i}$ and $|x| \leq r_i$. Then

$$|f_i(x)| \leq |f_i(x) - f_i(0)| + |f_i(0)| = c_i|x| + |f_i(0)|$$

$$\leq c_i r_i + |f_i(0)| = \left(\frac{c_i}{1 - c_i} + 1\right) |f_i(0)|$$

$$= \frac{c_i + 1 - c_i}{1 - c_i} |f_i(0)| = \frac{|f_i(0)|}{1 - c_i} = r_i$$

By setting $r = \max_i(r_i)$, we get our desired ball $\overline{B_0(r)}$.

Next, for X a set, define $f(X) = \bigcup_i f_i(X)$, and define $f^{(k)}(X)$ to be f composed with itself k times as in problem 4.0.1.

Then for each k, $f^{(k)}(C)$ is a compact set (cf. problem 4.0.2). Since $f(C) \subset C$, by repeated application of f to both sides, $f^{(k)}(C) \subset f^{(k-1)}(C)$. Thus,

$$C \supset f(C) \supset f^{(2)}(C) \supset ...$$

Forms a decreasing sequence of compact sets, so

$$F = \bigcap_k f^{(k)}(C)$$

is non-empty and compact, and it is clear that

$$f(F) = f\left(\bigcap_k f^{(k)}(C)\right) = \bigcap_k f^{(k+1)}(C) = \bigcap_k f^{(k)}(C) = F$$

Showing existence.

□

Problem 4.0.3. Prove uniqueness of the above theorem. That is, prove that if $(f_1, ..., f_m)$ is the attractor of two non-empty, compact sets F_1, F_2, then $F_1 = F_2$.

[Hint: at some point, it may be useful to prove the following alternative description of closed sets: a set C is closed if and only if C contains all points x such that for all $r > 0$, $B_x(r) \cap C \neq \emptyset$]

The set D is called the **attractor** of the IFS. Sometimes we will abuse notation and refer to something as an IFS when really we mean the attractor of it. No confusion should arise since attractors are unique.

Example. Let

$$f_1(x) = \frac{1}{3}x$$

$$f_2(x) = \frac{1}{3}x + \frac{2}{3}$$

be functions $\mathbb{R} \to \mathbb{R}$. Then the Cantor set is the attractor of the IFS (f_1, f_2).

Problem 4.0.4. For each of the following fractals, find an IFS whose attractor is that fractal and explain why it is the attractor.

1. the Sierpinski carpet
2. the Koch Curve
3. the Minkowski Sausage

4.1 Dimension of IFS's

Clearly, the Hausdorff dimension of an attractor of an IFS whose domain and codomain is $\mathbb{R}^n$ is at most n. Unfortunately, it is not always true that there is a nice way to glean the exact Hausdorff dimension from the functions which make up an IFS. However, in practice, most IFSs we care about satisfy the following condition:

Definition 4.1.1. An IFS $(f_1, ..., f_m)$ satisfies the **Open Set Condition (OSC)** if there exists a bounded, non-empty open set $U \subset \mathbb{R}^n$ such that for each $i \neq j$,

$$f_i(U) \cap f_j(U) = \emptyset$$

and

$$U \supset \bigcup_i f_i(U)$$

Given this condition, there is the following remarkable result:

Theorem 4.1.2. *Let $(f_1, ..., f_m)$ be an IFS, with c_i the contraction ratio for f_i. Then the unique real number s such that*

$$\sum_{i=1}^{m} c_i^s = 1$$

is the Hausdorff dimension of the attractor for the IFS.

Proof. Elementary but long and technical, so ommitted. □

Problem 4.1.1. Let $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f_1(x) = \frac{1}{2}x$$

$$f_2(x) = \frac{1}{2}x + (\frac{1}{2}, 0)$$

$$f_3(x) = \frac{1}{2}x + \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$$

Show that this IFS satisfies the open set condition, and calculate its attractor's Hausdorff dimension with proof.

Problem 4.1.2. Verify that the Hausdorff dimension of the Cantor Set is $\frac{\log 2}{\log 3}$, and find the Hausdorff dimension of the Koch Curve.

