1. One hundred concentric circles are labelled $C_1, C_2, C_3, ..., C_{100}$. Each circle C_n is inscribed within an equilateral triangle whose vertices are points on C_{n+1}. Given C_1 has a radius of 1, what is the radius of C_{100}?

   Proposed by: Amber Bajaj

   Answer: $\boxed{2^{99}}$

   Solution: Consider the nth circle of radius r_n, inscribed within an equilateral triangle, which is inscribed within the $n+1$th circle of radius r_{n+1}. The triangle intersects the smaller circle at 3 points, splitting the circle into sectors of 120 degrees. Additionally, the interior angles of the triangle are 60 degrees. Marking these interior angles and radii r_n, we get a kite, whose height is r_{n+1}. Splitting the kite in half, we find a 30-60-90 triangle, finding that $\sin(30°) = r_n/r_{n+1}$. This implies $r_{n+1} = r_n/\sin(30°) = 2r_n$. Thus, for all $r_n, r_{n+1} = 2r_n$. Since $r_1 = 1 = 2^0$, then $r_2 = 2^1, r_3 = 2^2, \ldots, r_{100} = 2^{99}$.

2. An infinite geometric sequence with common ratio r sums to 91. A new sequence starting with the same term has common ratio r^3. The sum of the new sequence produced is 81. What was the common ratio of the original sequence?

   Proposed by: Jaeho Lee

   Answer: $\boxed{\frac{1}{9}}$

   Solution: Let a be the first term and r the common ratio of the original infinite geometric series, so that its sum is given by

$$\frac{a}{1-r} = 91.$$

This implies $a = 91(1-r)$. When the common ratio is cubed, the new series has a sum of

$$\frac{a}{1-r^3} = 81.$$

Substituting $a = 91(1-r)$ into the equation for the new series gives

$$\frac{91(1-r)}{1-r^3} = 81.$$

Since $1-r^3$ factors as $(1-r)(1+r+r^2)$, we can cancel the $(1-r)$ term (assuming $r \neq 1$) to obtain

$$\frac{91}{1+r+r^2} = 81.$$

Multiplying both sides by $1+r+r^2$ yields

$$91 = 81(1+r+r^2),$$

or equivalently

$$1 + r + r^2 = \frac{91}{81}.$$

Multiplying both sides by 81, we get

$$81r^2 + 81r + 81 = 91,$$

which simplifies to

$$81r^2 + 81r - 10 = 0.$$

This factors as $(9r-1)(9r+10) = 0$, so $r = \frac{1}{9}$ or $r = -\frac{10}{9}$. Since the series converges only when $|r| < 1$, the common ratio must be $\frac{1}{9}$.

3. Let A, B, C, D, and E be five equally spaced points on a line in that order. Let F, G, H, and I all be on the same side of line AE such that triangles AFB, BGC, CHD, and DIE are equilateral with side length 1. Let S be the region consisting of the interiors of all four triangles. Compute the length of segment AI that is contained in S.

   Proposed by: Max Liu

   Answer: $\boxed{\dfrac{\sqrt{13}}{2}}$

   Solution: Ignore point E. Then $ADIF$ is a parallelogram, and if we rotate the diagram by 180 degrees about the center of the parallelogram, we find that A goes to I, B goes to H, C goes to G, and E goes to F. By symmetry, the length of segment IA that is contained in the interior of triangles IDH, HCG, and GBF is equal to the length of segment AI that is contained in AFB, BGC, and CHD. Therefore, the answer is just half the length of AI. By the law of cosines on triangle $ADI, AI = \sqrt{3^2 + 1^3 + 3 \cdot 1} = \sqrt{13}$, so the answer is $\frac{\sqrt{13}}{2}$.

4. If $5f(x) - xf\left(\frac{1}{x}\right) = \frac{1}{17}x^2$, determine $f(3)$.

   Proposed by: Amber Bajaj

   Answer: $\boxed{\dfrac{1}{9}}$

   Solution: To determine $f(3)$, we substitute $x = 3$ and $x = \frac{1}{3}$ into the equation and solve using elimination.

