NOVEMBER 17, 1998

1A *Time: 4 minutes*

How many different natural numbers between 10 and 200 have the sum of their digits equal to 6, if zero is not a digit of any of the numbers?

1B *Time: 5 minutes*

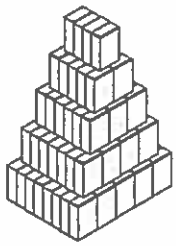
The perimeter of a rectangle is 26 units. Each of the length and width of the rectangle is measured in natural numbers. What is the largest area in square units that the rectangle could have?

1C *Time: 5 minutes*

Bryan can buy candy canes at 4 for 50¢ and can sell them at 3 for 50¢. How many canes must Bryan sell in order to make a profit of $5.00?

1D *Time: 5 minutes*

A supermarket clerk makes a solid pyramid out of identical cereal boxes. The top five layers are shown. What is the total number of cereal boxes in the top five layers as shown?



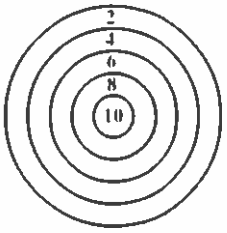
1E *Time: 6 minutes*

A normal duck has two legs. A lame duck has one leg. A sitting duck has no legs. Donald has 33 ducks. He has two more normal ducks than lame ducks and two more lame ducks than sitting ducks. How many legs in all do the 33 ducks have?

2A *Time: 5 minutes*

Roberta throws five darts at the target shown. Each dart lands in a region of the target, scoring the points shown. Of the following total scores, list all that are *not* possible:

6, 14, 17, 38, 42, 58



2B *Time: 4 minutes*

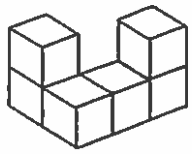
Kelly made two purchases. She gave one cashier $20 for a compact disc and received $6 change. Then she gave another cashier $15 for a bracelet and received $3 change. After these purchases she had $28. How many dollars did she have before buying the compact disc and the bracelet?

2C *Time: 5 minutes*

On a 100 cm measuring stick, marks are made at 19, N, and 99 cm, from left to right. The distance between the marks at N and 99 cm is three times the distance between the marks at N and 19 cm. What number is N?

2D *Time: 5 minutes*

A cube has 6 faces: top, bottom, and all 4 sides. The object shown is made of six congruent cubes. Not all faces are visible. All outer faces of the object including the bottom are painted blue. How many faces of the cubes are painted blue?



2E *Time: 7 minutes*

Assume that a post office issues only 3¢ and 8¢ stamps and all postage is in whole numbers of cents. What is the largest amount of postage in cents which *cannot* be made using only 3¢ and 8¢ stamps?

3A *Time: 4 minutes*

Suppose Sandy writes every whole number from 1 to 100 properly and without skipping any numbers. How many times will Sandy write the digit "2"?

3B *Time: 4 minutes*

Paul has half as many pieces of candy as Jennifer. Jennifer has half as many as Charles. Charles has 12 times as many as Susan. Susan has 4 pieces. How many pieces do Charles and Paul have altogether?

3C *Time: 5 minutes*

Consider all pairs of natural numbers whose sum is less than 11. The two members of a pair could be either the same as each other or different. How many different products are possible if the two numbers are multiplied?

3D *Time: 6 minutes*

Mary's rectangular garden and Kevin's rectangular garden each have the same area, 36 square meters. Each side is measured in whole meters. Mary's garden is 1 m wider than Kevin's garden, but Kevin's garden is 3 m longer than Mary's garden. How wide is Mary's garden, in meters?

3E *Time: 7 minutes*

Each of eight traffic lights on Main Street shows green for 2 minutes, then switches to other colors. The traffic lights turn green 10 seconds apart, from the first light to the eighth light. From the time that the first light turns green until it switches to another color, for how many seconds will all eight lights show green at the same time?