Problem 4.1.3. Use 4.1.2 to give a proof that the Cantor set is uncountable.

Example. Here is a method to show that the Hausdorff-measure $\mathcal{H}^s(C)$ of the Cantor set is less than or equal to 1 (with $s = \frac{\log 2}{\log 3}$, as shown in problem 4.1.2). Since, in the definition of Hausdorff measure, we use an infimum of δ-coverings, as $\delta \to 0$, all we have to do is demonstrate that for any δ, we can provide a δ-covering with the sum of the scaled diameters arbitrarily close to 1.

In fact, our covers are going to be exactly the disjoint intervals we used in the construction of C! In the construction, C_n is a finite, disjoint union of 2^n closed intervals, each of length 3^{-n}. Thus, each C_n comes from a 3^{-n}-cover; so let that cover be indexed as $\{E_i\}$. Then

$$\sum_i |E_i|^{\log 2/\log 3} = 2^n (3^{-n})^{\log 2/\log 3} = 2^n 2^{-n} = 1$$

Since for every $\delta > 0$, there exists some k such that $3^{-k} < \delta$, we have a δ-cover of size 1 for every δ. Thus, $\mathcal{H}^s(C) \leq 1$. In fact, $\mathcal{H}^s(E) = 1$, but that is harder to show.

Problem 4.1.4. Let $n, m \in \mathbb{N}$. Show that there is a k and a set $E \subset \mathbb{R}^k$ such that the Hausdorff dimension of E is $\frac{n}{m}$. Conclude that for any $0 \leq r \in \mathbb{R}$, there exist sets with Hausdorff dimensions arbitrarily close to r.

It is important to note, however, that many self-similar fractals *do not* satisfy the open set condition.

Problem 4.1.5. Construct an example of an IFS which does not satisfy the OSC, and where no two functions in the IFS are equal.

4.2 Mass Distribution Principle

Definition 4.2.1. A **Mass Distribution** on a compact set $F \subset \mathbb{R}^n$ is a measure μ on F such that $0 < \mu(F) < \infty$.

Theorem 4.2.2 (Mass Distribution Principle)**.** *Let μ be a mass distribution on F, and let s be the Hausdorff dimension of F. Suppose there exist $c, \epsilon > 0$ such that for all $U \subset F$ with $|U| \leq \epsilon$,*

$$\mu(U) \leq c|U|^s$$

Then

$$\mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$$

Proof. Let $\{U_i\}$ be an ϵ-covering of F. By countable subadditivity and the assumptions of the theorem,

$$0 < \mu(F) \leq \mu\left(\bigcup_i U_i\right) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s$$

For $\delta < \epsilon$, we take infima over all δ-coverings $\{U_i\}_i$,

$$0 < \frac{\mu(F)}{c} \leq \mathcal{H}^s_\delta(F)$$

As we draw $\delta \to 0$, we get

$$0 < \frac{\mu(F)}{c} \leq \mathcal{H}^s(F)$$

□

Problem 4.2.1. Prove that the real line has Hausdorff dimension 1, but $\mathcal{H}^1(\mathbb{R}) = \infty$. [Hint: work over $[0, 1]$].

The Mass Distribution principle is one of the few systematic ways to provide non-trivial lower bounds on the Hausdorff measure of a set.

Example. Let's describe a mass-distribution μ on the cantor set C. Remember that the Cantor set is constructed in steps, with the kth step consisting of 2^k disjoint intervals, each of length 3^{-k}. Define

$$\mu(I_k \cap C) = 2^{-k}$$

For I_k one of those kth level intervals; take a second to make sure that this is well-defined. for $U \subset C$, we define $\mu(U)$ to be the value obtained by approximating U it with smaller and smaller $I_k \cap C$: that is,

$$\mu(U) = \inf \left\{ \sum_i \mu(E_i) : U \subset \bigcup_i E_i, E_i \text{ is an interval in the construction} \right\}$$

