Girls in Math 2026 Team Round Solutions

Yale Math Competitions

February 2026

1. If a, b, and c are positive real numbers satisfying $(a+b)(b+c) = 9, (b+c)(c+a) = 12, (c+a)(a+b) = 27$, then $a+b+c$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

   Proposed by: Patrick Lu

   Answer: $\boxed{29}$

   Solution: Dividing the second equation by the first equation (this is possible since our values are positive), we get $\frac{27}{9} = 3 = \frac{c+a}{b+c}$, so substituting this into the second equation $(b+c)(c+a) = 12 \implies 3(b+c)^2 = 12 \implies b+c = 2$ since a, b, c are positive. Therefore $c+a = 6$, and from the first equation we get $a+b = \frac{9}{b+c} = \frac{9}{2}$. $a+b+c = \frac{(a+b)+(b+c)+(c+a)}{2} = \frac{2+6+\frac{9}{2}}{2} = \frac{25}{4}$, so the answer is $\boxed{29}$.

2. Compute $\displaystyle\sum_{n=1}^{2026} \left\lfloor \log_2 \left\lceil \frac{n}{2} \right\rceil \right\rfloor$.

   Proposed by: Madhav Krishnaswamy

   Answer: $\boxed{16208}$

   Solution: We see that $\sum_{n=1}^{2026} \left\lfloor \log_2 \left\lceil \frac{n}{2} \right\rceil \right\rfloor = \lfloor \log_2 1 \rfloor + \lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 2 \rfloor + \ldots + \lfloor \log_2 1013 \rfloor + \lfloor \log_2 1013 \rfloor$. So the sum is equal to $2 \cdot \sum_{n=1}^{1013} \lfloor \log_2 n \rfloor = 2(1 \cdot 0 + 2 \cdot 1 + 4 \cdot 2 + \ldots + 256 \cdot 8 + 502 \cdot 9) = \boxed{16208}$

3. Beatrice writes the sequence $2, 1, 3, 4, \ldots,$ where each term after the second term is the sum of the previous two terms. Find the remainder when the product of the first 2026 numbers is divided by 20.

   Proposed by: Shane Lee

   Answer: $\boxed{12}$

   Solution: The sequence is periodic with period 12 when taken mod 20: 2, 1, 3, 4, 7, 11, 18, 9, 7, 16, 3, 19. The product of these numbers is 4 mod 20. Since $2026 = 12 \cdot 168 + 10$, the answer is 4^{168} times the product of the first 10 numbers. 4^{168} is 16 mod 20 and the product of the first 10 numbers is 12 mod 20, thus the answer is $16 * 12 \equiv 12$.

Alternatively, the sequence has period 4 when taken mod 5: 2, 1, 3, 4. The product is 4, so the product after the first 2024 numbers is 1. The next 2 makes the product 2 mod 5, and because the product is 0 mod 4 it must be $\boxed{12}$ mod 20 by the Chinese Remainder Theorem.

4. In an 8-person tournament, there are 4 first-round matches, each with a skilled player vs. a unskilled player. Skilled players have a $\frac{3}{4}$ chance of beating unskilled players. Players of the same type (unskilled vs. unskilled, skilled vs. skilled) have a $\frac{1}{2}$ chance of beating each other. The winner of each round will play the winner of the other game on their side of the bracket in Round 2. Round 3 is the match between the final two players. The probability that the winner of the tournament is unskilled is $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.

