Girls in Math

Team: _______________________ **Team ID:** ___________

2024 Team Round

1. ________
2. ________
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11. ________
12. ________

— 45 minutes
— no calculators
— positive integer answers

1. To get from HLH13 to HLH17, Tim the Turtle can either walk 500 feet north and then 1200 feet west, or descend x feet into a tunnel, walk the distance between the two buildings in a straight line, and then ascend x feet. The two paths turn out to take the same time. If Tim descends at half the speed as compared to walking, how deep below ground is the tunnel, in feet?

2. Jane is jogging down a street at 5 miles per hour, leaving a trail of breadcrumbs. She begins next to the Hudson River and ends 10 miles away. Tina the Speedy Turtle begins at the same point, and runs at 20 miles per hour whenever there are breadcrumbs, and only 15 miles per hour without. How long of a head-start, in minutes, should Jane get so that Tina reaches the destination 40 minutes ahead of Jane?

3. Gim, a variant of Nim, is played by two players. On each player's turn, they flip a fair coin. On heads, they add 1 to the commutative sum, and on tails, they add two. If the sum exceeds 6 on their turn, they win. The probability that the first player wins is $\frac{n}{m}$ in simplest form. Find $n+m$.

4. Define $f(x) = 2 + \dfrac{(3x)!(4x)!}{13^{13}} + 5^x + x^6$. What is the smallest positive integer x such that 13 divides $f(x)$?

5. In Brick Roll, a 2 by 1 by 1 rectangular prism starts with a square face on the ground. A red dot is placed at the center of the top face. Define a "roll" as a rotation over one of the brick's edges, where the pivot edge does not slip and the brick lands on a new face. The "direction" of a roll corresponds to the edge which it is pushed over (i.e. the direction towards which it moves). After the brick is rolled north, east, and north (in that order), the total distance the red dot travels along its trajectory can be represented as a fraction in the form $\frac{a+\sqrt{b}+\sqrt{c}}{d}\pi$ such that b and c are not perfect squares and $\gcd(a,d)=1$. Find $a+b+c+d$.

6. A $3\times 3\times 3$ cube is divided into 27 unit cubes. Remove 11 of these cubes such that the remaining cubes form one connected component (two cubes are only connected if they share a face). What is the maximum possible surface area remaining?

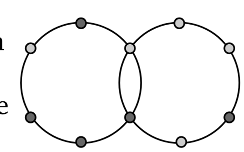
7. Find the number of sequences $a_1, a_2, \dots, a_{20}$ such that $a_1 = 10$, $a_{20} = 37$, and

$$|a_n - a_{n-1}| = 2 - \frac{\left|a_{n-1} - \frac{37}{2}\right|}{\frac{37}{2} - a_{n-1}}$$

8. There are 10 cakes that Cindy and Katherine will divide them amongst themselves in 10 rounds. In the ith round, Cindy cuts the ith cake into two pieces, and Katherine then either chooses to take or pass. If she chooses to take, she first picks a piece for herself, and Cindy gets the other. If she chooses to pass, Cindy first picks a piece, and then Katherine gets the other. Katherine must pass one round during the game. If both people play optimally, maximizing the amount of cake they get, the number of cakes Katherine ends up with is $\frac{m}{n}$ in simplest form. Find $m+n$.

9. Let S be the set of all polynomials of the form $(n+1)x^n + nx^{n-1} + \ldots + 2x + 1$ for some positive integer n, and let R be the set of all complex roots of polynomials in S. Let a and b be real numbers such that $a \le |r| \le b$ for all elements r in R. The minimum possible value of $b-a$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10. 5 red marbles and 5 blue marbles are randomly arranged to form two adjacent circles, such that each circle has 6 marbles (and 2 marbles are shared by each circle). An example configuration is shown to the right. Let a "move" be any rotation of one of the circles, such that each marble in the circle is moved by one spot. Let B represent the number of blue marbles in the righthand circle (including the shared marbles). Barbara wants to maximize B, and conducts two moves after seeing the initial random arrangement. Assuming she acts optimally, the expected value of B after Barbara makes her moves is $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m+n$.



11. The area of the hexagon in the complex plane whose vertices are the roots of the polynomial

$$ix^6 + \frac{37}{64}\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)x^3 + \frac{27}{64}$$

can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are integers, a and c are relatively prime, and b is not divisible by the square of any prime. Find $a+b+c$.

12. There is a 1000×1000 grid of cats of distinct sizes. Each cat feels safe if they are adjacent to exactly one cat that's bigger than it. What is the maximum number of cats that feel safe? (Cats are only adjacent if they are horizontally or vertically adjacent.)