Let $|U| < 1$; there exists some k such that $3^{-(k+1)} \leq |U| < 3^{-k}$. Then $U \subset I_k \cap C$ for I_k one of the intervals generated in the kth step (as discussed above). Then by problem 2.0.3,

$$\mu(U) \leq 2^{-k} = (3^{\log 2/\log 3})^{-k}$$
$$= (3^{-k})^{\log 2/\log 3} = (3 \cdot 3^{-(k+1)})^{\log 2/\log 3}$$
$$\leq (3|U|)^{\log 2/\log 3}$$

From problem 4.1.2, the dimension of C is precisely $\log 2/\log 3$, so from the Mass-Distribution principle,

$$\mathcal{H}^s(F) \geq 3^{-\log 2/\log 3} = \frac{1}{2}$$

Theorem 4.2.3. *Let $s = \frac{\log 2}{\log 3}$. Let C be the Cantor Set. Then*

$$\frac{1}{2} \leq \mathcal{H}^s(C) \leq 1$$

Proof. examples 4.1 and 4.2. □

It turns out that in fact $\mathcal{H}^s(C) = 1$ using a slightly more involved technique, but our bounds here are nothing to shake a stick at.

Problem 4.2.2. Find a mass distribution on the Cantor set such that the hypothesis of the Mass Distribution Principle is not met. More specifically, that for all $c, \epsilon > 0$, there is some $U \subset C$ where $|U| \leq \epsilon$, but

$$\mu(U) > c|U|^s$$

The example for the Cantor set provides a way to come up with a **natural measure** on any attractor of an IFS that satisfies OSC. Let our IFS be $(f_1, ..., f_m)$ with contraction ratios $c_1, ..., c_m$, and attractor E. Then let s be the dimension of E, as determined via the OSC. Lastly, denote $f_{i_1,i_2,...,i_m}(X)$ to be $f_{i_1}(f_{i_2}(...(f_{i_m}(X))))$. Then we can define

$$\mu(f_{i_1,i_2,...i_m}(E)) = c_{i_1}^s c_{i_2}^s ... c_{i_m}^s$$

Notice that $\mu(E) = 1$. The crucial fact to verify is that

$$\sum_{k=1}^{m} \mu(f_k(E)) = \mu\left(\bigcup_{k=1}^{m} f_k(E)\right) = \mu(E)$$

And indeed this follows from the definition of the attractor and of μ.

Lastly, we extend this to Borel subsets of E. The easiest way is indeed the right way to do it: if $A \subset E$, then let $J_k = \{f_{i_1,i_2,...i_k}(E) : f_{i_1,i_2,...i_k}(E) \subset A\}$. That is, J_k is all the level k copies of E that fit inside of A.

$$\mu(A) = \lim_{k\to\infty} \sum_{E \in J_k} \mu(E)$$

This is fancy notation for saying in order to get $\mu(A)$, we simply add the measure of all the sets which we can pack into A that we've already defined the measure for.

Theorem 4.2.4. *For E the attractor of the IFS $(f_1, ...f_m)$ satisfying the OSC, the natural measure described above is a mass distribution.*

Even though extension to Borel sets makes the natural measure an actual measure, notice that in example 4.2, we avoided having to use that part of the definition by passing immediately to the intervals using sub-additivity. It would be wise for you to do the same.

Theorem 4.2.5. *If an attractor E satisfies the OSC, with dimension s, then $\mathcal{H}^s(E) > 0$.*

Proof. Too long. □

Problem 4.2.3. Give an explicit, non-trivial (greater than 0) lower bound on the Hausdorff measure of the Cantor dust $C \times C \subset \mathbb{R}^2$, the product of the Cantor set with itself, in the appropriate dimension.

