P U M ∴ C

2025 Team Round

The Team Round consists of 15 questions and one bonus question. Your team has **50 minutes** to complete the Team Round. Each problem is worth 5 points. Do not discuss or distribute these problems after the test until **January 16th, 2026**.

Bonus: A *Rayo-string* is a string consisting of the symbols (,), $=$, $\in$, $\neg$, $\wedge$, $\exists$, and x_i for any positive integer i, such that

- $x_i = x_j$ and $x_i \in x_j$ are Rayo-strings for any i, j;
- If φ and ψ are Rayo-strings, then $(\neg\varphi)$ and $(\varphi \wedge \psi)$ and $\exists x_i(\varphi)$ are Rayo-strings;
- No other strings are Rayo-strings.

For example, $\exists x_2((\neg x_1 = x_2))$ is a Rayo-string, but $\exists x_2(\neg x_1 = x_2)$ is not a Rayo-string. Given any Rayo-string, we can interpret it as a mathematical statement: x_i are sets; $x_i = x_j$ means that x_i is equal to x_j; $x_i \in x_j$ means that x_i is an element of x_j; $(\neg\varphi)$ means that φ is false; $(\varphi \wedge \psi)$ means that both φ and ψ are true; $\exists x_i(\varphi)$ means that φ is true for some set x_i.

Let N be the smallest known positive integer for which there exists a Rayo-string of length N which is equivalent to the statement $x_1 = \{\varnothing, \{\varnothing\}\}$. Submit a guess M for N to get

$$\left\lfloor 5e^{-\frac{(M-N)^2}{4N}} \right\rfloor$$

bonus points.

Proposed by Zongshu Wu

Solution: The best known Rayo string equivalent to $x_1 = \{\varnothing, \{\varnothing\}\}$ is

$$((\neg\exists x_2(\exists x_3(\exists x_4((x_4 \in x_3 \wedge (x_3 \in x_2 \wedge x_2 \in x_1))))))$$
$$\wedge \exists x_2(\exists x_3((x_3 \in x_2 \wedge (x_2 \in x_1 \wedge x_3 \in x_1))))).$$

The first half says that there is no chain of inclusions $x_4 \in x_3 \in x_2 \in x_1$. If you've read Section 4.3 of the Power Round (on the von Neumann universe), then you will see that this is equivalent to

$$x_1 \in V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\},$$

but of course this can also be shown directly. The second half says that we do indeed have a chain $x_3 \in x_2 \in x_1$ such that $x_3 \in x_1$. The only element of V_3 satisfying this is $\{\varnothing, \{\varnothing\}\}$, as desired.

This Rayo string has $\boxed{56}$ symbols. It was found by Rachel Hunter in 2022, improving upon the previous bound of 59 symbols. No one has since found a shorter one.

1. Find the sum of all primes p such that p divides $2^p + 4^p + \cdots + 510^p$.

   Proposed by Rohit Agarwal

   Solution: Using Fermat's little theorem,

   $$2^p + 4^p + \cdots + 510^p \equiv 2 + 4 + \cdots + 510 \pmod p$$

   Then, the sum yields $2 \cdot \frac{(255)256}{2} = 255 \times 256 = 2^8 \times 3 \times 5 \times 17$. Thus, $2, 3, 5$ and 17 all work, yielding sum $\boxed{27}$.

2. A regular octahedron is a polyhedron with 6 vertices, 12 edges, and 8 triangular faces, where each vertex connects to exactly 4 others and there are 3 pairs of opposite vertices. Each vertex is assigned one of 8 colors. Adjacent vertices (connected by an edge) must have different colors, and opposite vertices must have the same color. Additionally, for each triangular face, the three vertices must use at least 3 different colors. Two colorings are considered the same if one can be obtained from the other by rotating the octahedron in 3D space. How many ways can the octahedron be colored?

Proposed by Thomas Jou

Solution: Opposite vertices share a color, so the 3 opposite pairs form a K_3 requiring three distinct colors. The three opposite pairs must all be different colors, so you'd pick any of the 8 colors of the first pair, then any of the remaining 7 for the second, then any of the remaining 6 for the third and thus $8 \cdot 7 \cdot 6 = \boxed{336}$.

3. Suppose $x = \sqrt{2 - \sqrt{2 + \sqrt{2 - x}}}$. Find x.

Proposed by Andrew Li

Solution: The RHS is between 0 and $\sqrt{2}$, so we may make the substitution $x = 2\cos(\theta)$ and find a $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$ that works. Repeatedly applying half angle formulas, we have

$$\sqrt{2 - \sqrt{2 + \sqrt{2 - 2\cos(\theta)}}} = \sqrt{2 - \sqrt{2 + 2\sin(\theta/2)}} = \sqrt{2 - \sqrt{2 + 2\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}}$$
$$= \sqrt{2 - 2\cos\left(\frac{\pi}{4} - \frac{\theta}{4}\right)} = 2\sin\left(\frac{\pi}{8} - \frac{\theta}{8}\right).$$

So $2\cos(\theta) = 2\sin\left(\frac{\pi}{8} - \frac{\theta}{8}\right)$, so $\theta = \frac{\pi}{2} - \left(\frac{\pi}{8} - \frac{\theta}{8}\right)$ gives a solution, so $x = \boxed{2\cos\left(\frac{3\pi}{7}\right)}$.