4A *Time: 4 minutes*

Each of PQ and RS represents a 2-digit number. Different letters represent different digits, chosen from 6, 7, 8, and 9. What is the largest product that PQ × RS can have?

4B *Time: 4 minutes*

Exactly 10 disks are in a bowl. Each is marked with a different natural number selected from 1 through 10. Gina and Monique each selects 5 disks. Two of Gina's disks are marked 2 and 8. Two of Monique's disks are marked 7 and 9. What is the largest sum that Gina's disks can have?

4C *Time: 5 minutes*

A "fast" clock gains time at the same rate every hour. It is set to the correct time at 10 AM. When the fast clock shows 11 AM the same day, the correct time is 10:52 AM. When the fast clock shows 3:30 PM that day, what is the correct time?

4D *Time: 6 minutes*

All counting numbers are arranged in the triangular pattern as shown by the first four rows. What is the first number in the 13th row?

```
          1
        2   3   4
      5   6   7   8   9
   10  11  12  13  14  15  16
```

. . . and so on.

4E *Time: 6 minutes*

A 14-digit number **N** is created by writing 8 as both the first and last digits and then placing the 3-digit number 793 between the two 8s four times. What is the remainder when **N** is divided by 7?

ANSWERS AND SOLUTIONS

Note: Number in parentheses indicates percent of all competitors with a correct answer.

CONTEST 1 — NOVEMBER 17, 1998

Answers: [A] 9 [B] 42 (*sq. units*) [C] 120 (*canes*) [D] 100 (*boxes*) [E] 37 (*legs*)

28% correct

1A. *Strategy: List 2- and 3-digit numbers separately.*

2-digit numbers: 15, 24, 33, 42, 51

3-digit numbers: 114, 123, 132, 141

According to conditions of the problem, 6, 60, 105, 150 must be excluded.

There are 9 numbers.

Follow-Ups: (1) If we allow 0 as a digit, and also eliminate the lower and upper limits (10 and 200), what patterns occur in the solution?

(2) What patterns occur if the digit-sum is 2, 3, 4, 5, etc.?

17%

1B *Strategy: Make a table of all possible lengths and widths and compute the areas.*

The perimeter is 26, so the sum of the length and width (called the semiperimeter) is 13.

WIDTH	LENGTH	AREA
1	13 - 1 = 12	1 × 12 = 12
2	13 - 2 = 11	2 × 11 = 22
3	13 - 3 = 10	3 × 10 = 30
4	13 - 4 = 9	4 × 9 = 36
5	13 - 5 = 8	5 × 8 = 40
6	13 - 6 = 7	6 × 7 = 42

No other rectangles are possible.

The largest area that the rectangle might have is 42 sq units.

Follow-Up: What is the largest whole number area if the perimeter is 18, 22, 30, 24? Find the rule.

[Remind students that a square is a special kind of rectangle.]

5%

1C *Strategy: Assume he buys and sells 12 canes in order to avoid using fractions.*

12 canes cost Bryan 3×50¢ = $1.50.

12 canes sell for 4×50¢ = $2.00.

12 canes bring a profit of $2.00 - $1.50 = $.50.

Since $5.00 = $.50 × **10**, multiply 12 canes by **10** = 120 canes.

Thus **Bryan must sell 120 candy canes.**

Suggestion: To help your students answer the question that was asked, train them to use the original words of the question in writing their answers.

58%

1D *Strategy: Break up the problem by layers, and make a table.*

LAYER	# of ROWS	×	# of BOXES per ROW	=	# of BOXES per LAYER
Top	1	×	4	=	4
2	2	×	5	=	10
3	3	×	6	=	18
4	4	×	7	=	28
5	5	×	8	=	40
			TOTAL	=	100

In all there are 100 cereal boxes.

52%

1E METHOD 1. *Strategy: Compare the number of each type of duck to the number of sitting ducks.*
Donald has 2 more normal ducks than lame ducks and 2 more lame ducks than sitting ducks, so he has 4 more normal ducks than sitting ducks. Suppose we remove 2 lame ducks and 4 sitting ducks. This gives him the same number of each type of duck. He now has a total of 33 – 2 – 4 = 27 ducks. Thus he now has 27 ÷ 3 = 9 of each type of duck.