   Proposed by: Noah Beckish

   Answer: $\boxed{2233}$

   Solution: At least one unskilled person must win in the first round in order to have a chance of winning the whole tournament. Let p_1 be the event that exactly one unskilled player wins in the first round. Then, the sole unskilled player (one of any of the four unskilled players initially) must win in rounds 2 and 3 to win the tournament. $P(p_1) = 4\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3$. Let p_4 be the event all four unskilled players win in Round 1. This guarantees a unskilled player wins. Thus, $P(p_4) = \left(\frac{1}{4}\right)^4$. For p_3, $P(p_3) = 4\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^3\left(1 - \left(\frac{3}{4}\right)^2\right)$, which discounts the chance of the sole remaining skilled player winning out. Exactly two unskilled players winning is trickier; if they are on the same side of the bracket, then they will play each other. This occurs in two of the six ways to arrange two skilled and two unskilled players. Thus, a unskilled player will make the final round. Let this event be $p_{2,same-side}$. $P(p_{2,same-side}) = 2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3$. In the event the two unskilled players are on opposite sides of the bracket after Round 1, let's call this $p_{2,opp-sides}$, then the bracket simplifies to be a 4-player version of the initial problem. A bad player winning this bracket has probability $1 - \left[\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)\right] = 1 - 54/64 = 10/64 = 5/32$. Thus, $P(p_{2,opp-sides}) = 4\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{10}{64}\right)$. Then, we have that the final probability is just $P(p_1 \vee p_{2,same-side} \vee p_{2,opp-side} \vee p_3 \vee p_4) = 4\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3 + 2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 4\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{10}{64}\right) + 4\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^3\left(1 - \left(\frac{3}{4}\right)^2\right) + \left(\frac{1}{4}\right)^4 = \frac{185}{2048}$. Therefore, $m = 185, n = 2048$, so $m + n = \boxed{2233}$.

5. Let ABC be a triangle with circumcircle Ω. Let D and E be the feet of the altitudes from B and C, respectively. Extend $\overline{DE}$ to intersect Ω at distinct points K and L. Let M be the midpoint of $\overline{KL}$. If $\angle B = 40^\circ$ and $\angle C = 70^\circ$, what is the measure of $\angle BAM$, in degrees?

   Proposed by: Patrick Lu

   Answer: $\boxed{20}$

   Solution: Let F be the foot of the perpendicular from A. Suppose K lies on minor arc $\widehat{AC}$ and L lies on minor arc $\widehat{AB}$. Note that quadrilateral $BCDE$ is cyclic, since $\angle BDC = \angle BEC = 90^\circ$. Thus

$$\widehat{AK} + \widehat{BL} = 2\angle BEL = 2\angle BCD = \widehat{BA} = \widehat{BL} + \widehat{LA},$$

implying $\widehat{AK} = \widehat{LA}$. Thus A is the midpoint of $\widehat{KL}$ and $AM \perp KL$. Thus $\triangle AME \sim \triangle AFC$ and $\angle BAM = \angle CAF = 180° - 90° - \angle C = \boxed{20°}$.

6. Define a sequence of polynomials by $F_1(x) = (x-1)(x-2)\cdots(x-2025)$, $F_2(x) = \left(x - 1^2\right)\left(x - 2^2\right) \cdots \left(x - 2025^2\right)$, and for $n \geq 3$, $F_n = F_{n-1}^2 + F_1^2$ if n is odd and $F_n = F_{n-1}^2 F_2^2$ if n is even. How many distinct real roots does F_{2025} have?

   Proposed by: Noah Ripke

   Answer: $\boxed{45}$

   Solution: Let $A = \{1, \cdots, 2025\}$ and $B = \{1^2, \cdots, 2025^2\}$. By the trivial inequality, $a_2(x)^2 + a_1(x)^2 \geq 0$ and it is exactly 0 when $a_2^2(x) = a_1^2(x) = 0$, thus $a_3(x)$ has roots $A \cap B$. Then $a_4(x) = a_3^2(x)a_2^2(x)$ so $a_4(x) = 0$ when $a_3(x) = 0$ or $a_2(x) = 0$ so $a_4(x)$ has roots $(A \cap B) \cup B = B$. In general, we find that for even n, a_n has roots B and for odd n, a_n has roots $A \cap B$. Since 2025 is odd, we get that a_{2025} has roots $A \cap B = \{1^2, \cdots, 45^2\}$ so the answer is $\boxed{45}$.

7. Cat and Claire are discussing Claire's favorite two-digit number.
   `Cat:` "If you told me the smaller of the two digits of your number and the number of factors of your number, but not which one was which, would I know your number?"
   `Claire:` "No, but if I told you which one was which, you would!"
   `Cat:` "What if you instead told me the larger of the two digits of your number and the number of factors of your number, but not which one was which?"
   `Claire:` "You still wouldn't know, but if I told you which one was which you would!"
   `Cat:` "Aha! Now I know your number!"
   What is Claire's favorite number?