[Hint: first find an IFS whose attractor is the Cantor Dust to compute the Hausdorff dimension, then proceed as in example 4.2]

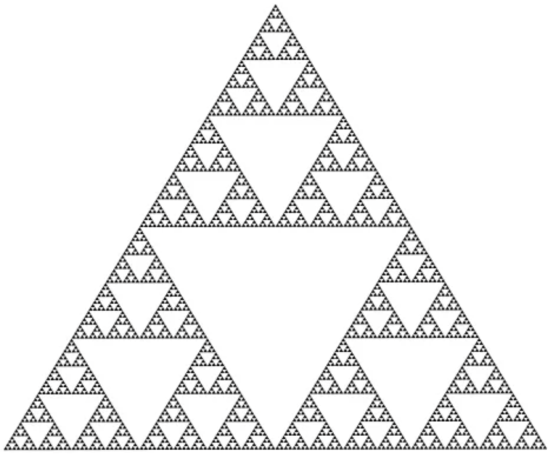
Problem 4.2.4. Show that the open set V which satisfies OSC for the Cantor set C can be made to have intersection $C \cap V \neq \emptyset$ (this is called the **Strong Open Set Condition**). In fact, show that V can even be made to contain C entirely. How much of this is true for the Sierpinski Carpet as well?

Problem 4.2.5. Construct an example of a set with Hausdorff dimension 1 but with 0 measure. [Hint: can you construct a cantor-like set for each dimension $s < 1$?]

Problem 4.2.6. Describe a mass distribution for the Sierpinski carpet inspired by its construction in the interlude. Use it (again mimicking the example for the cantor set) to give a lower bound on the Hausdorff measure in the appropriate dimension.

5 Sierpinski Triangle

The Attractor of the IFS described in problem 4.1.1 is called the **Sierpinski Triangle** (sometimes the Sierpsinki Gasket) and will be denoted as $\blacktriangle$. It looks like this:



For clarity, we restate here the IFS for $\blacktriangle$:

$$S_1(x) = \frac{1}{2}x$$

$$S_2(x) = \frac{1}{2}x + \left(\frac{1}{2}, 0\right)$$

$$S_3(x) = \frac{1}{2}x + \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$$

This quickly gives us that the Hausdorff dimension of $\blacktriangle$ is $s = \frac{\log 3}{\log 2}$.

Since $\blacktriangle$ satisfies the OSC, we can provide the natural measure on it: we will again describe it here for clarity. let Δ be the equilateral triangle with vertices $(0,0), (1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. It is clear that $\blacktriangle \subset \Delta$. Thus also $S_1(\blacktriangle) \subset S_1(\Delta)$ and so on. Let $I_n = (i_1, i_2, ..., i_n)$ be a list with each $i_k \in \{1,2,3\}$, and set $S_{I_n}(\Delta) = S_{i_1}(S_{i_2}(...(S_{i_n}(\Delta))))$. Then for any such I_n, we define $\mu(S_{I_n}(\Delta) \cap \blacktriangle) = 3^{-n}$.

Problem 5.0.1. Show that $\frac{1}{6} \leq \mathcal{H}^s(\blacktriangle) \leq 1$.

Now we will describe a refinement on the Mass-Distribution principle that allows us, in theory, to get arbitrarily close upper and lower bounds on $\mathcal{H}^s(\blacktriangle)$.

Definition 5.0.1. Let μ be the natural measure on $\blacktriangle$. Define $\mathcal{T}_n = \{S_{I_n}(\blacktriangle) : I_n = (i_1, ..., i_n), i_k \in \{1,2,3\}\}$ be the collection of all 3^n self-similar scalings of size 2^{-n} of the Sierpinski triangle. Now let $\{\Delta_i\}_i \subset \mathcal{T}_n$ be a non-empty collection of those scalings.

So lastly, define

$$a_n = \min_{\{\Delta_i\}_i \subset \mathcal{T}_n} \left\{ \frac{|\bigcup_i \Delta_i|^s}{\mu(\bigcup_i \Delta_i)} \right\}$$

The intuition behind this idea is that we are finding a "densest" subset possible at any given level.

> **Problem 5.0.2.** Find a_1 and a_2. (Take your time with this and draw some pictures! Doing this problem slowly will help a lot with the other problems in this section)

Theorem 5.0.2. $\mathcal{H}^s(\blacktriangle) \leq a_n$.