   Substitute $x = 3$

$$5f(3) - 3f\left(\frac{1}{3}\right) = \frac{1}{17}(3^2) = \frac{9}{17}. \tag{1}$$

   Substitute $x = \frac{1}{3}$

$$5f\left(\frac{1}{3}\right) - \frac{1}{3}f(3) = \frac{1}{17}\left(\frac{1}{3}\right)^2 = \frac{1}{153}. \tag{2}$$

Eq (1) $\times 5$:

$$25f(3) - 15f\left(\frac{1}{3}\right) = \frac{45}{17}.$$

Eq (2) $\times 3$:

$$15f\left(\frac{1}{3}\right) - f(3) = \frac{3}{153} = \frac{1}{51}.$$

Add the two equations:

$$25f(3) - f(3) = \frac{45}{17} + \frac{1}{51}.$$
$$24f(3) = \frac{8}{3}.$$
$$f(3) = \frac{1}{9}.$$

5. How many ways are there to arrange $1, 2, 3, 4, 5, 6$ such that no two consecutive numbers have the same remainder when divided by 3?

   Proposed by: An Cao

   Answer: $\boxed{240}$

   Solution: Let a, b, and c represent different equivalence classes modulo 3. The following are all possible configurations mod 3:

   $$(a, b, a, c, b, c), (a, b, c, a, b, c), (a, b, c, a, c, b), (a, b, c, b, a, c), (a, b, c, b, c, a)$$

   For each configuration, there are $3! = 6$ ways to choose (a, b, c) as a permutation of $(0, 1, 2)$ and $2^3 = 8$ ways to order the pair of numbers of each modulo class. In total, there are $5 \cdot 6 \cdot 8 = \mathbf{240}$ ways.

6. Joshua is playing with his number cards. He has 9 cards of 9 lined up in a row. He puts a multiplication sign between two of the 9s and calculates the product of the two strings of 9s. For example, one possible result is $999 \times 999999 = 998999001$. Let S be the sum of all possible distinct results (note that 999×999999 yields the same result as 999999×999). What is the sum of digits of S?

   Proposed by: Alice Wang

   Answer: $\boxed{72}$

   Solution: Note that each number in the equation can be expressed as $10^n - 1$ for some n. Thus the product of the equation is in the form of

   $$(10^n - 1)(10^m - 1) = 10^{n+m} - 10^n - 10^m + 1$$

   where (n, m) are integer partitions of 9, so we always have $n + m = 9$. Since we only want to sum up distinct results, we just sum over $(n, m) = (1, 8), (2, 7), (3, 6)$, and $(4, 5)$. It follows that the total sum of all possible results is

   $$4 \cdot 10^9 - \sum_{n=1}^{4} 10^n - \sum_{m=5}^{8} 10^m + 4.$$

It is easy to calculate this to be 3888888894. The digit sum of this number is 72.

7. Bruno the Bear is tasked to organize 16 identical brown balls into 7 bins labeled 1-7. He must distribute the balls among the bins so that each odd-labeled bin contains an odd number of balls, and each even-labeled bin contains an even number of balls (with 0 considered even). In how many ways can Bruno do this?

   Proposed by: Sarah Bao

   Answer: $\boxed{924}$

   Solution: Recall that an even number is an integer of the form $n = 2k$, and an odd number is an integer of the form $n = 2k + 1$. We use this fact to ensure that each bin ends up with the desired parity.

   Begin by placing 1 ball in each odd-labeled bin (accounting for the "+1" in $2k + 1$). So now all odd-labeled bins have 1 ball (an odd number), and all even-labeled bins have 0 balls (an even number). With the remaining 12 balls, pair them up into 6 pairs. Then, no matter how we distribute these 6 pairs into the bins, the parity of each bin will remain the same. Thus, the problem reduces to distributing 6 identical pairs among 7 distinct bins, which can be counted using stars and bars, giving us $\binom{6+6}{6} = \binom{12}{6} = \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7}{6\cdot5\cdot4\cdot3\cdot2\cdot1} = 924$ ways.

8. Let $f(n)$ be the number obtained by increasing every prime factor in f by one. For instance, $f(12) = (2+1)^2(3+1) = 36$. What is the lowest n such that 6^{2025} divides $f^{(n)}(2025)$, where $f^{(n)}$ denotes the nth iteration of f?