   Proposed by: Benjamin Wu

   Answer: $\boxed{64}$

   Solution: For reference, here are all two-digit numbers, listed by number of factors:

   - 2: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97
   - 3: 25, 49
   - 4: 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95
   - 5: 16, 81
   - 6: 12, 18, 20, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 92, 98, 99
   - 7: 64
   - 8: 24, 30, 40, 42, 54, 56, 66, 70, 78, 88
   - 9: 36

   (Note that we can immediately rule out numbers with 10+ factors or numbers that contain a 0, since Cat will immediately know that that number refers to the number of factors and not one of the digits.)

Observe that if Cat knows the smallest (or largest) digit and the number of factors, she will be able to identify the number. That means within each set, we can rule out numbers that share a smallest/largest digit with another number. This leaves only:

- 3: 25, 49
- 6: 63
- 7: 64
- 8: 54
- 9: 36

Now, we can inspect each option. The number cannot be 25, since if told the pair $5, 3$ in the second prompt, Cat knows there are no numbers with 5 factors with largest digit 3. The number cannot be 49, since if told the pair $9, 4$ in the second prompt, Cat knows there are no numbers with 9 factors with largest digit 4. The number cannot be 63, since if told the pair $6, 6$, Cat immediately knows the number is 63 (since only 63 has 6 factors and has largest digit 6). The number cannot be 54, since if told the pair $4, 8$, Cat would know there are no numbers with 4 factors and smallest digit 8. Likewise, the number cannot be 36, since if told the pair $3, 9$, Cat would know there cannot be a number with 3 factors and smallest digit 9. The only option remaining is $\boxed{64}$, which indeed satisfies the conditions (in the first prompt, Cat knows the number would be either $87, 77$, or 64, and in the second prompt, Cat knows the number would be either $27, 57, 74, 77$ or 64; both sets of possibilities would be uniquely reduced to just 64 if the ordering were given).

8. Equilateral triangle $\triangle ADG$, where the vertices are labeled clockwise, has side length 27. A circle is drawn such that it intersects each side of the equilateral triangle twice. Clockwise from A, label these intersections B, C, E, F, H, and I. If $AI = 6$, $GF = 7$, $DC = 15$, and the other six edges each have integer length, find $BC \cdot EF \cdot HI$.

   Proposed by: Alex Wa

   Answer: $\boxed{128}$

   Solution: Let $DE = y$. By Power of a Point around D, we have $DC \cdot DB = DE \cdot DF$, which implies $15(15 + BC) = DE(DG - FG) = 20 \cdot DE$. Taken modulo 4, then BC must be 1, 5, or 9 (since DE is at most $27 - 7 = 20$). If $BC = 1$, using Power of a Point on A, D, G gives $AB = 11, BC = 1, DE = 12, EF = 8, GH = 5, HI = 16$, which can be verified as the only possibility. This gives $BC \cdot EF \cdot HI = \boxed{128}$.

9. Find the remainder when $\displaystyle\prod_{k=0}^{2026}(k^2+1)$ is divided by 2027.

   Proposed by: Yusuf Sheikh

   Answer: $\boxed{2028}$

Solution: Let $p = 2027$. Say we have a solution (x, y) to $x^2 + y^2 \equiv 1 \pmod{p}$, and further assume that $x \neq -1 \pmod{p}$. Then note that

$$\left(\frac{y}{1+x}\right)^2 + 1 \not\equiv 0 \pmod{p}$$

as $p \equiv 3 \pmod{4}$ and therefore there exists no t with $t^2 \equiv -1 \pmod{p}$. Further, note that any solution to $t^2 + 1 \not\equiv 0 \pmod{p}$ corresponds to the solution $(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ with

$$\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 \equiv 1 \pmod{p}$$

(which can be seen from simply expanding the products). We further have $x \not\equiv -1 \pmod{p}$ as if we did have $x \equiv -1 \pmod{p}$, we would have

$$t^2 - 1 \equiv t^2 + 1 \pmod{p} \iff 2 \equiv 0 \pmod{p},$$

impossible. Thus the solutions to $t^2 + 1 \not\equiv 0 \pmod{p}$ correspond to solutions to $x^2 + y^2 \equiv 1 \pmod{p}$ with $x \neq -1$. Then again, as there exists no t with $t^2 \equiv -1 \pmod{p}$, any t works and we thus have $p = 2027$ solutions. Further, taking $x \equiv -1 \pmod{p}$ gives $y = 0$ and one extra solution, for a total of $\boxed{2028}$.