4. There exist positive integers a, b, c such that $a^3 + b^3 + c^3 = 3438$. What is $a + b + c$?

Proposed by Aiden Sharp and Andrew Li

Solution: Note that $9 \mid 3438$. Also, for any integer n, $n^3 \equiv 0, \pm 1 \pmod 9$. Thus, one of a, b, c must be divisible by 3. WLOG, assume $3 \mid c$. Since $16^3 = 4096 > 3438$, $c \leq 15$. That leaves only five possibilities for c: $15, 12, 9, 6$, or 3.

We can quickly check these from the top down:

- If $c = 15$, $c^3 = 3375 \implies a^3 + b^3 = 63$. No two positive cubes sum to 63.
- If $c = 12$, $c^3 = 1728 \implies a^3 + b^3 = 1710$. The max possible cube under 1710 is $11^3 = 1331$. $1710 - 1331 = 379$ (not a cube), and $10^3 = 1000 \implies 1710 - 1000 = 710$ (not a cube).
- If $c = 9$, $c^3 = 729 \implies a^3 + b^3 = 2709$. The max possible cube under 2709 is $13^3 = 2197$. $2709 - 2197 = 512 = 8^3$.

We have a match! The integers are $(13, 9, 8)$.

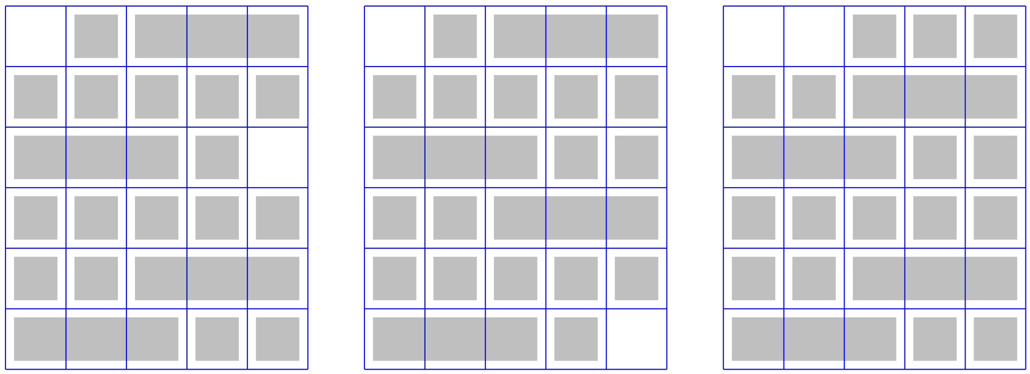
Thus, $a + b + c = 13 + 9 + 8 = 30$.

5. Zappy Zack wants to place rectangular chargers on a 6×5 power grid. He has (1×3) chargers and (1×1) chargers. The top leftmost outlet doesn't work, so he cannot plug any chargers into it. Furthermore, any (1×1) charger must be adjacent to a (1×3) charger (diagonals do not count). He also cannot rotate any chargers. Zack places chargers into the slots such that

he maximizes the total number of chargers he places down. How many ways could he have done this?

Proposed by Rohit Agarwal

Solution: From some careful experimentation, one can observe that to place down the maximum number of chargers, if one is forced to miss a square, they must miss at least one more square elsewhere. Further, they then must use exactly 4 (1×3) chargers. From these observations, one can deduce that these are the only $\boxed{3}$ possible configurations.



6. Let $\triangle ABC$ be a triangle with $\text{AB} = 14$, $\text{AC} = 19$, incenter I, and incircle tangent to side $\overline{BC}$ at D. Let H' be the orthocenter of triangle $\triangle BIC$. Given that $\frac{DH'}{DI} = 10$, find the length of BC.

   Proposed by Ana Boiangiu

   Solution: Note that $DI \cdot DH' = DB \cdot DC$, so $k \cdot r^2 = (s-b)(s-c)$, where $k = 10$ and r is the inradius. Plugging this in to Heron's formula and setting it equal to the inradius area formula, we get $s = \sqrt{10s(s-a)}$. Squaring both sides and setting $s = \frac{33+a}{2}$ yields $a = \boxed{27}$.

7. For each positive integer n, let $\zeta_{n0}, \zeta_{n1}, \ldots, \zeta_{n(n-1)}$ denote the n distinct complex solutions to $z^n = 1$. Compute

$$\lim_{n\to\infty} \max_{T\subseteq\{0,1,\ldots,n-1\}} \frac{1}{n} \left| \sum_{i\in T} \zeta_{ni} \right|.$$

   Proposed by Kiran Reddy

   Solution: Consider first some fixed n. For convenience let $\zeta_{nk} = e^{\frac{2\pi}{n}ki}$. Suppose $\frac{1}{n}\left|\sum_{i\in T}\zeta_{ni}\right|$ is maximal for some set T and define $r = \sum_{i\in T}\zeta_{ni}$. Treat r and each ζ_{ni} as a vector in $\mathbb{R}^2$. Then T must include each ζ_{ni} for which the dot product between ζ_{ni} and r is positive (otherwise we could add such a vector to T to increase the quantity), and must exclude each ζ_{ni} for which the dot product is negative (otherwise we could remove such a vector to T to increase the quantity). This dot product condition equivalently says that each ζ_{ni} which strictly lies on the same side as r of the line perpendicular to r through the origin, and excludes each ζ_{ni} which lies strictly on the opposite side. So T consists of a semicircle of $1, \zeta_{n1}, \ldots, \zeta_{n\lfloor n/2\rfloor}$. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \left| \sum_{k=0}^{\lfloor n/2\rfloor} (e^{\frac{2\pi}{n}i})^k \right| = \lim_{n\to\infty} \frac{1}{n} \left| \frac{1 - e^{\frac{2\pi}{n}i(\lfloor n/2\rfloor+1)}}{1 - e^{\frac{2\pi}{n}i}} \right|$$

As $n \to \infty$, the $+1$ and the fractional part of $\lfloor n/2 \rfloor$ vanish in the limit, so we can substitute $n/2$ for the exponent in the numerator.

$$1 - e^{\frac{2\pi}{n} i(n/2)} = 1 - e^{\pi i} = 1 - (-1) = 2$$

For the denominator, we use trig identities

$$|1-e^{i\theta}| = \sqrt{(1-\cos\theta)^2 + \sin(\theta)^2} = \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = 2\sin(\theta/2).$$

Thus, the magnitude of the denominator is $2\sin(\frac{\pi}{n})$.