Originally, he must have had 9 sitting ducks, 9 + 2 = 11 lame ducks, and 11+ 2 = 13 normal ducks. The 9 sitting ducks have 0 legs, the 11 lame ducks have 11 legs and the 13 normal ducks have 26 legs. Hence, **the 33 ducks have a total of 0 + 11 + 26 = 37 legs.**

METHOD 1A. *Strategy: Use algebra.(Increasingly algebra is being introduced into the elementary grades. We will begin to show the use of algebra in solutions as an alternate method.)*
Suppose Donald has S sitting ducks. Then he has S + 2 lame ducks, and S + 4 normal ducks.

1. Then: $S + (S + 2) + (S + 4) = 33$
2. Remove parentheses: $S + S + 2 + S + 4 = 33$
3. Combine like terms: $3S + 6 = 33$
4. Subtract 6 from each side of the equation: $3S = 27$
5. Divide each side of the equation by 3 to isolate S: $S = 9$
6. Add 2 to get the number of lame ducks: $S + 2 = 11$
7. Add another 2 to get the number of normal ducks: $11 + 2 = 13$
8. Thus, 9 + 11 + 13 = 33 ducks have a total of 0 + 11 + 26 = 37 legs.

METHOD 2. *Strategy: Compare the number of legs to the number of ducks.*
Pair each sitting duck with a normal duck. In this group there are exactly as many legs as ducks. Remember that four normal ducks are not paired.

Next add the lame ducks to the group, which then consists of 29 ducks with a total of 29 legs. Now add the unpaired four normal ducks to the group. This results in 33 ducks with 29 + 8 = 37 legs. Hence, **the 33 ducks have a total of 37 legs.**

METHOD 3. *Strategy: Make a table beginning with 1 sitting duck.*

	SITTING	LAME	NORMAL	TOTAL # of DUCKS	
	1	3	5	9	(This total lacks 24 ducks)
(Add 8 ducks to each column)	9	11	13	33	

The 33 ducks have a total of 0 + 11 + 26 = 37 legs.

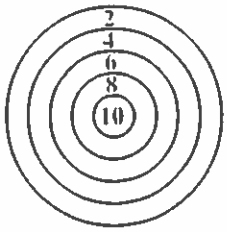
CONTEST 2 — DECEMBER 15, 1998

Answers: [A] 6, 17, 58 [B] 54 ($54) [C] 39 [D] 26 (faces) [E] 13(¢)

49%

2A *Strategy: Try to duplicate each total separately.*

The minimum score is 5 × 2 = 10. Therefore 6 is not possible.
The maximum score is 5 × 10 = 50. Therefore 58 is not possible.
Since each of the target scores are even, 17 is not possible.
The remaining scores are all possible. One way is shown for each:
14 = 2+2+2+2+4; 38 = 10+10+10+6+2; 42 = 10+10+10+10+2.
Thus, **6, 17, and 58 are not possible.**



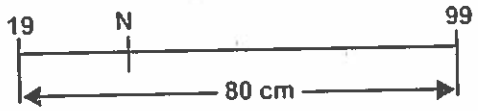
40%

2B *Strategy: Do one thing at a time.*

The CD costs 20 – 6 = $14 and the bracelet costs 15 – 3 = $12, so Kelly spent 14 + 12 = $26.
Then $26 spent plus $28 remaining = $54 to start.
Kelly had $54 before buying the CD and the bracelet.