   Proposed by: Ethan Bove

   Answer: $\boxed{20}$

   Solution: We factor 2025 as $3^4 \cdot 5^2$ and write out the first few values of $f^{(n)}(2025)$:

$$\begin{aligned}
f^1(2025) &= 4^4 \cdot 6^2 = 2^{10} \cdot 3^2\\
f^2(2025) &= 3^{10} \cdot 4^2 = 2^4 \cdot 3^{10}\\
f^3(2025) &= 3^4 \cdot 4^{10} = 2^{20} \cdot 3^4\\
f^4(2025) &= 3^{20} \cdot 4^4 = 2^8 \cdot 3^{20}\\
f^5(2025) &= 3^8 \cdot 4^{20} = 2^{40} \cdot 3^8.
\end{aligned}$$

   We can see that $f^n(2025) = 2^{2\cdot2^{n/2}}3^{5\cdot2^{n/2}}$ when n is even and $f^n(2025) = 2^{5\cdot2^{(n+1)/2}}3^{2^{(n+1)/2}}$. We need each exponent to be at least 2025.
   If n is even, we need $2 \cdot 2^{n/2} \geq 2025$ and $5 \cdot 2^{n/2} \geq 2025$, which yields $n = 20$. If n is odd, we need $5 \cdot 2^{(n+1)/2} \geq 2025$ and $2^{(n+1)/2} \geq 2025$, which yields $n = 21$. Therefore, the answer is 20.

9. How many positive integer divisors of 63^{10} do not end in a 1?

   Proposed by: Sebastian Weinberger

   Answer: $\boxed{173}$

   Solution:

$$63^{10} = 3^{20}7^{10}$$

Note that $3^n \pmod{10}$ follows the cycle

$$1 \to 3 \to 9 \to 7$$

and $7^m \pmod{10}$ follows the cycle

$$1 \to 7 \to 9 \to 3$$

Thus $3^n 7^m = 1 \pmod{10} \iff n = m \pmod 4$. We consider each case:

$$n = m = 0 \pmod 4$$

There are 6 choices for n and 3 choices for m for a total of $6 \cdot 3 = 18$ divisors.

$$n = m = 1 \pmod 4$$

There are 5 choices for n and 3 choices for m for a total of $5 \cdot 3 = 15$ divisors.

$$n = m = 2 \pmod 4$$

There are 5 choices for n and 3 choices for m for a total of $5 \cdot 3 = 15$ divisors.

$$n = m = 3 \pmod 4$$

There are 5 choices for n and 2 choices for m for a total of $5 \cdot 2 = 10$ divisors. Thus there are a total of

$$18 + 15 + 15 + 10 = 58$$

divisors ending in 1. Thus the number of divisors not ending in 1 is

$$(20 + 1)(10 + 1) - 58 = 173.$$

10. Bruno is throwing a party and invites n guests. Each pair of party guests are either friends or enemies. Each guest has exactly 12 enemies. All guests believe the following: the friend of an enemy is an enemy. Calculate the sum of all possible values of n. (Please note: Bruno is not a guest at his own party)

Proposed by: Amber Bajaj

Answer: $\boxed{100}$

Solution: Suppose a and b are friends. Let c be friends with a. Then c must also be friends with b; otherwise, c would be friends with one of a's enemies, a contradiction. Therefore, all of a's friends (except for b) must be friends with b. Similarly, all of b's friends (except for a) must be friends with a.

Using this observation, we can deduce that people must form "friend groups," where everyone in the group is only friends with everyone else in the group, and everyone has the same 12 enemies in common. It follows that each friend group has $n - 12$ people. Since friend groups are disjoint, the total number of people is equal to $(n - 12) \times$ (number of friend groups).

This tells us that $n - 12$ divides n, so $n - 12$ divides $n - (n - 12) = 12$. The possible values of $n - 12$ are $1, 2, 3, 4, 6$, and 12, which correspond to $n = 13, 14, 15, 16, 18$, and 24. The answer is 100.

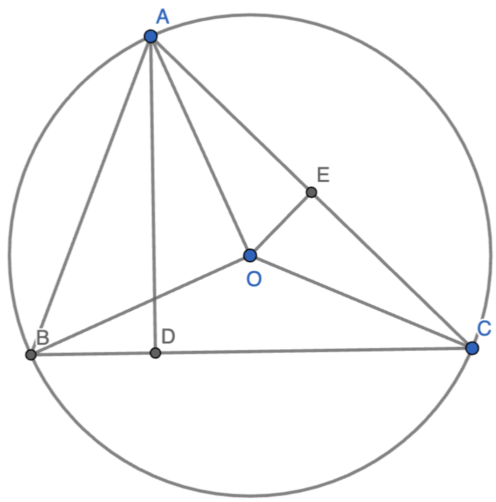
To construct an assignment of friends and enemies, we divide n into $\frac{n}{n-12}$ friend groups, each containing $n - 12$ friends. People in different friend groups are enemies. One can check that this satisfies the constraint.