Substituting these magnitudes back into our limit:

$$\lim_{n\to\infty} \frac{1}{n} \frac{2}{2\sin(\frac{\pi}{n})} = \lim_{n\to\infty} \frac{1}{n\sin(\frac{\pi}{n})}$$

To evaluate this elementarily, we can multiply the numerator and denominator by π to leverage the standard trigonometric limit $\lim_{x\to 0} \frac{x}{\sin x} = 1$:

$$\lim_{n\to\infty} \frac{1}{\pi} \cdot \frac{\frac{\pi}{n}}{\sin(\frac{\pi}{n})} = \frac{1}{\pi} \cdot 1 = \frac{1}{\pi}$$

Thus, the maximum possible magnitude of the sum approaches $\boxed{\frac{1}{\pi}}$.

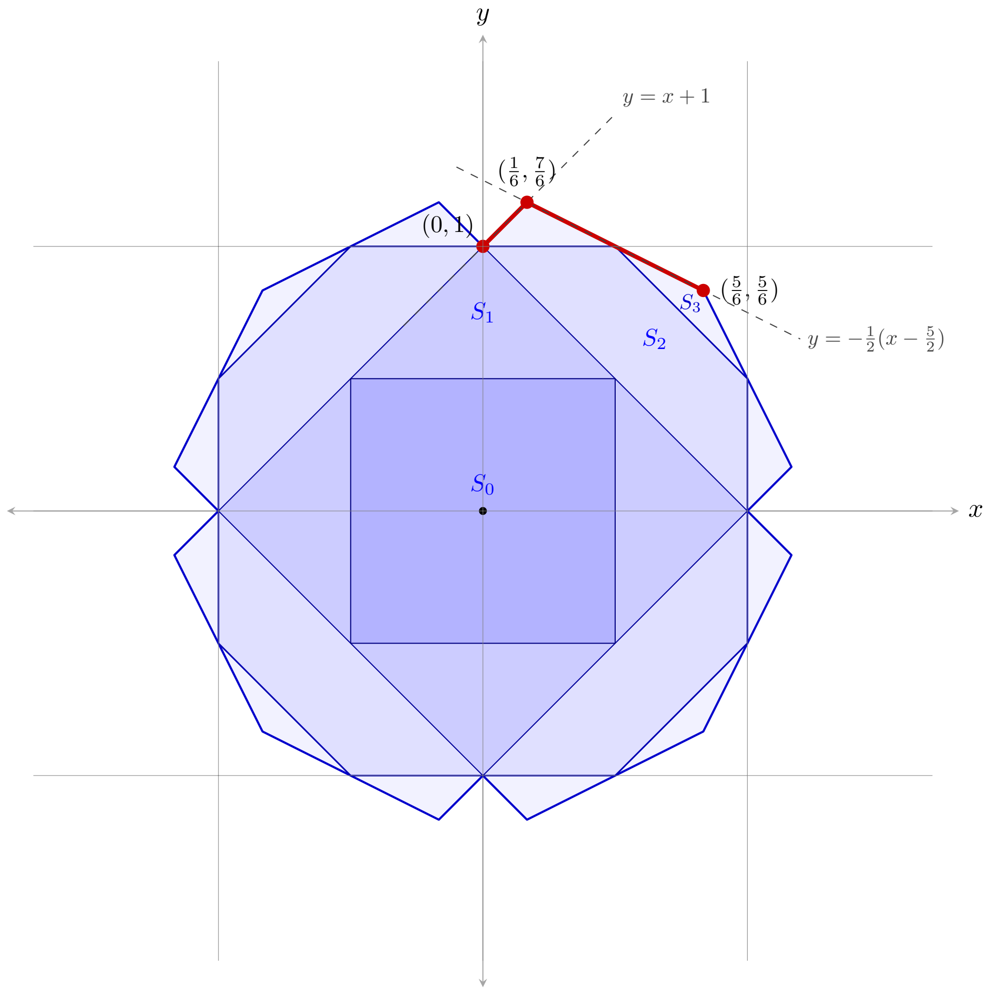
8. Compute the number of ordered pairs (a, b) of positive integers satisfying $ab \le 100$ and $\gcd(a, b) + \operatorname{lcm}(a, b) = a + b$.

   Proposed by Thomas Jou

   Solution: With $a = gd, b = ge, \gcd(d, e) = 1$, we have $\operatorname{lcm}(a, b) = gde$. The condition becomes $g + gde = gd + ge \Rightarrow 1 + de = d + e \Rightarrow (d-1)(e-1) = 0$. So $d = 1$ or $e = 1$. Count ordered pairs with $a|b$. Let $b = ka$. Then $ab = ka^2 \le 100$, so for each $a \le 10$ there are $\lfloor \frac{100}{a^2} \rfloor$ choices of k. Thus $\sum_{a=1}^{10} \lfloor \frac{100}{a^2} \rfloor = 100 + 25 + 11 + 6 + 4 + 2 + 2 + 1 + 1 + 1 = 153$. By symmetry, the same number have $b|a$. Pairs with $a = b$ (ten of them) are double counted and thus $N = 2 \cdot 153 - 10 = \boxed{296}$.

