BrUMO

Individual Round

Rules

For the individual round, you will have 80 minutes to solve 15 short-answer questions. You are allowed to use writing utensils and scratch paper, but you may not use books, notes, calculators, slide rules, abaci, or any other computational aids. Similarly, you may not use graph paper, rulers, protractors, compasses, architectural tools, or any other drawing aids. You must complete this round individually, and we ask that you leave a seat between you and your neighbors.

Scoring

For each problem you solve correctly, you will be awarded $e^{n/45} + \max(9 - \lfloor \ln N \rfloor, 4)$ points, where n is the problem number and N is the number of contestants who solved the problem. There is no penalty for guessing.

Answer Format

Acceptable answers include integers, fractions, decimals, exponents, factorials, binomial coefficients, trigonometric functions, and radicals. You may also use any basic arithmetic operations, such as addition, subtraction, multiplication, and division, to express your answers. Summation or product notation will not be accepted.

Answers must be simplified in order to receive credit. Only exact answers will be accepted; in particular, decimal approximations will not receive credit. If a problem calls for a list (e.g. "Find all solutions x . . . "), you will be awarded full credit for a complete list in any order, but no partial credit for an incomplete list, as well as no credit for a list with extra elements.

Protests

If you have a protest, email brumo@brown.edu after the contest. All individual round protests must be submitted through email by 2 pm EST.

Reminders

Please fill in your full name, team name, and ID on the answer sheet. Write your answers clearly in the corresponding box on the answer sheet. If you need scratch paper, please raise your hand and we will bring it to you.

1. One hundred concentric circles are labelled $C_1, C_2, C_3, ..., C_{100}$. Each circle C_n is inscribed within an equilateral triangle whose vertices are points on C_{n+1}. Given C_1 has a radius of 1, what is the radius of C_{100}?

2. An infinite geometric sequence with common ratio r sums to 91. A new sequence starting with the same term has common ratio r^3. The sum of the new sequence produced is 81. What was the common ratio of the original sequence?

3. Let A, B, C, D, and E be five equally spaced points on a line in that order. Let F, G, H, and I all be on the same side of line AE such that triangles AFB, BGC, CHD, and DIE are equilateral with side length 1. Let S be the region consisting of the interiors of all four triangles. Compute the length of segment AI that is contained in S.

4. If $5f(x) - xf\left(\frac{1}{x}\right) = \frac{1}{17}x^2$, determine $f(3)$.

5. How many ways are there to arrange $1, 2, 3, 4, 5, 6$ such that no two consecutive numbers have the same remainder when divided by 3?

6. Joshua is playing with his number cards. He has 9 cards of 9 lined up in a row. He puts a multiplication sign between two of the 9s and calculates the product of the two strings of 9s. For example, one possible result is $999 \times 999999 = 998999001$. Let S be the sum of all possible distinct results (note that 999×999999 yields the same result as 999999×999). What is the sum of digits of S?

7. Bruno the Bear is tasked to organize 16 identical brown balls into 7 bins labeled 1-7. He must distribute the balls among the bins so that each odd-labeled bin contains an odd number of balls, and each even-labeled bin contains an even number of balls (with 0 considered even). In how many ways can Bruno do this?

8. Let $f(n)$ be the number obtained by increasing every prime factor in f by one. For instance, $f(12) = (2+1)^2(3+1) = 36$. What is the lowest n such that 6^{2025} divides $f^{(n)}(2025)$, where $f^{(n)}$ denotes the nth iteration of f?

9. How many positive integer divisors of 63^{10} do not end in a 1?

10. Bruno is throwing a party and invites n guests. Each pair of party guests are either friends or enemies. Each guest has exactly 12 enemies. All guests believe the following: the friend of an enemy is an enemy. Calculate the sum of all possible values of n. (Please note: Bruno is not a guest at his own party)

11. In acute $\triangle ABC$, let D be the foot of the altitude from A to BC and O be the circumcenter. Suppose that the area of $\triangle ABD$ is equal to the area of $\triangle AOC$. Given that $OD = 2$ and $BD = 3$, compute AD.

12. Alice has 10 gifts $g_1, g_2, \ldots, g_{10}$ and 10 friends $f_1, f_2, \ldots, f_{10}$. Gift g_i can be given to friend f_j if

$$i - j = -1, 0, \text{ or } 1 \pmod{10}$$

How many ways are there for Alice to pair the 10 gifts with the 10 friends such that each friend receives one gift?

13. Let $\triangle ABC$ be an equilateral triangle with side length 1. A real number d is selected uniformly at random from the open interval $(0, 0.5)$. Points E and F lie on sides AC and AB, respectively, such that $AE = d$ and $AF = 1 - d$. Let D be the intersection of lines BE and CF.

    Consider line ℓ passing through both points of intersection of the circumcircles of triangles $\triangle DEF$ and $\triangle DBC$. O is the circumcenter of $\triangle DEF$. Line ℓ intersects line $\overleftrightarrow{BC}$ at point P, and point Q lies on AP such that $\angle AQB = 120°$. What is the probability that the line segment $\overline{QO}$ has length less than $\frac{1}{3}$?

14. Define sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_1 = \frac{\pi}{3}$ and $a_{n+1} = \cot^{-1}(\csc(a_n))$ for all positive integers n. Find the value of
$$\frac{1}{\cos(a_1)\cos(a_2)\cos(a_3)\cdots\cos(a_{16})}.$$

15. Define $\{x\}$ to be the fractional part of x. For example, $\{20.25\} = 0.25$ and $\{\pi\} = \pi - 3$. Let $A = \displaystyle\sum_{a=1}^{96}\sum_{n=1}^{96}\left\{\frac{a^n}{97}\right\}$, where $\{x\}$ denotes the fractional part of x. Compute A rounded to the nearest integer.