Girls in Math 2026 Individual Round Solutions

Yale Math Competitions

February 2026

1. For any real numbers a, b, define $a \triangleright b = 2a - b$. Evaluate $(311 \triangleright 131) + (131 \triangleright 113) + (113 \triangleright 311)$.

   Proposed by: Aron Fekete

   Answer: $\boxed{555}$

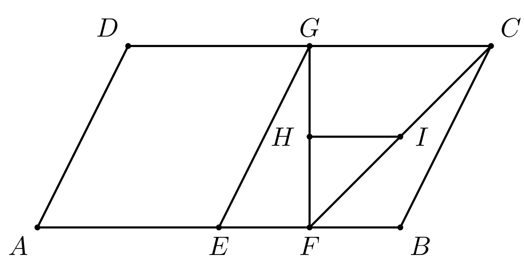
   Solution: In general, $(x \triangleright y) + (y \triangleright z) + (z \triangleright x) = (2x + 2y + 2z) - (x + y + z) = x + y + z$, so the answer is $311 + 131 + 113 = \boxed{555}$.

2. Parallelogram $ABCD$ has area 128. E, F, G, H and I are the midpoints of $\overline{AB}, \overline{BE}, \overline{CD}, \overline{FG}$, and $\overline{CF}$ respectively. What is the area of quadrilateral $HICG$?

   Proposed by: Aron Fekete

   Answer: $\boxed{24}$

   Solution:



   Let $[P]$ denote the area of polygon P. We have that $[EBCG] = \frac{128}{2} = 64$, so the triangle with the same base and height will give $[FCG] = 32$. Since FH and FI are half-lengths, by similar triangles we have that $[FHI] = 8$, so we subtract to obtain our desired answer of $[HICG] = 32 - 8 = \boxed{24}$.

3. Brenda has tasked you with buying the necessary materials to build the wire frame of a table. Each inch of wire costs you \$4. The wire will be used for the four legs of the table, as well as the four edges of the tabletop. The volume contained by the tabletop will be a rectangular

prism, and Brenda wants this volume to be at least 500 in^3. What's the minimum price required to buy the necessary materials?

Proposed by: Shane Lee

Answer: $\boxed{240}$

Solution: If x, y, z are the length, height, and width of the table, respectively, we want to minimize $4(2x + 4y + 2z)$. By AM-GM, $x + 2y + z \geq \sqrt[3]{2xyz} = 10$. We achieve the minimum when $x = 2y = z$, so we should set $x = z = 10$ and $y = 5$ to get a total cost of \$$\boxed{240}$.

4. Find the sum of all real values of x that satisfy the equation $\log_3(x + 5) - 2\log_9(x - 1) = 1$.

   Proposed by: Dieter Yang

   Answer: $\boxed{4}$

   Solution: Observe that $\log_3(x+5) = \log_9(x^2+10x+25)$ and $2\log_9(x-1) = \log_9(x^2-2x+1)$. Therefore we have that $\log_3(x+5)-2\log_9(x-1) = \log_9\left(\frac{x^2+10x+25}{x^2-2x+1}\right) = 1$, which implies that $x^2 + 10x + 25 = 9(x^2 - 2x + 1)$ or $8x^2 - 28x - 16 = 0$, which has roots $x = 4, -\frac{1}{2}$. However, note that $-\frac{1}{2}$ is extraneous, so the only solution is $x = \boxed{4}$.

5. For a finite set of integers A, we define the difference set $A - A := \{a_i - a_j \mid a_i, a_j \in A\}$ to be the set of all differences of two (possibly not distinct) elements of A. Let N be the minimum of $|A - A|$ over all subsets $A \subset \mathbb{Z}$ with $|A| = 99$, and let M be the maximum of $|A - A|$ over all subsets $A \subset \mathbb{Z}$ with $|A| = 99$. Find $N + M$.