9. For all $(a, b) \in \mathbb{Z}^2$ (excluding the origin), Biao draws the perpendicular bisector of the line segment from $(0, 0)$ to (a, b). An ant starts at $(0,\ 0)$ and walks around on $\mathbb{R}^2$. Let S_n be the set of points that the ant can reach by crossing up to n of Biao's perpendicular bisectors. Thus S_0 is the square whose vertices are $(\pm\frac{1}{2}, \pm\frac{1}{2})$ and S_1 is the square whose vertices are $(0, 1), (1, 0), (0, -1), (-1, 0)$. Find the perimeter of S_3.

   Proposed by Atharva Pathak



Solution: By symmetry, S_3 is a 16-gon with 8-fold rotational and reflective symmetry. We can calculate the total perimeter by focusing on just one octant.

In this octant, the outer boundary of S_3 is formed by two specific perpendicular bisectors:

(a) The perpendicular bisector of the segment from $(0,0)$ to $(-1,1)$, which is the line $y = x+1$.

(b) The perpendicular bisector of the segment from $(0,0)$ to $(1,2)$, which is the line $y = -\frac{1}{2}(x - \frac{5}{2})$.

We find the vertices of this boundary segment by calculating the intersections of these bisectors with each other and with the boundaries of our octant ($x = 0$ and $y = x$):

- Intersection of $y = x + 1$ and the y-axis ($x = 0$): $(0, 1)$
- Intersection of $y = x + 1$ and $y = -\frac{1}{2}x + \frac{5}{4}$: $(\frac{1}{6}, \frac{7}{6})$
- Intersection of $y = -\frac{1}{2}x + \frac{5}{4}$ and the diagonal $y = x$: $(\frac{5}{6}, \frac{5}{6})$

Now, we calculate the lengths of the two segments connecting these vertices:

$$d_1 = \sqrt{\left(\frac{1}{6} - 0\right)^2 + \left(\frac{7}{6} - 1\right)^2} = \sqrt{\frac{1}{36} + \frac{1}{36}} = \frac{\sqrt{2}}{6}$$

$$d_2 = \sqrt{\left(\frac{5}{6} - \frac{1}{6}\right)^2 + \left(\frac{5}{6} - \frac{7}{6}\right)^2} = \sqrt{\frac{16}{36} + \frac{4}{36}} = \frac{\sqrt{20}}{6}$$

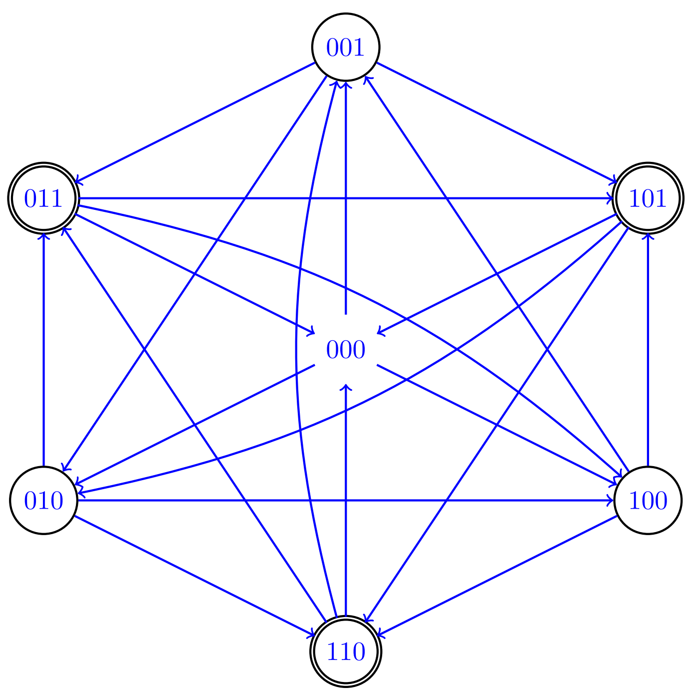
The perimeter within this octant is the sum of these lengths, $\frac{1}{6}(\sqrt{2}+\sqrt{20})$. Because the region is symmetric across all 8 octants, the total perimeter of S_3 is:

$$8 \cdot \frac{1}{6}(\sqrt{2} + \sqrt{20}) = \boxed{\frac{4}{3}(\sqrt{2} + 2\sqrt{5})}$$

10. How many ways can the integers from 0 to 6 be arranged in a cycle such that the difference from one integer to the next is a power of 2 when taken (mod 7)?

Proposed by Azalea Li

Solution: Let us consider the integers in binary. Observe that $100_2 + 100_2 \equiv 001_2 \bmod 7$, so any carries on the 4s bit effectively wraps around to the unit bit. This cues us that some nice threefold symmetries may be in play, especially if we group the integers according to how many set bits (i.e. how many bits are 1) they have. Also, $111_2 \equiv 000_2 \bmod 7$. We consider the following transition diagram:



Note that we can group integers according to how many set bits they have, which is denoted in the above diagram by the number of circles for the corresponding node. Importantly, 000 can only be preceded by a node with two set bits, and succeeded with a node with one set bit; of the four remaining integers to place in the sequence exactly two have one set bit and the other two have two set bits. In addition, for each integer with two set bits, there is only one way it can be succeeded by an integer with one set bit and one way it can be succeeded by an integer with two set bits. For each integer with one set bit, there is also one way it can be succeeded by another integer with one set bit.