11. In acute $\triangle ABC$, let D be the foot of the altitude from A to BC and O be the circumcenter. Suppose that the area of $\triangle ABD$ is equal to the area of $\triangle AOC$. Given that $OD = 2$ and $BD = 3$, compute AD.

    Proposed by: Max Liu

    Answer: $\boxed{3 + 2\sqrt{2}}$

    Solution:



Let E be the foot of the altitude from O to AC. We prove that $\triangle ABD \sim \triangle AOE$.

Consider the circumcircle of $\triangle ABC$. Since $\angle ABC$ subtends minor arc AC, we know that $\angle ABC = \frac{1}{2} = \angle AOC = \angle AOE$. We also have $\angle ADB = \angle AEO = 90°$, so these triangles are similar.

Since the ratio of the area of $\triangle AOE$ to the area of $\triangle ABD$ is $\frac{1}{2}$, their side lengths have ratio $\frac{\sqrt{2}}{2}$. Let the circumradius of $\triangle ABC$ be R. Then, $AO = BO = R$. Since $\frac{AO}{AB} = \frac{\sqrt{2}}{2}$, we have $AB = R\sqrt{2}$.

We can see that AOB is a $45-45-90$ right triangle. Then, quadrilateral $AODB$ is cyclic, so by Ptolemy's theorem, $AB \cdot OD + AO \cdot BD = AD \cdot BO$. Plugging the known lengths in, we have

$$R\sqrt{2} \cdot 2 + R \cdot 3 = AD \cdot R.$$

Dividing by R yields $AD = 3 + 2\sqrt{2}$.

12. Alice has 10 gifts $g_1, g_2, \ldots, g_{10}$ and 10 friends $f_1, f_2, \ldots, f_{10}$. Gift g_i can be given to friend f_j if

$$i - j = -1, 0, \text{ or } 1 \pmod{10}$$

How many ways are there for Alice to pair the 10 gifts with the 10 friends such that each friend receives one gift?

Proposed by: Joshua Kou

Answer: $\boxed{125}$

Solution: Assume every subscript is given mod 10. Let $R(i)$ denote the recipient of gift g_i. For any number c, observe that if $R(c) = c+1$ and $R(c+1) = c+2$ then $R(c+2)$ has no choice except $c+3$ and so on. Consequently $R(i) = i+1$ for all i. The same occurs if $R(c) = c-1$ and $R(c-1) = c-2$, then consequently $R(i) = i-1$ for all i. Thus there are two cases where the indices of the recipients are all shifted one forward or all shifted one backward. In all other cases, there are only gift swaps, where $R(c) = c+1$ and $R(c+1) = c$, and fixed points, where $R(c) = c$.

We can think of the problem as tiling a circle with 10 spaces using pieces of length 1 and 2, where a pieces of length 1 corresponds to a fixed point, and a piece of length 2 corresponds to a gift swap. Let F_n denote the nth Fibonacci number. Note that in the linear case, there are F_{n+1} ways to tile n spaces with pieces of length 1 and 2, which can be observed by recursion. To reduce the circular case to linear cases, we do casework on whether $R(1) = 10$, concluding that there are F_9 possibilities when $R(1) = 10$ and F_{11} possibilities when $R(1) \neq 10$. Thus the final answer is

$$2 + F_9 + F_{11} = \boxed{125}$$

13. Let $\triangle ABC$ be an equilateral triangle with side length 1. A real number d is selected uniformly at random from the open interval $(0, 0.5)$. Points E and F lie on sides AC and AB, respectively, such that $AE = d$ and $AF = 1 - d$. Let D be the intersection of lines BE and CF.

Consider line ℓ passing through both points of intersection of the circumcircles of triangles $\triangle DEF$ and $\triangle DBC$. O is the circumcenter of $\triangle DEF$. Line ℓ intersects line $\overleftrightarrow{BC}$ at point P, and point Q lies on AP such that $\angle AQB = 120°$. What is the probability that the line segment $\overline{QO}$ has length less than $\frac{1}{3}$?