   Proposed by: Henry Greenwold

   Answer: $\boxed{9900}$

   Solution: For subsets $|A| \subset \mathbb{Z}$ with $|A| = k \geq 2$, the minimum of $|A - A|$ is achieved when A is an arithmetic progression. We will show this later by lower bounding the minimum of $|A - A|$ by $2k - 1$. Taking $A = \{0, 1, \ldots, k - 1\}$, we observe $A - A\{-(k - 1), \ldots, k - 1\} \implies |A - A| = 2k - 1$. This shows that the arithmetic sequence actually obtains the lower bound and hence the arithmetic sequence is indeed the minimum. Thus, $N = 2k - 1$. Observe that the lower bound of $|A - A|$ is always $2k - 1$ because we are guaranteed $2 \times (k - 1)$ different values from $a_1 - a_k, a_2 - a_k, \ldots, a_{k-1} - a_k$, and the times two because we can also obtain the negatives of all of these values. Finally, we can also obtain 0 from $a_1 - a_1$. This gives a lower bound of at least $2k - 1$ differences. The maximum of A is achieved when every pair of distinct elements $a_i, a_j \in A$ yields a distinct difference $a_i - a_j$. We can achieve this by taking $A = \{1, k, k^2, \ldots, k^{k-1}\}$. In this case $A - A$ contains the $k(k - 1)$ distinct differences $a_i - a_j$, as well as $0 = a_i - a_i$. It follows that $M = |A - A| = k(k - 1) + 1 = k^2 - k + 1$. We then have $N + M = (2k - 1) + (k^2 - k + 1) = k^2 + k = k(k + 1)$. Plugging in $k = 99$, we have $N + M = 99(100) = \boxed{9900}$

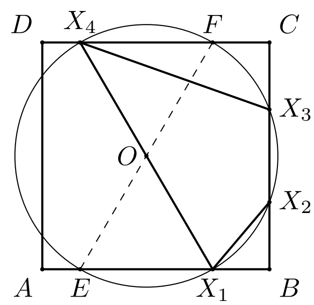
6. $ABCD$ is a square with side length 40. Points E, F are chosen on $\overline{AB}, \overline{CD}$, respectively, such

that $AE = 1$ and $CF = 9$. Point X_1 is chosen on $\overline{AB}$, points X_2 and X_3 are chosen on $\overline{BC}$, and point X_4 is chosen on $\overline{CD}$ so that $\angle EX_iF = 90^\circ$ for $i = 1, 2, 3, 4$. Given that the polygon $X_1X_2X_3X_4$ is non-degenerate, find its area.

Proposed by: Henry Burton

Answer: $\boxed{648}$

Solution:



Let circle ω have center O and diameter $\overline{EF}$. It has diameter $\sqrt{(39-9)^2+40^2} = 50$, so its radius is 25. The right angles condition just means that each X_i is on ω. Let M be the midpoint of $\overline{BC}$. We have that $BE = 39$ and $CF = 9$, so $MO = \frac{39+9}{2} = 24$, and using the Pythagorean Theorem, $MX_2 = MX_3 = 7$. Additionally, $BX_1 = CF = 9$, and $CX_4 = BE = 39$. Using the fact that $[X_1X_2X_3X_4] = [X_1BCX_2] - [X_1BX_2] - [X_3CX_4]$, the desired area is $40 \cdot 24 - \frac{1}{2} \cdot 13 \cdot 9 - \frac{1}{2} \cdot 13 \cdot 39 = \boxed{648}$

7. Let N be the number of functions f from the integers $\{1, 2, \cdots, 25\}$ to integers in $\{1, 2, \cdots, 25\}$ such that $f(i) \neq f(j)$ for all $i \neq j$ and $f(i) + j \equiv i + f(j) \pmod 5$ for all $1 \leq i < j \leq 25$. How many divisors does N have?