From here, we can proceed with casework. We consider the possible sequences according to the set bits of each integer in the sequence, and can WLOG assume that the first two integers of the cycle are 000 and 001 so long as we multiply by 3 later.

- Case 1: 0, 1, 1, 1, 2, 2, 2

  Note that the sequence is fixed to be $000, 001, 010, 100$, then we have two choices, then the sequence is fixed again until we wrap around to 000 for both choices. There are two solutions.
- Case 2: 0, 1, 1, 2, 1, 2, 2

  The sequence starts as $000, 001, 010, ?, 100$, where the fifth term must be 100 by process of elimination. Therefore, the fourth term is 011. The sixth and seventh terms are 101 and 110 and must appear in that order, for one solution.
- Case 3: 0, 1, 1, 2, 2, 1, 2

  The sequence starts as $000, 001, 010, ?, ?, 100$. Working backwards, the fourth and fifth terms must be 110 and 011 respectively. so the seventh term is 101 for one solution.
- Case 4: 0, 1, 2, 1, 1, 2, 2

  The sequence starts $000, 001, ?, 010, 100$. The third term is 101, and the last two must be $110, 011$ in that order for one solution.
- Case 5: 0, 1, 2, 1, 2, 1, 2

  The sequence must either start as $000, 001, 011, 100, ?, 010$ or $000, 001, 101, 010, ?, 100$. We work backwards to find the fifth and seventh terms to be 101 and 110 for the first subcase, and 011 and 110 for the second subcase, for two solutions.
- Case 6: 0, 1, 2, 2, 1, 1, 2

  The sequence must start as $000, 001, ?, ?, 010, 100$. Working backwards, the third and forth terms must be $011, 101$. Thus, the seventh term is 110, for one solution.

The total number of solutions is $3(2 + 1 + 1 + 1 + 2 + 1) = \boxed{24}$.

11. Let $\mathcal{A}$ be a quadrilateral on a sphere, whose sides are all sections of great circles. All the sides of $\mathcal{A}$ are congruent, and all the corners of $\mathcal{A}$ are 120°. Find the proportion of the sphere's surface area enclosed by the quadrilateral.

    Proposed by Joshua Wolford

    Solution: Imagine a regular cube inscribed within the sphere. If we project the edges of this cube radially outward onto the surface of the sphere, the straight edges of the cube map to sections of great circles. The 6 square faces of the cube "inflate" to become 6 congruent regular spherical quadrilaterals that completely tile the sphere.

    Because 3 faces of a cube meet at each of its 8 vertices, the corresponding spherical quadrilaterals will also meet in groups of 3 at each vertex on the sphere. For these 3 identical spherical quadrilaterals to seamlessly surround a vertex, their interior angles must sum to 360°. Thus, the interior angle of each spherical quadrilateral is $360°/3 = 120°$.

    Since our quadrilateral $\mathcal{A}$ has exactly these properties, it is congruent to one of these 6 identical faces. Therefore, the proportion of the surface area it encloses is exactly $\boxed{1/6}$.

12. In a game, each of 6 players get a card on which is printed a nonempty subset of $\{1, 2, 3, 4, 5\}$. Each round, an integer from 1 through 5 is selected at random, and any player who has this number on their card circles it. The player who first circles all numbers on their card wins.

    It is known that every player has a nonzero chance of winning, and there is no possibility of a tie. It is also known that three of the cards are $\{1, 2, 3\}$, $\{2, 3, 4\}$, and $\{3, 4, 5\}$. If each player multiplies the numbers on their card together, what is the sum of these 6 products?

    Proposed by Alex Divoux

    Solution: First, we establish two necessary conditions for the set of cards:

    (a) No card can be a subset of another. If $C_i \subset C_j$, then C_j can never win; all of its numbers cannot be called before all of C_i's numbers are called. This violates the rule that every player has a nonzero chance of winning.

    (b) For any two cards C_i, C_j and any element $x \in C_i \cap C_j$, there must exist a third card $C_k \subseteq (C_i \cup C_j) \setminus \{x\}$. Otherwise, if the drawn sequence consists exactly of the elements in $(C_i \cup C_j) \setminus \{x\}$ followed by x, both C_i and C_j would be completed simultaneously on the draw of x, resulting in a tie.

    We are given the cards $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, and $C = \{3, 4, 5\}$. We can use our two conditions to deduce the remaining three cards:

    Consider the intersection of A and B, which contains 3. By our tie-breaker lemma, there must be some card $D \subseteq (A \cup B) \setminus \{3\} = \{1, 2, 4\}$. By our first condition, D cannot be a subset of A (so $D \neq \{1, 2\}, \{1\}, \{2\}, \emptyset$) and D cannot be a subset of B (so $D \neq \{2, 4\}, \{4\}$). To avoid these subsets while remaining within $\{1, 2, 4\}$, the only possible subset is $D = \{1, 4\}$.

    Similarly, consider the intersection of B and C, which contains 4. There must be a card $E \subseteq (B \cup C) \setminus \{4\} = \{2, 3, 5\}$. To avoid being a subset of B or C, E must contain both 2 and 5. The minimal valid subset is $E = \{2, 5\}$.

    Now we have five cards: $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{1, 4\}$, and $\{2, 5\}$. Let's check the intersection of our new card $D = \{1, 4\}$ and $C = \{3, 4, 5\}$. They intersect at 4. Thus, there must be a card $F \subseteq (D \cup C) \setminus \{4\} = \{1, 3, 5\}$. Checking this against all previous cards, $\{1, 3, 5\}$ does not contain and is not contained by any of them, making it our valid sixth card.