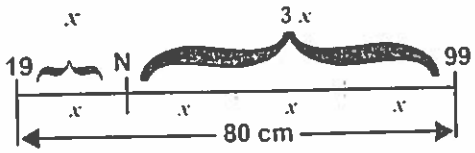
22%

2C METHOD 1: *Strategy: Use the definition of "three times".*



The interval from 19 to 99 is 80 cm. Since the distance from 19 to N is three times the distance from N to 99, picture the latter distance as consisting of three parts, each equal to the former distance. Then the entire interval from 19 to 99 consists of 1 + 3 = 4 equal parts. Therefore each part contains 80 ÷ 4 = 20 cm. Then, **N is 19 + 20 = 39.** Check: 3 × 20 = 60 and 99 – 60 = 39.

METHOD 2: *Strategy: Algebra. (Note: As stated after Olympiad 1, algebraic solutions will be offered hereafter as an alternative method.)*



From the diagram: $x + 3x = 80$
$4x = 80$
$x = 20$ Then N = 19 + x = 19 + 20 = 39.

24%

2D METHOD 1: *Strategy: Subtract the number of unpainted faces from the total.*

Each cube has 6 faces, so 6 cubes have 36 faces. There are 5 places where 2 cubes touch face-to-face. Thus a total of 10 faces are *not* painted blue.
Then 36 – 10 = **26 faces of the cubes are painted blue.**

Method 2 on next page.

(2D) **METHOD 2:** *Strategy: Separate the faces by category.*

Number of faces that are painted blue:

top faces - visible	4	
bottom faces - not visible	4	
vertical faces - visible	9	
vertical faces - not visible	9	Total = 26

26 faces of the cubes are painted blue.

Alternately, look at the faces from each of the six directions: from the top 4 faces are visible, from the bottom 4 faces, from the left 4 vertical faces, from the right 4 faces, from the front 5 faces, and from the back 5 faces. A total of 26 faces of the cubes are painted blue.

METHOD 2A: The 4 cubes on the bottom each have 4 painted sides while the two cubes on the top each have 5 exposed sides. Thus, $(4 \times 4) + (2 \times 5) = 16 + 10 = 26$.

4%

2E *Strategy: Check each amount until 3 consecutive values are found.*

Suppose we find three consecutive numbers that can be made. Then the next three consecutive numbers can also be made. All we need to do is add a 3¢ stamp to each. In this way, we can make every amount larger than the first three consecutive amounts.

Small numbers that can be made: 3, 6, 8, 9, 11, 12.
The next numbers that can be made: 14, 15, 16. ($14 = 2\times3 + 8$, $15 = 5\times3$ and $16 = 2\times8$.)

Since we can make 14, 15, and 16, we can make 17, 18, and 19 by adding 3 to each. Then we can make 20, 21, and 22 by adding 3 again to each. In fact, we can make any amount above 13 by adding some multiple of 3 to 14, 15, or 16.

Thus the largest number which cannot be made by some combination of 3 and 8 is 13. **The largest postage which cannot be made is 13¢.**

Follow-Ups: (1) Suppose bolts are sold in packages of 4 and 7 each. How many consecutive numbers of bolts are needed in order to guarantee that all numbers above a certain amount can be made? What is the largest number of bolts which cannot be bought by a combination of packages?

(2) Suppose the packages contain 5 and 6 bolts, or 4 and 5 bolts. Find the general rule for determining consecutive numbers.

CONTEST 3 — JANUARY 12, 1999

Answers: [A] 20 [B] 60 (*pieces*) [C] 19 [D] 4 (*cm*) [E] 50 (*seconds*)

43%

3A *Strategy: Examine each digit separately.*

The digit "2" appears in the ones' place 10 times: 2, 12, 22, . . . , 92.
The digit "2" appears in the tens' place 10 times: 20, 21, 22, . . . , 29.
The digit "2" appears no other times. **Sandy wrote the digit "2" 10 + 10 = 20 times.**

76%

3B *Strategy: Work backwards from Susan's 4 pieces.*
Susan has 4 pieces.
Charles has 12 × 4 = 48 pieces.
Jennifer has ½ × 48 = 24 pieces.
Paul has ½ × 24 = 12 pieces.
Charles and Paul have 48 + 12 = 60 pieces altogether.

5%

3C METHOD 1: *Strategy: Make a table and cross out duplicates.*
Let P and Q represent the two natural numbers.