   Proposed by: Aron Fekete

   Answer: $\boxed{672}$

   Solution: Rearranging $i + f(j) \equiv f(i) + j \pmod 5$, we get that $f(i) - i \equiv f(j) - j \pmod 5$. In other words, $f(i) - i \equiv c \pmod 5$ is constant. For each choice of $c \in \{0, 1, 2, 3, 4\}$, we have 5 elements for each modular class being mapped to five elements of some other modular class, giving $(5!)^5$ possible permutations for each c, and since there are 5 choice of c, $N = 5(5!)^5 = 2^{15}3^55^6$, so $\tau(N) = 16 \cdot 6 \cdot 7 = \boxed{672}$.

8. For a positive integer n, let $\Omega(n)$ denote the number of prime factors of n, counted with multiplicity. What is the smallest positive integer n for which $2^{\Omega(n)} < n < 2.2^{\Omega(n)}$?

   Proposed by: Aron Fekete

   Answer: $\boxed{48}$

Solution: First, we show that n will have the form $3 \cdot 2^k$ for some k. Observe that $2^{\Omega(n)}$ is equivalent to taking n and replacing all its prime factors with 2, which can only decrease it, so the inequality $2^{\Omega(n)} < n$ just implies that n is ont a power of 2. Next, if n had any prime factors other than 3, they can all be replaced with 3, which gives the same $\Omega(n)$ but a smaller n, so such an n would be suboptimal. If n had more than one power of 3, we could replace all but one with 2, and this would also decrease n without changing $\Omega(n)$. Now, it suffices to find the minimum k such that $2^{k+1} < 3 \cdot 2^k < 2.2^{k+1}$. The left inequality is trivially true, and so we need $\frac{3}{2.2} < 1.1^k$. We can calculate that $k = 4$ is the smallest that works, so the answer is $3 \cdot 2^4 = \boxed{48}$.

9. Let S be the set of prime numbers less than 2026. For $p \neq q$, p and q are considered `connected` if $pq \equiv k \mod 2026$, for some integer k with $0 < k < 2026$. The degree of p is the number of $q \in S$ such that p and q are connected. For what value of k is the maximum degree of an element in S maximized?

   Proposed by: Nathan Abebe

   Answer: $\boxed{1013}$

   Solution: If p is connected to q and r, then $pq \equiv pr \pmod{2026}$, so $p(q - r)$ is a multiple of 2026, but since all our primes are less than 2026, $|q - r| < 2026$. Therefore, we must have that p is a divisor of 2026 in order for p to have degree more than 1. Therefore the maximum degree will either be at 2 or 1013. $2 \cdot 3, 2 \cdot 5, 2 \cdot 7$ are all different mod 2026, so 2 can have degree at most $|S| - 2$. Our maximum occurs with $k = 1013$, which is connected to all odd primes (so $|S| - 1$), and so $k = \boxed{1013}$ is optimal.

10. Triangle ABC is inscribed in a circle centered at O with radius 5. It is given that $BC = 6$ and the three distances from O to the sides of ABC form the side lengths of a right triangle. The area of ABC can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

    Proposed by: Andrew Brahms

    Answer: $\boxed{79}$

    Solution: Let the sides of the triangle have lengths a, b, c with $a = BC = 6$. Let d_a, d_b, d_c be the distances from O to the sides a, b, c. Then we are given that the radius of the circumcircle of $\triangle ABC$ is $R = 5$. We can see that $\triangle OBC$ is isosceles with $OB = OC = 5$ and $BC = 6$, and that d_a is the height (from O) of this triangle. Simply using the Pythagorean Theorem, we see that

$$d_a^2 = OB^2 - \left(\frac{BC}{2}\right)^2 = 5^2 - 3^2 = 16,$$

    so $d_a = 4$. Also, we note the well-known identity that for a triangle with vertices A, B, C and circumradius R, we have $d_a = R|\cos A|$, $d_b = R|\cos B|$, $d_c = R|\cos C|$. We just computed that $d_a = 4$, so we see that $|\cos A| = \frac{4}{5}$. We are given that the lengths $4, 5|\cos B, 5|\cos C|$

form a right triangle. If 4 is not the hypotenuse of this triangle, then we see that