    We have found our 6 distinct cards. Summing their products yields $6 + 24 + 60 + 4 + 10 + 15 = \boxed{119}$.

13. Consider the equations

$$\begin{aligned} x + y + z &= A, \\ x^2 + y^2 + z^2 &= B, \\ x^3 + y^3 + z^3 &= C. \end{aligned}$$

    An ordered triple $(A, B, C) \in \mathbb{R}^3$ is called "nice" if the corresponding system above has exactly three real solutions for x. For fixed $(A, B) \in \mathbb{R}^2$, the set of real C such that the system is "nice" is a finite union of finite intervals; denote the sum of their lengths as $\ell(A, B)$. Find $\sum_{A=3}^{5} \sum_{B=3}^{5} \ell(A, B)$.

    Proposed by Rohit Agarwal

    Solution: By Vieta's formulas, x, y, and z are the roots of the polynomial $f(t) = t^3 - e_1 t^2 + e_2 t - e_3 = 0$, where e_i represents the ith elementary symmetric polynomial of the roots. Using Newton's sums, we can express these as:

$$e_1 = A, \quad e_2 = \frac{A^2 - B}{2}, \quad e_3 = \frac{A^3 - 3AB + 2C}{6}$$

To understand the roots of this cubic, we can apply a simple shift to eliminate the t^2 term. Let $t = s + \frac{A}{3}$. Substituting this into our polynomial gives:

$$f\left(s+\frac{A}{3}\right) = \left(s+\frac{A}{3}\right)^3 - A\left(s+\frac{A}{3}\right)^2 + \frac{A^2-B}{2}\left(s+\frac{A}{3}\right) - e_3 = 0$$

Expanding and combining the linear terms gives:

$$s^3 + s\left(\frac{A^2}{3} - \frac{2A^2}{3} + \frac{A^2-B}{2}\right) - M = 0$$

$$s^3 - \left(\frac{3B-A^2}{6}\right)s - M = 0$$

where M is a constant that absorbs all the constant terms, including e_3 (and thus C). Let $D = \frac{3B-A^2}{6}$. Our equation simplifies to $s^3 - Ds - M = 0$.

For this depressed cubic to have three real roots, the function $h(s) = s^3 - Ds$ cannot be strictly increasing. If $D \leq 0$, then for any real numbers $u > v$, we have $u^3 > v^3$ and $-Du \geq -Dv$. Thus, $h(u) > h(v)$, meaning $h(s)$ is strictly increasing and $s^3 - Ds = M$ can only ever have exactly one real solution.

Therefore, for the system to be "nice," we must have $D > 0$:

$$\frac{3B-A^2}{6} > 0 \implies 3B > A^2$$

Given $A, B \in \{3,4,5\}$, the only pairs that satisfy this are $(A,B) = (3,4)$ and $(3,5)$.

When $D > 0$, the function $h(s) = s^3 - Ds$ "turns around," meaning there is a bounded interval of values for M that results in three real roots. The endpoints of this interval occur exactly where $s^3 - Ds - M = 0$ factors with a double root (the local extrema).

Let r be a double root and k be the third root. Since the s^2 coefficient is 0, the sum of the roots is $2r + k = 0$, which means $k = -2r$. We can construct the factored polynomial:

$$(s-r)^2(s+2r) = (s^2 - 2rs + r^2)(s+2r) = s^3 - 3r^2 s + 2r^3 = 0$$

Matching coefficients with $s^3 - Ds - M = 0$, we get:

$$3r^2 = D \implies r = \pm\sqrt{\frac{D}{3}}$$

$$M = -2r^3 = \mp 2\left(\frac{D}{3}\right)^{3/2}$$

Thus, for the equation to have three real roots, M must lie strictly between the minimum and maximum bounds, $-2(D/3)^{3/2}$ and $2(D/3)^{3/2}$. The length of the interval for M is the difference between these bounds, which is $4(D/3)^{3/2}$.

Since M represents the shifted constants, we know $M = e_3 - K = \frac{A^3-3AB+2C}{6} - K$, where K is a constant independent of C. Because C is multiplied by $\frac{2}{6} = \frac{1}{3}$ in this expression, an interval of length ΔM corresponds to an interval for C that is exactly 3 times as long.

Therefore, the length of the interval for C is:

$$\ell(A,B) = 3 \times 4\left(\frac{D}{3}\right)^{3/2} = 12\left(\frac{3B-A^2}{18}\right)^{3/2}$$

Now we evaluate this for our two valid pairs: For $(A,B)=(3,4)$, we have $\frac{3B-A^2}{18}=\frac{12-9}{18}=\frac{1}{6}$:

$$\ell(3,4)=12\left(\frac{1}{6}\right)^{3/2}=\frac{12}{6\sqrt{6}}=\frac{2}{\sqrt{6}}=\frac{\sqrt{6}}{3}$$

For $(A,B)=(3,5)$, we have $\frac{3B-A^2}{18}=\frac{15-9}{18}=\frac{1}{3}$:

$$\ell(3,5)=12\left(\frac{1}{3}\right)^{3/2}=\frac{12}{3\sqrt{3}}=\frac{4}{\sqrt{3}}=\frac{4\sqrt{3}}{3}$$

Summing the lengths of these intervals gives the final answer:

$$\sum_{A=3}^{5}\sum_{B=3}^{5}\ell(A,B)=\frac{\sqrt{6}}{3}+\frac{4\sqrt{3}}{3}=\boxed{\frac{\sqrt{6}+4\sqrt{3}}{3}}.$$

14. A 24-cell is a region that can be formed by the following procedure: taking convex hull of 8 vertices formed by permuting integer coordinates $(\pm1,0,0,0)$ and 16 vertices at all half-integer coordinates $(\pm1/2,\pm1/2,\pm1/2,\pm1/2)$. If we rotate and scale the 24-cell so that each coordinate has absolute value at most 1 in such a way that the 24-cell has greatest hypervolume, how many vertices (x,y,z,w) of the 24-cell satisfy $x=1$?