Proof. The original proof for this is easy, but a tiny-bit more than elementary, so we provide a more elementary proof sketch instead.

For each δ, we will provde a covering based on a_n such that $\mathcal{H}^s_\delta(\blacktriangle) \leq a_n$. We will start with showing it for $\mathcal{H}^s_1(\blacktriangle)$.

Since a_n is the finite minimum over non-empty subsets of $\mathcal{T}_n$, there is some collection $\{\Delta_i\}_i, i \geq 1$, which minimizes a_n. Call this collection U, with $\#U = k$. Lastly, let $\mathcal{U} = \bigcup_{\Delta_i \in U} \Delta_i$.

If $\mathcal{U} = \blacktriangle$, we are done; it is already a cover for $\blacktriangle$, and a calculation shows the theorem is satisfied.

Otherwise, let $V = \mathcal{T}_n \setminus U$; $\#V = 3^n - k$. Each element of V will be covered in a scaled down way like we covered $\blacktriangle$ itself: if $S_{I_n}(\blacktriangle) = \Delta'_m \in V$, then scale down each element of U and place it in V at the $2n$-level of the construction of $\blacktriangle$. That is, if we denote our scaled-down U as U' and notating $\mathcal{U}'$ similarly, then $|\mathcal{U}'| = 2^{-n}|\mathcal{U}|$. There will be $\#V = 3^n - k$ of these U's, one for each missed element of $\mathcal{T}_n$ that U does not hit. Our covering so far consists of $\mathcal{U}$, and the $3^n - k$ sets congruent to $\mathcal{U}'$. Now, since $\mathcal{U} \neq \blacktriangle$, our new covering also doesn't cover $\blacktriangle$, so we create a U'' by scaling down U again, this time so that $|\mathcal{U}''| = 2^{-n}|\mathcal{U}''| = 2^{-2n}|\mathcal{U}|$, and there will be $(\#V)^2$ of these, and so on.

Letting this process run to infinity, we get a cover of $\blacktriangle$ consisting of 1 copy of $\mathcal{U}$, $3^n - k$ copies of $\mathcal{U}'$, $(3^n - k)^2$ copies of $\mathcal{U}''$, and so on; thus,

$$\mathcal{H}^s_1(\blacktriangle) \leq \sum_{i=0}^{\infty} (3^n - k)^i |\mathcal{U}^{(i)}|^s$$

$$= \sum_{i=0}^{\infty} (3^n - k)^i (2^{-in}|\mathcal{U}|)^s$$

$$= |\mathcal{U}|^s \sum_{i=0}^{\infty} \left(\frac{3^n - k}{3^n}\right)^i = \frac{|\mathcal{U}|^s}{1 - \frac{3^n - k}{3^n}} = \frac{|\mathcal{U}|^s}{\frac{k}{3^n}} = \frac{|\mathcal{U}|}{\mu(\mathcal{U})} = a_n$$

Where we use the definition of $\mathcal{H}^s_\delta$ and $s = \log 3/\log 2$, the definition of a_n, and the geometric series formula.

Now, for $\delta < 1$, there exists some $m \in \mathbb{N}$ such that $2^{-m} < \delta$. This part is fairly easy. Instead of starting with the cover U, simply start with 3^m copies of U scaled down by 2^{-m}, one at each level m tile of $\blacktriangle$. Then perform the process as stated above for each of the scaled down copies of U. At each stage we will simply get 3^n extra copies of $U^{(n)}$, and each of them will have a diameter scaled by 2^{-n}. So we get

$$\mathcal{H}^s_\delta(\blacktriangle) \leq 3^m \left[\sum_{i=0}^{\infty} (3^n - k)^i (2^{-n}|\mathcal{U}^{(n)}|)^s\right]$$

$$= 3^m 2^{-n\log 3/\log 2}\left[\sum_{i=0}^{\infty}(3^n-k)^i(|\mathcal{U}^{(n)}|)^s\right] = \sum_{i=0}^{\infty}(3^n-k)^i(|\mathcal{U}^{(n)}|)^s = a_n$$

Using the definition of s and the earlier calculation. Since the largest cover we used this time was size $2^{-n} < \delta$, this is a valid covering for $\mathcal{H}^s_\delta$, and taking the limit gives

$$\mathcal{H}^s(\blacktriangle) \le a_n$$

$\square$

Problem 5.0.3. Show that a_n decreases as n increases.