$$|(5\cos B)^2 - (5\cos C)^2| = 25|\cos^2 B - \cos^2 C| = 4^2 = 16.$$

But $\cos^2 B - \cos^2 C = \sin(B+C)\sin(C-B)$ by simply expanding the right-hand side. Then $|\sin(B+C)\sin(C-B)| = \frac{16}{25}$, but we see that $\sin(B+C) = \sin(\pi - B - C) = \sin(A) = \frac{3}{5}$ as $|\cos A| = \frac{4}{5}$. So we see that $\frac{3}{5}|\sin(C-B)| = \frac{16}{25}$, and hence $|\sin(C-B)| = \frac{16}{15}$. But $-1 \le \sin(C-B) \le 1$, absurd! Hence 4 must be the hypotenuse of the right triangle, and we see then that

$$25\cos^2 B + 25\cos^2 C = 16 \iff \cos^2 B + \cos^2 C = \frac{16}{25}.$$

We can also see, noting the usual identities $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, then $a = 6$ (given) with $b = 10\sin B$, $c = 10\sin C$. Also note that $\cos A = \pm\frac{4}{5}$ by our earlier work. Then by Law of Cosines, we have

$$36 = 100\sin^2 B + 100\sin^2 C - 200\sin B\sin C\cdot\cos A = 100\sin^2 B + 100\sin^2 C \pm 160\sin B\sin C.$$

As $\cos^2 B + \cos^2 C = \frac{16}{25}$, we can see that

$$\sin^2 B + \sin^2 C = (1-\cos^2 B) + (1-\cos^2 C) = 2 - \frac{16}{25} = \frac{34}{25}.$$

Then substituting, we find

$$36 = 100\cdot\left(\frac{34}{25}\right) \pm 160\sin B\sin C \iff \pm 160\sin B\sin C = -100,$$

and hence $\sin B\sin C = \pm\frac{5}{8}$. But $\sin B\sin C$ must be positive as $\sin B$ and $\sin C$ are, so $\sin B\sin C = \frac{5}{8}$. Then we can finally use the area formula

$$[ABC] = 2R^2\sin A\sin B\sin C$$

where $[ABC]$ denotes the area of $\triangle ABC$. Plugging in all our values, we see that

$$[ABC] = 2\cdot 5^2\cdot\frac{3}{5}\cdot\frac{5}{8} = \frac{75}{4},$$

and hence the answer is $75 + 4 = \boxed{79}$, as desired.

11. Define a `move` to be an operation on a binary string where we select four consecutive digits 0110 in the string and replace it with 1001. What is the maximal number moves can we perform to a length 100 binary string before no further moves are possible, across all 2^{100} starting strings?

    Proposed by: Owen Zhang

    Answer: $\boxed{625}$

    Solution: We first show the following lemma: Once a move is made at index i (acting on indices $i, i+1, i+2, i+3$), then no moves can ever be made at indices $i-1$ and $i+1$. It

suffices to show the $i+1$ case by symmetry. Let m be the sequence of moves where m_0 acts on i and m_2 acts on $i+1$ in the minimum number of moves afterward (across all i). Observe that after m_0, the string (from i to $i+5$) is 1001??. At some point, we need the string to be ?0110? for m_2, so we need a move to set $i+2$ from 0 to 1. Let m_1 be the last move to change $i+2$. Then, m_1 must have acted on either $i+2$ or $i-1$.
Suppose the first case (m_1 at $i+2$): then, after m_1, our string looks like ??1001. But for m_2 to be valid, we need $i+3$ to be set to 1. The only moves that can do so are either at i or $i+3$. By our minimality assumption, it cannot be i, since this will change $i+2$. However, it also cannot be $i+3$, since then we made the sequence of moves $m_1 = i+2$, followed by a move at $i+3$ (which is an adjacent pair) which happened before the m_0, m_2 pair, contradicting minimality.
Suppose the second case (m_1 at $i-1$): then, before m_1, we need something to set $i+1$ to 1, which is either a move at $i+1$ or $i-2$. However, by similar reasoning, $i+1$ is not valid since this means we made moves at $i, i+1$ before our m_0, m_2 pair, contradicting minimality. Likewise, we cannot move at $i-2$, since then we would need a move to set i to 1, which requires a move at i or $i-3$, either way which gives the same minimality contradiction.