Proposed by Joshua Wolford

Solution: Let's apply an algebraic transformation to the coordinates of the 24-cell. Consider the mapping that takes any point (x,y,z,w) and pairs up the coordinates like this:

$$(x,y,z,w)\to(x+y,x-y,z+w,z-w)=\sqrt{2}\left(\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}x-\frac{1}{\sqrt{2}}y,\frac{1}{\sqrt{2}}z+\frac{1}{\sqrt{2}}w,\frac{1}{\sqrt{2}}z-\frac{1}{\sqrt{2}}w\right)$$

Let's see where the original 24 vertices end up under this mapping:

(a) The 8 permutations of $(\pm1,0,0,0)$ map to points like $(\pm1,\pm1,0,0)$ and $(0,0,\pm1,\pm1)$.

(b) The 16 vertices of the form $(\pm1/2,\pm1/2,\pm1/2,\pm1/2)$ map to points where the pairs sum or subtract to 0 or ±1. For example, $(1/2,1/2,1/2,-1/2)$ maps to $(1,0,0,1)$, and $(1/2,-1/2,1/2,1/2)$ maps to $(0,1,1,0)$.

If you map all 24 vertices, you get exactly the 24 permutations of $(\pm1,\pm1,0,0)$.

Note that if you remove the $\sqrt{2}$, the first two coordinates are exactly what you get from a rotation by $45°$ in the $x-y$ plane, and the same for the last two coordinates and the $z-w$ plane. Because this transformation is just a rigid geometric rotation followed by a uniform scaling (it multiplies all distances by $\sqrt{2}$), the 24 permutations of $(\pm1,\pm1,0,0)$ represent a perfectly valid, scaled and rotated 24-cell.

Notice that in this new coordinate representation, the maximum absolute value of any coordinate is exactly 1. This means the 24-cell fits perfectly flush inside the bounding hypercube $[-1,1]^4$.

To prove this configuration guarantees the *greatest* possible hypervolume, we can look at the "thickness" of the 24-cell. Let's find the center of the facet ($3d$ "face") that touches the $x=1$ wall. The vertices on this wall are those with an x-coordinate of 1. There are exactly 6 such vertices: $(1,\pm1,0,0)$, $(1,0,\pm1,0)$, and $(1,0,0,\pm1)$. The average of these 6 coordinates (the centroid of the facet) is exactly $(1,0,0,0)$.

Because the center of this facet is exactly at a distance of 1 from the origin (which is the closest point of approach), this 24-cell tightly wraps an inscribed hypersphere of radius 1. The bounding hypercube $[-1,1]^4$ also contains an inscribed hypersphere of radius 1. If we were to scale the 24-cell to be slightly larger even rotating it would not save it–its inscribed hypersphere would have a radius strictly greater than 1, meaning the 24-cell would inevitably poke outside the hypercube! Thus, this configuration is strictly maximal.

Since we are in the maximal configuration, the question simply asks how many vertices satisfy $x = 1$. These are exactly the vertices we just found on that boundary wall: $(1, \pm 1, 0, 0)$, $(1, 0, \pm 1, 0)$, and $(1, 0, 0, \pm 1)$.

There are 3 positions for the ± 1 coordinate, and 2 choices for its sign, yielding $3 \times 2 = \boxed{6}$ vertices.

15. Let x be the unique real number which satisfies

$$x = \frac{\sqrt{2}}{\sqrt{\sqrt{x+1}+\sqrt{x}} - \sqrt{\sqrt{x}+\sqrt{x-1}}}.$$

Find $\lfloor x \rfloor$.

Proposed by Daniel Carter

Solution: We begin by rationalizing the denominator of the right-hand side. Multiplying the numerator and denominator by $\sqrt{\sqrt{x+1}+\sqrt{x}} + \sqrt{\sqrt{x}+\sqrt{x-1}}$, we get:

$$x = \frac{\sqrt{2}\left(\sqrt{\sqrt{x+1}+\sqrt{x}} + \sqrt{\sqrt{x}+\sqrt{x-1}}\right)}{\sqrt{x+1}-\sqrt{x-1}}$$

Rationalizing the new denominator by multiplying the numerator and denominator by $\sqrt{x+1}+\sqrt{x-1}$ yields:

$$x = \frac{\sqrt{2}}{2}\left(\sqrt{\sqrt{x+1}+\sqrt{x}} + \sqrt{\sqrt{x}+\sqrt{x-1}}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)$$

Note that the presence of $\sqrt{x-1}$ requires $x \geq 1$.