Proposed by: Joey Chun

Answer: $\boxed{\dfrac{1}{3}}$

Solution: The length condition tells us that $BF = AE = d$, so $\triangle BCF \cong \triangle ABE$. This gives us $\angle BCF = \angle ABE$, so $\angle BCF + \angle DBC = \angle BCF + (60° - \angle ABE) = 60°$. Then $\angle BDC = 180° - 60° = 120°$, so $\angle EDF = 120°$.

Since $\angle EDF + \angle EAF = 180°$, we know that $AEDF$ is cyclic. Let Q' be the intersection of the circumcircle of $\triangle ABC$ and the circumcircle of $AEDF$ other than A. By the radical axis theorem, $\overline{AQ'}, l$, and $\overline{BC}$ intersect at P. This reveals that Q' lies on AP.

Since Q' is on minor arc AB on the circumcircle of $\triangle ABC$, we know that $\angle AQ'B = 180° - \angle C = 120°$. This proves that $Q' = Q$, so we know that OQ is just the circumradius of $AEDF$. When we let R be the circumradius of $AEDF$, our goal is to find when $R < \frac{1}{3}$.

Now the problem becomes solvable with trigonometry bash. By the law of cosines, $EF = \sqrt{d^2 + (1-d)^2 - d(1-d)} = \sqrt{3d^2 - 3d + 1}$. By the law of sines, $\frac{EF}{\sin A} = 2R$, so $R = \sqrt{\frac{3d^2 - 3d + 1}{3}}$. We compute when $R < \frac{1}{3}$:

$$\sqrt{\frac{3d^2 - 3d + 1}{3}} < \frac{1}{3}$$

$$\begin{aligned}3d^2 - 3d + 1 &< \frac{1}{3}\\ 9d^2 - 9d + 2 &< 0\\ (3d-1)(3d-2) &< 0.\end{aligned}$$

This means that $R < \frac{1}{3}$ when $d \in (\frac{1}{3}, \frac{2}{3})$.

(Alternatively, a more efficient approach avoids some computation by observing that R is a decreasing function of d, and that when $d = \frac{1}{3}$, we have $R = \frac{1}{3}$ because $\triangle AEF$ becomes a $30-60-90$ triangle. This implies that $R < \frac{1}{3}$ when $d > \frac{1}{3}$.)
Since d is chosen from uniformly at random from $(0, 0.5)$, the probability that $R < \frac{1}{3}$ is $\frac{1}{3}$.

14. Define sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_1 = \frac{\pi}{3}$ and $a_{n+1} = \cot^{-1}(\csc(a_n))$ for all positive integers n. Find the value of

$$\frac{1}{\cos(a_1)\cos(a_2)\cos(a_3)\cdots\cos(a_{16})}.$$

Proposed by: Leo Zhang

Answer: $\boxed{7}$

Solution: Notice that $a_{n+1} = \cot^{-1}(\csc(a_n))$, so $\cot(a_{n+1}) = \csc(a_n)$. Therefore, we have $\cot^2(a_{n+1}) = \csc^2(a_n)$. Thus, $\cot^2(a_{n+1}) = \cot^2(a_n) + 1$. So we can get

$$\cot^2(a_n) = (n-1) + \cot^2(a_1),$$

for $\forall n \in \mathbb{N}$. Because $a_1 = \frac{\pi}{3}$, $\cot^2(a_n) = (n-1) + \cot^2(\frac{\pi}{3}) = n - 1 + \frac{1}{3} = \frac{3n-2}{3}$. By the definition of a_n, we have $a_n \in (0, \pi)$. Thus, $\csc(a_n) > 0$. Since $\cot(a_{n+1}) = \csc(a_n)$, $\cot(a_{n+1}) > 0$. Together with, $\cot^2(a_n) = \frac{3n-2}{3}$, we have $\cot(a_n) = \sqrt{\frac{3n-2}{3}}$. Now, notice that

$$\begin{aligned}\cos(a_1)\cos(a_2)\cos(a_3)\cdots\cos(a_{16}) &= \frac{\cot(a_1)}{\csc(a_1)} \cdot \frac{\cot(a_2)}{\csc(a_2)} \cdots \frac{\cot(a_{16})}{\csc(a_{16})}\\ &= \frac{\cot(a_1)}{\csc(a_{16})}\\ &= \frac{\cot(a_1)}{\cot(a_{17})}\\ &= \frac{\sqrt{3}}{3} \cdot \sqrt{\frac{3}{49}}\\ &= \frac{1}{7}\end{aligned}$$