Since $a_n > 0$ is clear, it immediately follows that $L = \lim_{n\to\infty} a_n$ exists and $L \ge 0$. Combining it with the previous theorem, we get that $\mathcal{H}^s(\blacktriangle) \le L$ (and therefore also that $L > 0$).

Theorem 5.0.3. $\mathcal{H}^s(\blacktriangle) = \lim_{n\to\infty} a_n$.

Proof. This follows from the remark above by showing that $L \le H^s(\blacktriangle)$. We first remember that the only good way we have of creating lower bounds is the mass distribution principle, we thus want to show that for any set $U \subset \blacktriangle$, $\mu(U) \le \frac{1}{L}|U|^s$. This will imply that $\mathcal{H}^s(\blacktriangle) \ge L$ by mass distribution.

$$\mu(U) = \lim_{n\to\infty}\mu\left(\bigcup_{S_{I_n}(\blacktriangle)\subset U} S_{I_n}(\blacktriangle)\right)$$

$$\le \lim_{n\to\infty}\frac{1}{a_n}\left|\bigcup_{S_{I_n}(\blacktriangle)\subset U} S_{I_n}(\blacktriangle)\right|^s \le \lim_{n\to\infty}\frac{1}{a_n}|U|^s = \frac{1}{L}|U|^s$$

The questionable implications here are the first equality and the first inequality. The first comes from our definition of the natural measure, and the inequality comes from our definition of a_n.

$\square$

Problem 5.0.4. Find some $A < 0.9$ such that $\mathcal{H}^s(\blacktriangle) \le A$ and prove it.
[Hint: find a covering that shows that $a_k < 0.9$ for some k]

The best known current bounds on $\mathcal{H}^s(\blacktriangle)$ are proven using a continuation of the methods developed during this Power Round:

Theorem 5.0.4. $0.77 \le \mathcal{H}^s(\blacktriangle) \le 0.82$

The area is an active field of research.

Problem 5.0.5. Generalize the construction in this section to the Minkowski Sausage M (with dimension m), and use it to prove that $\mathcal{H}^m(M) \leq 0.8$.

Problem 5.0.6. Find a_3 (use of a computer program is allowed for the computation itself, although the code must be provided and justified).

Problem 5.0.7. Use the following fact about $\blacktriangle$ to show that $\mathcal{H}^s(\blacktriangle) \geq 0.3$: for all n,

$$H^s(\blacktriangle) \geq a_n e^{-\frac{16\sqrt{3}}{3} s \left(\frac{1}{2}\right)^n}$$

[Hint: find a_4 (use of a computer program is allowed for the computation itself, although the code must be provided and justified)]

This concludes the Power Round. Congratulations!

Team Number:

PUMaC 2023 Power Round Cover Sheet

In years past, this page was for hand-turned in submissions. Now it is for reference for you guys! So that you know how many points each problem is worth.

Problem Number	Points	Attempted?
2.0.1	5	
2.0.2	5	
2.0.3	5	
2.0.4	10	
2.0.5	5	
2.1.1	15	
2.1.2	5	
2.1.3	10	
2.1.4	15	
2.1.5	10	
2.1.6	5	
2.1.7	10	
2.1.8	10	
2.1.9	5	
4.0.1	15	
4.0.2	10	
4.0.3	15	
4.0.4	10	
4.1.1	5	
4.1.2	5	
4.1.3	5	
4.1.4	15	
4.1.5	10	

Problem Number	Points	Attempted?
4.2.1	10	
4.2.2	10	
4.2.3	5	
4.2.4	5	
4.2.5	20	
4.2.6	15	
5.0.1	10	
5.0.2	5	
5.0.3	5	
5.0.4	10	
5.0.5	25	
5.0.6	40	
5.0.7	50	