To establish the upper bound ($x < 256$), one can show that for any positive $A \neq B$, $\sqrt{A}+\sqrt{B} < 2\sqrt{\frac{A+B}{2}}$. Applying this to the factors in our equation:

$$\sqrt{x+1}+\sqrt{x-1} < 2\sqrt{\frac{2x}{2}} = 2\sqrt{x}$$

Next, we apply this same property to the other factor:

$$\sqrt{\sqrt{x+1}+\sqrt{x}} + \sqrt{\sqrt{x}+\sqrt{x-1}} < 2\sqrt{\frac{\sqrt{x+1}+2\sqrt{x}+\sqrt{x-1}}{2}} < 2\sqrt{\frac{4\sqrt{x}}{2}} = 2\sqrt{2}x^{1/4}$$

Now, multiply these upper bounds back into our rationalized equation for x:

$$x < \frac{\sqrt{2}}{2}\left(2\sqrt{2}x^{1/4}\right)\left(2\sqrt{x}\right) = 4x^{3/4}$$

Dividing both sides by $x^{3/4}$ (which is valid since $x \geq 1$), we get:

$$x^{1/4} < 4 \implies x < 256$$

To find the lower bound ($x > 255$), we use the fact that for any $z \in (0,1)$, $\sqrt{1-z} > 1-z$. This means $\sqrt{x^2-1} = x\sqrt{1-\frac{1}{x^2}} > x\left(1-\frac{1}{x^2}\right) = x - \frac{1}{x}$.
Using this to bound our first factor from below:

$$\left(\sqrt{x+1}+\sqrt{x-1}\right)^2 = 2x + 2\sqrt{x^2-1} > 4x - \frac{2}{x} = 4x\left(1-\frac{1}{2x^2}\right)$$

Taking the square root and applying the $\sqrt{1-z} > 1-z$ trick again:

$$\sqrt{x+1}+\sqrt{x-1} > 2\sqrt{x}\sqrt{1-\frac{1}{2x^2}} > 2\sqrt{x}\left(1-\frac{1}{2x^2}\right) = 2\sqrt{x} - \frac{1}{x^{3/2}}$$

Now, let $u = \sqrt{x+1}+\sqrt{x}$ and $v = \sqrt{x}+\sqrt{x-1}$. We want a lower bound for $(\sqrt{u}+\sqrt{v})^2 = u+v+2\sqrt{uv}$. First, $u+v = 2\sqrt{x}+\sqrt{x+1}+\sqrt{x-1} > 4\sqrt{x} - \frac{1}{x^{3/2}}$. Second, $uv = x + \sqrt{x^2-1} + \sqrt{x^2+x} + \sqrt{x^2-x}$.
One can show that $(\sqrt{1+t}+\sqrt{1-t})^2 = 2+2\sqrt{1-t^2} > 4-2t^2 = 4(1-\frac{t^2}{2})$. Taking the square root gives $\sqrt{1+t}+\sqrt{1-t} > 2\sqrt{1-\frac{t^2}{2}} > 2-t^2$. Letting $t = 1/x$, we get $\sqrt{x^2+x}+\sqrt{x^2-x} > x\left(2-\frac{1}{x^2}\right) = 2x - \frac{1}{x}$. Combining our terms for uv:

$$uv > x + \left(x - \frac{1}{x}\right) + \left(2x - \frac{1}{x}\right) = 4x - \frac{2}{x} = 4x\left(1-\frac{1}{2x^2}\right)$$

$$2\sqrt{uv} > 2\sqrt{4x}\sqrt{1-\frac{1}{2x^2}} > 4\sqrt{x}\left(1-\frac{1}{2x^2}\right) = 4\sqrt{x} - \frac{2}{x^{3/2}}$$

Adding $u+v$ and $2\sqrt{uv}$ gives our lower bound for $(\sqrt{u}+\sqrt{v})^2$:

$$(\sqrt{u}+\sqrt{v})^2 > \left(4\sqrt{x} - \frac{1}{x^{3/2}}\right) + \left(4\sqrt{x} - \frac{2}{x^{3/2}}\right) = 8\sqrt{x} - \frac{3}{x^{3/2}} = 8\sqrt{x}\left(1-\frac{3}{8x^2}\right)$$

$$\sqrt{u}+\sqrt{v} > \sqrt{8\sqrt{x}}\left(1-\frac{3}{8x^2}\right) = 2\sqrt{2}x^{1/4} - \frac{3\sqrt{2}}{4x^{7/4}}$$

Finally, multiply the lower bounds of the two factors back into the equation for x:

$$x > \frac{\sqrt{2}}{2}\left(2\sqrt{2}x^{1/4} - \frac{3\sqrt{2}}{4x^{7/4}}\right)\left(2\sqrt{x} - \frac{1}{x^{3/2}}\right)$$

$$x > \left(2x^{1/4} - \frac{3}{4x^{7/4}}\right)\left(2\sqrt{x} - \frac{1}{x^{3/2}}\right) = 4x^{3/4} - \frac{2}{x^{5/4}} - \frac{3}{2x^{5/4}} + \frac{3}{4x^{13/4}} > 4x^{3/4} - \frac{7}{2x^{5/4}}$$

Dividing by $x^{3/4}$ leaves us with:

$$x^{1/4} > 4 - \frac{7}{2x^2}$$

Assume for the sake of contradiction that $x \le 255$. Since $x \ge 1$, the function $h(x) = x^{1/4} + \frac{7}{2x^2}$ is strictly increasing. Thus, its maximum on the interval $[1, 255]$ occurs at $x = 255$. One can show that $255^{1/4} = 256^{1/4}\left(1-\frac{1}{256}\right)^{1/4} < 4\left(1-\frac{1}{1024}\right) = 4 - \frac{1}{256} \approx 3.9961$. Additionally, $\frac{7}{2(255)^2} = \frac{7}{130050} < 0.0001$. Therefore, $255^{1/4} + \frac{7}{2(255)^2} < 3.9962 < 4$. This contradicts the strict inequality $x^{1/4} + \frac{7}{2x^2} > 4$. Therefore, it must be true that $x > 255$.

Since $255 < x < 256$, we have $\lfloor x \rfloor = \boxed{255}$.