As a result,

$$\frac{1}{\cos(a_1)\cos(a_2)\cos(a_3)\cdots\cos(a_{16})} = \boxed{7}.$$

15. Define $\{x\}$ to be the fractional part of x. For example, $\{20.25\} = 0.25$ and $\{\pi\} = \pi - 3$. Let $A = \sum_{a=1}^{96}\sum_{n=1}^{96}\left\{\frac{a^n}{97}\right\}$, where $\{x\}$ denotes the fractional part of x. Compute A rounded to the nearest integer.

Proposed by: Max Liu

Answer: $\boxed{4529}$

Solution: We use the fact that the multiplicative order of an element modulo 97 must divide $\phi(97) = 96 = 3 \cdot 2^5$. Based on this factorization, the order can be 1, 3, or an even number. We evaluate $\displaystyle\sum_{n=1}^{96} \left\{ \frac{a^n}{97} \right\}$ in each of these three cases.

Case 1: $\text{ord}_n(a) = 1$. In this case, a must be 1, so the summation equals $96 \cdot \frac{1}{97} = \frac{96}{97}$.

Case 2: $\text{ord}_n(a) = 3$. We compute

$$\left\{ \frac{a^1}{97} \right\} + \left\{ \frac{a^2}{97} \right\} + \left\{ \frac{a^3}{97} \right\} = \left\{ \frac{a^1}{97} \right\} + \left\{ \frac{a^2}{97} \right\} + \left\{ \frac{1}{97} \right\}.$$

Since a has order 3, we must have $a^3 - 1 \equiv 0 \pmod{97}$, so $(a-1)(a^2+a+1) \equiv 0 \pmod{97}$. Since $a \neq 1$, we must have $a^2 + a + 1 \equiv 0 \pmod{97}$ since 97 is prime. Therefore, the sum of the three fractional terms must be an integer.

The sum is greater than 0 because each of the terms is positive, and the sum is less than 2 because $\frac{96}{97} + \frac{96}{97} + \frac{1}{97} < 2$. Therefore, the sum must be 1. Since $a^4 = a, a^5 = a^2, a^6 = 1, \ldots$, the sum of every group of three consecutive terms is 1. We can form 32 groups of 3 consecutive terms, so the total is 32. Since there are two elements with order 3, the total in this case is $2 \cdot 32 = 64$.

Case 3: $\text{ord}_n(a)$ is even. Let $\text{ord}_n(a) = 2k$. Then $a^{2k} \equiv 1 \pmod{97}$ implies that $(a^k+1)(a^k-1) \equiv 0 \pmod{97}$. Since we cannot have $a^k \equiv 1 \pmod{97}$, we must have $a^k \equiv -1 \pmod{97}$. It follows that

$$\left\{ \frac{a^i}{97} \right\} + \left\{ \frac{a^{k+i}}{97} \right\} = \left\{ \frac{a^i}{97} \right\} + \left\{ \frac{97 - a^i}{97} \right\} = 1.$$

Then, the sum of the first $2k$ terms is

$$\left(\left\{ \frac{a^1}{97} \right\} + \left\{ \frac{a^{k+1}}{97} \right\} \right) + \left(\left\{ \frac{a^2}{97} \right\} + \left\{ \frac{a^{k+2}}{97} \right\} \right) + \cdots + \left(\left\{ \frac{a^k}{97} \right\} + \left\{ \frac{a^{2k}}{97} \right\} \right) = k.$$

Since $a^{2k+1} = a^1, a^{2k+2} = a^2, \ldots$, the sum of every group of $2k$ consecutive terms is k. We can form $\frac{96}{2k}$ groups of $2k$ consecutive terms, so the total is $\frac{96}{2k} \cdot k = 48$. Since there are $96 - 1 - 2 = 93$ elements with even order, the total in this case is $93 \cdot 48 = 4464$.

Adding up all the cases, we have $A = 4528 + \frac{96}{97}$, so the answer is 4529.