P	Q	P × Q	
1	1, 2, 3, ...9	1, 2, 3, 4, 5, 6, 7, 8, 9	9 products
2	1, 2, 3, ...8	~~2, 4, 6, 8~~, 10, 12, 14, 16	4 new products
3	1, 2, 3, ..., 7	~~3, 6, 9, 12~~, 15, 18, 21	3 new products
4	1, 2, 3, 4, 5, 6	~~4, 8, 12, 16~~, 20, 24	2 new products
5	1, 2, 3, 4, 5	~~5, 10, 15, 20~~, 25	1 new product

TOTAL = **19 products are possible.**

METHOD 2: *Strategy: Eliminate the impossible.*
The smallest product possible is 1×1 = 1 and the largest is 5×5 = 25. From the 25 possibilities (1 through 25), eliminate those which are (a) prime numbers greater than 9 and (b) multiples of these primes.
(a) Eliminate 11, 13, 17, 19 and 23: (5 products)
(b) Eliminate 22 = 11×2: (1 product)
Then 25 – 5 – 1 = 19 products are possible.

Follow-Up: Two fair dice are tossed. What products are not possible? The answers illustrate both categories listed in method 2, and a third one: 32 is the product of two composites, one of which is larger than 6.

29%

3D METHOD 1: *Strategy: Make a table of all possible dimensions based on an area of 36.*
Find a row in which the width increases by 1 and the length decreases by 3. This occurs when we compare Row 4 against Row 3. Therefore **Mary's garden is 4 m wide.**

Area	= Width	× Length
36	= 1	× 36
36	= 2	× 18
36	= 3	× 12
36	= 4	× 9
36	= 6	× 6

[Kevin's garden is 3 m wide.]
[Mary's garden is 4 m wide.]
[A square is a special kind of rectangle]

METHOD 2: *Strategy: Make 2 tables: change the dimensions and then compare areas.*
Create Table 2 from Table 1 by increasing the width by 1, decreasing the length by 3, and then computing the area. Then row 3 of the table 2 is the only row in which the area is 36 sq m.

Kevin's Garden			*Mary's Garden*		
W	L	Area	W	L	Area
1 ×	36 =	36	2 ×	33 =	66
2 ×	18 =	36	3 ×	15 =	45
3 ×	12 =	36	4 ×	9 =	36
4 ×	9 =	36	5 ×	6 =	30
6 ×	6 =	36	7 ×	3 =	21

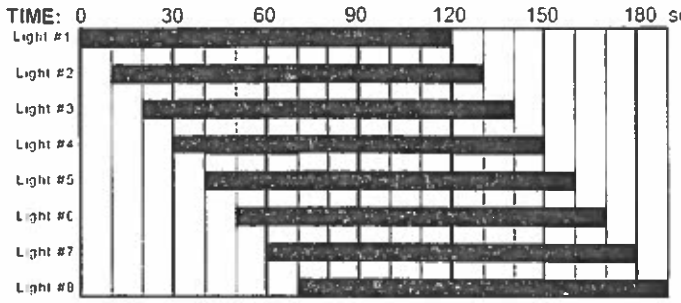
[Mary's garden is 4 m wide.]

23%

3E *Strategy: Track the situation, introducing one traffic light at a time.*

METHOD 1: Measure the time from when traffic light #1 turns green. Light #2 turns green 10 seconds later, light #3 turns green 10 seconds after light #2 and 20 seconds after light #1, light #4 turns green 30 seconds after light #1, and so on. Then **all eight lights will be green at the same time for the next 120 – 70 = 50 seconds**, until light #1 switches to another color after 120 seconds.

METHOD 2: *Strategy: Make a diagram*



Each bar represents a green light.

All lights show green from the 70th second to the 120th second (see shading).

All eight lights will show green at the same time for 50 seconds.