10. For integers $n, m > 1$, the set of all divisors of n^2 that are NOT divisors of $m^2 - 1$ is the same as the set of divisors of $7n - 10$ that are NOT divisors of $m^2 + m$, and this set is nonempty. What is the sum of all possible values of n?

    Proposed by: Aron Fekete

    Answer: $\boxed{5}$

    Solution: Call this set S, and consider the largest element of S. If n^2 was a divisor of $m^2 - 1$, then S would be empty so n^2 cannot be a divisor of $m^2 - 1$ thus $n^2 \in S$. However, all elements of S are divisors of n^2, so we get that n^2 is the largest element of S. Similarly, we get that $7n - 10$ is also the largest element of S so

    $$n^2 = 7n - 10 \implies n \in \{2, 5\}$$

    If $n = 5, m = 2$, then $m^2 - 1 = 3$ and $m^2 + m = 6$, so $S = \{5, 25\}$ and 5 works. However, we must be careful. If $n = 2$ and m is odd then $m^2 - 1$ is divisible by $4 = n^2$ so S is empty. If $n = 2$ and m is even then $m^2 + m$ is even and hence $2 \notin S$ but also $m^2 - 1$ is odd so $2 \in S$, a contradiction, meaning that $n = 2$ actually can not work. Hence, our sum is $\boxed{5}$.

11. Bella is decorating a new home for her pet bee, Betsy. The home is an 8×8 grid of cells. Bella has four distinct colors of paint to use. A set of four cells is called `colorful` if their centers form the corners of a rectangle with sides parallel to the grid edges, and all four cells are painted different colors. Bella wants to maximize the number of `colorful` sets to please

Betsy. What is the maximum number of colorful rectangles she can achieve?

Proposed by: Owen Zhang

Answer: $\boxed{384}$

Solution: For a given pair of rows i, j, let $a_{i,j}^{c_1,c_2}$ be the number of columns k such that squares $(i,k),(j,k)$ have colors c_1, c_2 in some order. Observe that the number of colorful rectangles is just:

$$\sum_{1 \le i,j, \le 8, i \ne j} a_{i,j}^{1,2} a_{i,j}^{3,4} + a_{i,j}^{1,3} a_{i,j}^{2,4} + a_{i,j}^{1,4} a_{i,j}^{2,3}$$

(essentially, we pick two rows and check how many columns cover all 4 colors). By AM-GM, we have $a_{i,j}^{1,2} a_{i,j}^{3,4} \le \frac{1}{4}(a_{i,j}^{1,2} + a_{i,j}^{3,4})^2$. Observe that $\sum_{i \ne j} a_{i,j}^{1,2} + a_{i,j}^{1,3} + a_{i,j}^{1,4} + a_{i,j}^{2,3} + a_{i,j}^{2,4} + a_{i,j}^{3,4}$ is just the number of pairs of cells in the same column that have different colors. Our goal is to bound this value.

For a given column j, let b_j^c be the number of cells in column j with color c. We have $b_j^1 + b_j^2 + b_j^3 + b_j^4 = 8$. The number of pairs of cells in a column that have different colors is:

$$\begin{aligned} b_j^1 b_j^2 + b_j^1 b_j^3 + b_j^1 b_j^4 + b_j^2 b_j^3 + b_j^2 b_j^4 + b_j^3 b_j^4 &= \frac{1}{2}\left(\left(\sum_c b_j^c\right)^2 - \sum_c (b_j^c)^2\right) \\ &= \frac{64 - \sum_c (b_j^c)^2}{2} \end{aligned}$$

Our goal is to minimize $\sum_c (b_j^c)^2$ subject to $\sum_c b_j^c = 8$, by Cauchy-Schwarz, this is minimized when $b_j^c = 2$, so the number of pairs of cells in a column that has different colors is at most $\frac{64-16}{2} = 24$. Summing across all columns gives $24 \cdot 8 = 192$, so:

$$\sum_{1 \le i,j, \le 8, i \ne j} a_{i,j}^{1,2} + a_{i,j}^{1,3} + a_{i,j}^{1,4} + a_{i,j}^{2,3} + a_{i,j}^{2,4} + a_{i,j}^{3,4} \le 192$$