Now, we can bound the number of moves that can happen at a given index. Without loss of generality, suppose i is even (the odd case follows similarly). Observe that a move sends 0110 to 1001, which means that the 1 at the even index moves left and the 1 at the odd index moves right, and vice versa for the 0s. Therefore, the maximum number of moves that can happen at this index is upper bounded by the number of 1s at indices with the same parity of i to the right of $i+2$ and the number of 1s at indices with the opposite parity to the left of $i+1$, as well as the number of 0s with opposite parity to the left, and the same parity to the right. Essentially, moving at an index i causes all 1s and 0s of a certain parity to move only in the same direction across the $(i+1)$-$(i+2)$ border. The only possible moves that can undo these moves occur at $i+1$ and $i-1$, however, the lemma demonstrates this is impossible.
Now, we calculate, that at index i, there are $i+1$ indices from $1, \cdots, i+1$ to the left of the boundary, and $99-i$ indices from $i+2, \cdots, 100$ to the right of the boundary. Since we are bounded by both the number of 0s and 1s of a certain parity on each side, our total number of moves is at most $\min\left(\left\lfloor \frac{i+1}{2} \right\rfloor, \left\lfloor \frac{99-i}{2} \right\rfloor\right)$ for $i = 1, 2, \cdots, 97$. This evaluates to $1, 1, 2, 2, \cdots, 24, 24, 25, 24, 24, \cdots, 2, 2, 1, 1$, respectively. However, recall that if we make a move at a given index, we can make no moves at adjacent indices. Therefore, we can never choose adjacent elements in this list. A simple inductive argument demonstrates that we should take all odd i, which gives $1, 2, \cdots, 25, \cdots, 2, 1$ for a sum of 625.
Indeed, we show that 625 is achievable. Write 01 as A and 10 as B. Observe every move swaps AB with BA, and if we start with the string $AA \cdots AB \cdots BB$, then we indeed can make 25^2 swaps with every pair of A, B, which gives the desired $\boxed{625}$.

12. Twenty points are evenly spaced on a unit circle. Three distinct points A, B, and C are chosen from them uniformly at random. The expected value of $AB^2 + BC^2 + CA^2$ can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

    Proposed by: Patrick Lu

    Answer: $\boxed{139}$

    Solution: Let ω be a primitive 20th root of unity. Without loss of generality, the twenty points are $1, \omega, \ldots, \omega^{19}$. Then

$$\begin{aligned}
\mathbb{E}[AB^2 + BC^2 + CA^2] &= \frac{1}{20 \cdot 19 \cdot 18} \sum_{\substack{0 \le i,j,k \le 19, \\ i \ne j \ne k}} |\omega^i - \omega^j|^2 + |\omega^j - \omega^k|^2 + |\omega^k - \omega^i|^2 \\
&= \frac{1}{20 \cdot 19 \cdot 18} \sum_{\substack{0 \le i,j,k \le 19, \\ i \ne j \ne k}} 6 - \omega^{i-j} - \omega^{j-i} - \omega^{j-k} - \omega^{k-j} - \omega^{k-i} - \omega^{i-k} \\
&= 6 - \frac{1}{20 \cdot 19 \cdot 18} \sum_{\substack{0 \le i,j,k \le 19, \\ i \ne j \ne k}} \omega^{i-j} \\
&= 6 - \frac{6}{20 \cdot 19} \sum_{\substack{0 \le i,j \le 19, \\ i \ne j}} \omega^{i-j} \\
&= 6 - \frac{6}{20 \cdot 19} \cdot 20 \cdot (-1) \\
&= \frac{120}{19},
\end{aligned}$$

giving an answer of $\boxed{139}$.