METHOD 3: *Strategy: Make an organized chart*

[Times On and Off = number of seconds measured from the time that Light #1 turns green]

Light #	Times On and Off
1	0 – 120
2	10 – 130
3	20 – 140
4	30 – 150
5	40 – 160
6	50 – 170
7	60 – 180
8	70 – 190

Note the 8-way overlap from the 70th second to the 120th second, a total of 50 seconds.

All eight lights will show green at the same time for 50 seconds.

CONTEST 4 — FEBRUARY 9, 1999

Answers: [A] 8352 [B] 31 [C] 2:46 (*PM*) [D] 145 [E] 4

31%

4A *Strategy: Put the largest digits in the places with the greatest place value.*

To form the largest product, the tens' digits must be 9 and 8. That leaves 7 and 6 for the ones' place. Test each arrangement: 97×86 = 8342; 96×87 = 8352. **The largest product is 8352.**

Follow-Up: Assign different sets of 4 digits. Try to discover a rule for choosing the tens' digits and then a rule for matching up the ones' digits with each tens' digit.

44%

4B *Strategy: Select the highest available values for Gina's disks.*

Gina must draw 10, 6, and 5, and Monique must draw 4, 3, and 1. **The largest sum that Gina's disks can have is 2 + 8 + 10 + 6 + 5 = 31.** The least sum that Monique's disks can have is 7 + 9 + 4 + 3 + 1 = 24.

Continued on next page.

Continued from page 11

(4B) *Follow-Ups: (1) If Gina and Monique play a new round, can their totals be equal? Can their totals form a simple ratio?*

(2) Suppose there are 100 numbered disks, each girl selects half the disks, and Gina draws #2 while Monique draws #98. What is the largest possible sum for Gina's disks? Can the students create a simple general method for this problem?

13%

4C METHOD 1: *Strategy: Compare the elapsed times proportionally.*

At 11 AM (fast clock time), the fast clock has gained 8 minutes.
From 10 AM to 3:30 PM (fast clock time), the fast clock shows 5½ hours of elapsed time.
Therefore, at 3:30 PM (fast clock time), the fast clock has gained 5½ hours × 8 minutes per hour = 44 minutes. Then: **The correct time is 3:30 – 0:44 = 2:46 PM.**

[Note: give credit if an answer omits the "PM"]

METHOD 2: *Strategy: Construct a table and note the differences in times.*

Fast Clock	Correct Time	
10 AM	10:00 AM	
11 AM	10:52 AM	**+52 minutes**
12 Noon	11:44 AM	**+52 minutes**
1 PM	12:36 PM	**+52 minutes**
2 PM	1:28 PM	**+52 minutes**
3 PM	2:20 PM	**+52 minutes**
3:30 PM	2:46 PM	**+½ of 52 = 26 minutes**

From 10:00 to 10:52 is 52 minutes. To 10:00 AM, add 52 minutes for each hour and 26 minutes for the half-hour. The correct time is 2:46 PM.

25%

4D METHOD 1: *Strategy: Look for a pattern and then extend it.*

Look at the last number in each row. It is a perfect square. That is, the last number in row 1 is 1 = 1×1; the last number in row 2 is 4 = 2×2; the last number in row 3 is 9 = 3×3; the last number in row 4 is 16 = 4×4; and so on. The last number in each row is the square of the row number, so the last number in row 12 is 12×12 = 144. Then **the first number in the 13th row is 145.**

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16
. . . And so on.

METHOD 2: *Strategy: Look for a pattern and then extend it.*

Look at the first number in each row: 1, 2, 5, 10, 17, . . .
Their differences are 1, 3, 5, 7, . . .
To the first number add the sum of the 12 differences:
(1) + (1+3+5+7+9+11+13+15+17+19+21+23) = 1 + 144 = 145.
The first number in the 13th row is 145.

Follow-Up: Examine the pattern below for a relationship between the middle number on the left side of each line and the value on the right side of that line.

$1 = 1^2$
$1+3 = 2^2$
$1+3+5 = 3^2$
$1+3+5+7 = 4^2$, etc.