The number of rectangles is bounded by:

$$\begin{aligned} R &= \sum_{1 \le i,j, \le 8, i \ne j} a_{i,j}^{1,2} a_{i,j}^{3,4} + a_{i,j}^{1,3} a_{i,j}^{2,4} + a_{i,j}^{1,4} a_{i,j}^{2,3} \\ &\le \sum_{1 \le i,j, \le 8, i \ne j} \frac{1}{4}\left((a_{i,j}^{1,2} + a_{i,j}^{3,4})^2 + (a_{i,j}^{1,3} + a_{i,j}^{2,4})^2 + (a_{i,j}^{1,4} + a_{i,j}^{2,3})^2\right) \\ &\le \sum_{1 \le i,j, \le 8, i \ne j} \frac{1}{4}\left(a_{i,j}^{1,2} + a_{i,j}^{1,3} + a_{i,j}^{1,4} + a_{i,j}^{2,3} + a_{i,j}^{2,4} + a_{i,j}^{3,4}\right)^2 \end{aligned}$$

Let $d_{i,j} = a_{i,j}^{1,2} + a_{i,j}^{1,3} + a_{i,j}^{1,4} + a_{i,j}^{2,3} + a_{i,j}^{2,4} + a_{i,j}^{3,4}$, so our goal is to maximize

$$\frac{1}{4} \sum_{1 \le i,j, \le 8, i \ne j} d_{i,j}^2$$

such that

$$\sum_{1 \le i,j, \le 8, i \ne j} d_{i,j} \le 192 \quad \text{and} \quad d_{i,j} \le 8$$

since $d_{i,j}$ is a count of the number of columns with different colors in rows i, j, so it is always at most 8. Then, we have:

$$\frac{1}{4} \sum_{1 \le i,j, \le 8, i \ne j} d_{i,j}^2 \le \frac{1}{4} \sum_{1 \le i,j, \le 8, i \ne j} 8 d_{i,j} \le 384$$

(which is achieved by setting 24 values of $d_{i,j} = 8$ and the rest to 0).

The below configuration does indeed attain 384 (there are 24 choices of subgrids to pick as corners, and each choice has 4^2 choices within each 2×2 subgrid).

1	1	2	2	3	3	4	4
1	1	2	2	3	3	4	4
2	2	1	1	4	4	3	3
2	2	1	1	4	4	3	3
3	3	4	4	1	1	2	2
3	3	4	4	1	1	2	2
4	4	3	3	2	2	1	1
4	4	3	3	2	2	1	1

12. Triangle ABC has incircle ω and incenter I. Let Ω be the circle outside ABC that is tangent to side $\overline{BC}$ and to the extensions of the other two sides, and let J be its center. The circles ω and Ω touch $\overline{BC}$ at D and E, respectively. Suppose lines $\overleftrightarrow{IE}$ and $\overleftrightarrow{JD}$ meet at a point on ω. If ω and Ω have radii 12 and 25, respectively, then the perimeter of $\triangle ABC$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Proposed by: Yibo Zhang

Answer: $\boxed{1763}$

Solution: Let P be the intersection of IE and JD. Note that $IP = ID$, so $\angle DPI = \angle IDP = \angle EJP$, and $\triangle EJP$ is isosceles. By the Pythagorean theorem, we have

$$DE = \sqrt{EI^2 - DI^2} = \sqrt{13^2 - 12^2} = 5,$$

and

$$IJ = \sqrt{(DI + EJ)^2 + DE^2} = \sqrt{37^2 + 5^2}.$$

Let F be the point of tangency between Ω and AB. Note that the perimeter of $\triangle ABC$ is equal to $2AF$, and we can compute AF from IJ as

$$\begin{aligned}
AF &= \sqrt{AJ^2 - JF^2} \\
&= \sqrt{\left(\frac{25}{13}IJ\right)^2 - JF^2} \\
&= \sqrt{25^2\left(\frac{37^2+5^2-13^2}{13^2}\right)} \\
&= \frac{875}{13}.
\end{aligned}$$

Then the perimeter is $2 \cdot \frac{875}{13} = \frac{1750}{13}$, giving an answer of $1750 + 13 = \boxed{1763}$.